Advances in Differential Equations and Control Processes

Published Online: July 2016
http://dx.doi.org/10.17654/DE017010057
Volume 17, Number 1, 2016, Pages 57-87 **ISSN: 0974-3243**

ON A NONLINEAR WAVE EQUATION WITH MIXED BOUNDARY CONDITIONS OF MANY-POINT TYPE

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Abstract

This paper is devoted to the study of a wave equation with mixed boundary conditions of many-point type. The existence and uniqueness of weak solutions are proved by applying the Galerkin method. In addition, the stability of the weak solution is considered.

1. Introduction

In this paper, we consider the following wave equation:

$$u_{tt} - u_{xx} + f(x, u) + \lambda(t, \| u(t) \|^2) u_t = F(x, t, \| u(t) \|^2), \tag{1.1}$$

associated with mixed boundary conditions

Received: February 8, 2016; Revised: April 21, 2016; Accepted: May 10, 2016
2010 Mathematics Subject Classification: 34A12, 34A30, 34D20.
Keywords and phrases: many-point boundary condition, Galerkin method, Schauder theorem, weak solutions, stability of the weak solutions.
Communicated by K. K. Azad

$$u(1, t) = 0, \tag{1.2}$$

$$u_x(0, t) = g(t) + h(u(\xi_1, t)) + \int_0^t k(t-s, u(\xi_1, s), ..., u(\xi_N, s))ds \tag{1.3}$$

and initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad (x, t) \in (0, 1) \times (0, T), \tag{1.4}$$

where $f, F, g, h, k, \lambda, u_0, u_1$ are given functions and $\xi_1, \xi_2, ..., \xi_N$ are constants such that $0 = \xi_1 < \xi_2 < \cdots < \xi_N < 1$.

The problems of wave equations have been studied by many authors, for example, we can see in [1-3, 5, 6, 10-16]. Below are some typical works.

Rabinowitz [14] gave a procedure to construct the time-periodic solution of the following problem

$$u_{tt} - u_{xx} + f(x, u) = 0, \quad (x, t) \in (0, l) \times \mathbb{R}, \tag{1.5}$$

together with the boundary and periodicity conditions

$$u(0, t) = u(l, t) = 0, \tag{1.6}$$

$$u(x, t+T) = u(x, t), \tag{1.7}$$

where l, T are positive constants and f is a given function.

Nguyen and Giang Vo [11] showed the asymptotic behavior of the solution of the following problem as $\varepsilon \to 0^+$,

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x, t), \tag{1.8}$$

$$u(1, t) = 0, \tag{1.9}$$

$$u_x(0, t) = \int_0^t k(t-s)u(0, s)ds + hu(0, t) + \varepsilon u_t(0, t) + g(t), \tag{1.10}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.11}$$

where $K, \lambda, h, \varepsilon$ are positive constants and f, g, k, u_0, u_1 are given functions.

In [10], Li studied the existence and uniqueness of weak solutions for the Emden-Fowler type wave equation

$$t^2 u_{tt} - u_{xx} = |u|^{p-2} u, \tag{1.12}$$

subject to zero boundary values and initial values

$$u(x, 1) = u_0(x), \quad u_t(x, 1) = u_1(x), \quad (x, t) \in (a, b) \times (1, T), \tag{1.13}$$

where $p > 2$ is a constant and u_0, u_1 are given functions.

Bergounioux et al. [2] considered the linear wave equation given by

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x, t), \tag{1.14}$$

$$u_x(0, t) = v(t), \quad -u_x(1, t) = pu(1, t) + qu_t(1, t), \tag{1.15}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.16}$$

where K, λ, p, q are given constants and f, u_0, u_1 are given functions. Also, u and v satisfy the following Cauchy problem:

$$\begin{cases} v''(t) + \mu^2 v(t) = hu_{tt}(0, t), \\ v(0) = v_0, \ v'(0) = v_1, \end{cases} \tag{1.17}$$

where $\mu > 0$, $h \geq 0$, v_0, v_1 are constants.

Also, Giang Vo [16] obtained the stability of the weak solution to the problem (1.1)-(1.4) in the case of $F = \lambda = 0$ and $k(t, u_1, ..., u_N) = k_1(t)k_2(u_1, ..., u_N)$, where k_1 and k_2 are given functions.

This paper consists of two parts. In Part 1, we show the existence and uniqueness of weak solutions of the problem (1.1)-(1.4) by applying the Galerkin method. Finally, in Part 2, we prove that this solution is stable in the sense of continuous dependence on the given data (f, F, g, h, k, λ). The results are considered as the relative generalization of the works [2, 5, 11, 15, 16].

2. Preliminaries

Firstly, we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, the scalar product and the norm in $L^2(0, 1)$.

Let $u(t), u'(t) = u_t(t), u''(t) = u_{tt}(t), u_x(t)$ and $u_{xx}(t)$ denote $u(x, t)$, $\frac{\partial u}{\partial t}(x, t), \frac{\partial^2 u}{\partial t^2}(x, t), \frac{\partial u}{\partial x}(x, t)$ and $\frac{\partial^2 u}{\partial x^2}(x, t)$, respectively.

Next, we define a closed subspace of the Sobolev space $H^1(0, 1)$ as follows:

$$H = \{u \in H^1(0, 1) : u(1) = 0\}, \tag{2.1}$$

with the following scalar product and norm:

$$\langle u, v \rangle_H = \langle u_x, v_x \rangle \text{ and } \| u \|_H = \| u_x \|. \tag{2.2}$$

Then it is easy to prove the following:

Lemma 2.1. *The embedding* $H \hookrightarrow C^0([0, 1])$ *is compact and*

$$\| v \|_{C^0([0,1])} \le \| v \|_H \le \| v \|_{H^1(0,1)} \le \sqrt{2} \| v \|_H, \quad \forall v \in H. \tag{2.3}$$

Lemma 2.2. *Let* $\varepsilon > 0$. *Then*

$$\| v \|^2_{C^0([0,1])} \le \varepsilon \| v_x \|^2 + \left(1 + \frac{1}{\varepsilon}\right) \| v \|^2, \quad \forall v \in H^1(0, 1). \tag{2.4}$$

Also, we have other lemmas.

Lemma 2.3 (See [16])**.** *Let* $m \in \mathbb{N}$ *and* $\mu_j = (2j - 1)\frac{\pi}{2}$, $j = \overline{1, m}$. *Then*

$$\left| \sum_{j=1}^{m} \frac{\sin \mu_j x}{\mu_j} \right| \le 1 + \frac{4}{\pi}, \quad \forall x \in \mathbb{R}. \tag{2.5}$$

Lemma 2.4 (See [9])**.** *Let u be the weak solution of the following problem*:

$$\begin{cases} u_{tt} - u_{xx} = \chi, \ 0 < x < 1, \ 0 < t < T, \\ u(1, t) = 0, \ u_x(0, t) = v(t), \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \\ u \in L^\infty(0, T; H^1(0, 1)), \ u_t \in L^\infty(0, T; L^2(0, 1)), \\ u(0, \cdot), \ v \in H^1(0, T) \text{ and } \chi \in L^2(0, 1) \times (0, T). \end{cases} \tag{2.6}$$

Then we have

$$\| u'(t) \|^2 + \| u_x(t) \|^2 + 2\int_0^t v(s) u'(0, s) ds$$

$$\geq \| u_1 \|^2 + \| u_{0x} \|^2 + 2\int_0^t \langle \chi(s), u'(s) \rangle ds, \quad \text{a.e. } t \in [0, T]. \tag{2.7}$$

Equality holds in case $u_0 = u_1 = 0$.

3. Existence and Uniqueness of Weak Solutions

We make the following assumptions:

(A_1) $u_0 \in H$ and $u_1 \in L^2(0, 1)$,

(A_2) $g \in W^{1,1}(0, T)$,

(A_3) $\lambda, D_2\lambda \in C^0([0, T] \times \mathbb{R}_+)$, there exists a positive function $\hat{\lambda} \in L^1(0, T)$ such that

$$| \lambda(t, u) | \leq \hat{\lambda}(t), \text{ a.e. } t \in [0, T], \quad \forall u \in \mathbb{R}_+,$$

(A_4) $f, D_2 f \in C^0([0, 1] \times \mathbb{R})$, there exist positive functions $f_1, f_2 \in L^1(0, 1)$ such that

$$\hat{f}(x, u) = \int_0^u f(x, s) ds \geq -f_1(x) u^2 - f_2(x), \quad \text{a.e. } x \in [0, 1], \forall u \in \mathbb{R},$$

(A_5) $F, D_2F, D_3F \in C^0([0, 1] \times [0, T] \times \mathbb{R}_+)$, there exist positive functions $F_1, F_2 \in L^1(0, T; L^2(0, 1))$ such that

$|F(x, t, u)| \leq F_1(x, t)u^{1/2} + F_2(x, t)$, a.e. $(x, t) \in [0, 1] \times [0, T]$, $\forall u \in \mathbb{R}_+$,

(A_6) $h \in C^2(\mathbb{R})$, there exist positive constants h_1 and h_2 such that

$$\hat{h}(u) = \int_0^u h(s)ds \geq -h_1 u^2 - h_2, \quad \forall u \in \mathbb{R},$$

(A_7) $k \in C^2([0, T] \times \mathbb{R}^N)$, there exist positive functions $k_1, k_2 \in W^{1,1}(0, T)$ such that

$$(|k| + |D_1 k|)(t, u_1, ..., u_N) \leq k_1(t)\sum_{i=1}^N |u_i| + k_2(t),$$

a.e. $t \in [0, T]$ and $\forall u_i \in \mathbb{R}$, $i = \overline{1, N}$.

Remark 3.1. (i) We consider the following function:

$$f(x, u) = |u|^{p-2}u - \alpha(x)|u|^{q-2}u - \beta(x)u - \gamma(x),$$

where $p > q \geq 2$ are constants and $\alpha, \beta, \gamma \in C^0([0, 1])$, $\gamma \not\equiv 0$. Then

$$\int_0^u f(x, s)ds = \frac{1}{p}|u|^p - \frac{1}{q}\alpha(x)|u|^q - \frac{1}{2}\beta(x)u^2 - \gamma(x)u.$$

Applying the Young inequality $ab \leq \varepsilon a^r + c_\varepsilon b^{r'}$, with

$$\begin{cases} a = |u|^q, \ b = \dfrac{1}{q}|\alpha(x)|, \ r = \dfrac{p}{q}, \\ r' = \dfrac{p}{p-q}, \ \varepsilon = \dfrac{1}{2p}, \ c_\varepsilon = \dfrac{1}{r'}(\varepsilon r)^{-\frac{r'}{r}}. \end{cases}$$

Then

$$\frac{1}{q}|\alpha(x)||u|^q \leq \frac{1}{2p}|u|^p + 2^{\frac{q}{p-q}}\left(\frac{1}{q} - \frac{1}{q}\right)|\alpha(x)|^{\frac{p}{p-q}}.$$

Therefore,

$$\int_0^u f(x, s)ds \geq -\frac{1}{2}(|\beta(x)|+1)u^2 - \left[2^{\frac{q}{p-q}}\left(\frac{1}{q} - \frac{1}{q}\right)|\alpha(x)|^{\frac{p}{p-q}} + \frac{1}{2}\gamma^2(x)\right].$$

Consequently, f satisfies (A_4).

(ii) (A_4) still holds if f satisfies the following condition:

(A_4') $f \in C^1([0, 1] \times \mathbb{R})$, $f(x, 0) = 0$ and $uf(x, u) \geq 0$, $\forall x \in [0, 1]$, $u \in \mathbb{R}$.

Since $\int_0^u f(x, s)ds \geq 0$, $\forall x \in [0, 1]$, $u \in \mathbb{R}$. But f in (i) does not satisfy (A_4').

Under the above assumptions, we obtain the following theorem:

Theorem 3.2. *Let* (A_1)-(A_7) *hold. Then the problem* (1.1)-(1.4) *has a unique weak solution u such that*

$$\begin{cases} u \in L^\infty(0, T; H), \quad u_t \in L^\infty(0, T; L^2(0, 1)), \\ u(\xi_i, \cdot) \in H^1(0, T), \quad i = \overline{1, N}. \end{cases} \tag{3.1}$$

Remark 3.3. If we replace the term u_t in equation (1.1) or the term $h(u(\xi_1, t))$ in equation (1.3) by the term $\varphi(u_t)$ or the term $h(u(\xi_i, t))$, with $i = \overline{2, N}$, φ is a nonlinear function, respectively, then we have no conclusion about the existence of weak solutions for the problem (1.1)-(1.4). These are open problems.

Proof of Theorem 3.2. The proof consists of steps 1-4.

Step 1. Galerkin approximation. We use a special orthonormal basis of H:

$$\varphi_k(x) = \sqrt{2/(1 + \mu_k^2)} \cos(\mu_k x), \quad \mu_k = (2k - 1)\frac{\pi}{2}, \quad k = 1, 2, \dots. \tag{3.2}$$

We find the approximate solution of the problem (1.1)-(1.4) in the form

$$u_m(x, t) = \sum_{k=1}^m \omega_{mk}(t)\varphi_k(x), \tag{3.3}$$

where the functions $\omega_{mk}(t)$ satisfy the following system of differential equations:

$$\langle u''_m(t), \varphi_k\rangle + \langle u_{mx}(t), \varphi_{kx}\rangle + v_m(t)\varphi_k(0) + \langle f(\cdot, u_m(t)), \varphi_k\rangle$$

$$+ \lambda(t, \|u_m(t)\|^2)\langle u'_m, \varphi_k\rangle = \langle F(\cdot, t, \|u_m(t)\|^2), \varphi_k\rangle, \quad k = \overline{1, m}, \tag{3.4}$$

with

$$\begin{cases} v_m(t) = \int_0^t k(t-s, u_m(\xi_1, s), \dots, u_m(\xi_N, s))ds + h(u_m(\xi_1, t)) + g(t), \\ u_m(0) = u_{0m} = \sum_{k=1}^m a_{mk}\varphi_k \to u_0 \text{ strongly in } H^1(0, 1), \\ u'_m(0) = u_{1m} = \sum_{k=1}^m b_{mk}\varphi_k \to u_1 \text{ strongly in } L^2(0, 1). \end{cases} \tag{3.5}$$

Therefore, this system of the equations is written in the form

$$\begin{cases} \omega''_{mk}(t) + \mu_k^2\omega_{mk}(t) = -\lambda(t, \|u_m(t)\|^2)\omega'_{mk}(t) \\ \qquad - \dfrac{1}{\|\varphi_k\|^2}[v_m(t)\varphi_k(0) + \langle f(\cdot, u_m(t)) - F(\cdot, t, \|u_m(t)\|^2), \varphi_k\rangle], \\ v_m(t) = \int_0^t k(t-s, u_m(\xi_1, s), \dots, u_m(\xi_N, s))ds + h(u_m(\xi_1, t)) + g(t), \\ \omega_{mk}(0) = a_{mk}, \quad \omega'_{mk}(0) = b_{mk}, \quad k = \overline{1, m}. \end{cases} \tag{3.6}$$

Setting $\rho_k(t) = \sin(\mu_k t)/\mu_k$, we easily see that

$$\omega_{mk}(t) = a_{mk}\rho'_k(t) + b_{mk}\rho_k(t) - \int_0^t \rho_k(t-s)\lambda(s, \|u_m(s)\|^2)\omega'_{mk}(s)ds$$

$$- \frac{2}{\varphi_k(0)}\int_0^t ds\int_0^s \rho_k(t-s)k(s-\tau, u_m(\xi_1, \tau), \dots, u_m(\xi_N, \tau))d\tau$$

$$- \frac{2}{\varphi_k(0)}\int_0^t \rho_k(t-s)[h(u_m(\xi_1, s)) + g(s)]ds$$

$$- \frac{2}{\varphi_k^2(0)}\int_0^t \rho_k(t-s)\langle f(\cdot, u_m(s)) - F(\cdot, s, \|u_m(s)\|^2), \varphi_k\rangle ds. \tag{3.7}$$

By (A_1)-(A_7), the system (3.7) has a solution $(\omega_{m1}, \dots, \omega_{mm})$ on an interval $[0, T_m]$ (see [7]). This implies that the solution $u_m(t)$ of the system

(3.4)-(3.5) exists on $[0, T_m]$. The following estimates allow us to take $T_m = T$ (see [4]).

Step 2. A priori estimates. In (3.4), we replace φ_k by

$$u'_m(t) = \sum_{k=1}^{m} \omega'_{mk}(t)\varphi_k,$$

integrating from 0 to *t*, we obtain

$$\begin{aligned}
E_m(t) \le\; & E_m(0) + 2g(0)u_{0m}(0) + 2\hat{h}(u_{0m}(0)) + 2\int_0^1 \hat{f}(x, u_{0m}(x))dx \\
& + 2\int_0^t \hat{\lambda}(s)E_m(s)ds - 2g(t)u_m(0, t) + 2\int_0^t g'(s)u_m(0, s)ds \\
& - 2\hat{h}(u_m(0, t)) - 2\int_0^1 \hat{f}(x, u_m(t))dx \\
& + 2\int_0^t \langle F(\cdot, s, \| u_m(s)\|^2), u'_m(s)\rangle ds \\
& - 2\int_0^t u'_m(0, s)ds\int_0^s k(s-\tau, u_m(\xi_1, \tau), \ldots, u_m(\xi_N, \tau))d\tau \\
=\; & E_m(0) + 2g(0)u_{0m}(0) + 2\hat{h}(u_{0m}(0)) + 2\int_0^1 \hat{f}(x, u_{0m}(x))dx \\
& + 2\int_0^t \hat{\lambda}(s)E_m(s)ds + I_1(t) + I_2(t) + \cdots + I_6(t),
\end{aligned} \tag{3.8}$$

where

$$E_m(t) = \| u'_m(t)\|^2 + \| u_{mx}(t)\|^2. \tag{3.9}$$

We shall estimate, respectively, the following terms on the right-hand side of (3.8).

Estimating $I_1(t)$. Using (3.9) and Lemma 2.1, we have

$$I_1(t) \leq \varepsilon E_m(t) + \frac{1}{\varepsilon}\| g \|^2_{L^\infty(0,\, T)}, \quad \forall \varepsilon > 0. \tag{3.10}$$

Estimating $I_2(t)$. By (A_2), then

$$I_2(t) = 2\int_0^t g'(s) u_m(0,\, s) ds \leq \int_0^t | g'(s) | E_m(s) ds + \| g' \|_{L^1(0,\, T)}. \tag{3.11}$$

Estimating $I_3(t)$. Using Lemma 2.2, we obtain

$$I_3(t) \leq 2h_1 \| u_m(t) \|^2_{C^0([0,1])} + 2h_2$$

$$\leq 2\varepsilon h_1 E_m(t) + 4h_1\left(1 + \frac{1}{\varepsilon}\right)\left(T\int_0^t E_m(s) ds + \| u_{0m} \|^2\right) + 2h_2. \tag{3.12}$$

Estimating $I_4(t)$. Using (A_4), we also obtain from Lemma 2.2 that

$$I_4(t) \leq 2\varepsilon \| f_1 \|_{L^1(0,1)} E_m(t) + 4t\left(1 + \frac{1}{\varepsilon}\right)\| f_1 \|_{L^1(0,1)} \int_0^t E_m(s) ds$$

$$+ 4\left(1 + \frac{1}{\varepsilon}\right)\| u_{0m} \|^2 \| f_1 \|_{L^1(0,1)} + 2\| f_2 \|_{L^1(0,1)}. \tag{3.13}$$

Estimating $I_5(t)$, by (A_5), we get

$$I_5(t) \leq 2\int_0^t (\| u_m(s) \| \langle F_1(s), | u_m'(s) | \rangle + \langle F_2(s), | u_m'(s) | \rangle) ds$$

$$\leq 2\int_0^t \| F_1(s) \| \| u_{mx}(s) \| \| u_m'(s) \| ds$$

$$+ \int_0^t \| F_2(s) \| \| u_m'(s) \|^2 ds + \| F_2 \|_{L^1(0,T;\, L^2(0,1))}$$

$$\leq \int_0^t (2\| F_1(s) \| + \| F_2(s) \|) E_m(s) ds + \| F_2 \|_{L^1(0,T;\, L^2(0,1))}. \tag{3.14}$$

Estimating $I_6(t)$. Using integration by parts, it follows from (A_7) that

$$I_6(t) = -2u_m(0, t)\int_0^t k(t-s, u_m(\xi_1, s), \dots, u_m(\xi_N, s))ds$$

$$+ 2\int_0^t u_m(0, s)k(0, u_m(\xi_1, s), \dots, u_m(\xi_N, s))ds$$

$$+ 2\int_0^t u_m(0, s)ds\int_0^s D_1k(s-\tau, u_m(\xi_1, \tau), \dots, u_m(\xi_N, \tau))d\tau$$

$$= J_1(t) + J_2(t) + J_3(t). \tag{3.15}$$

Also, we estimate the integrals in the right-hand side of (3.15) as follows:

$$J_1(t) \le 2(N+1)\sqrt{E_m(t)}\int_0^t k_1(t-s)\sqrt{E_m(s)}\,ds + 2\sqrt{E_m(t)}\int_0^t k_2(t-s)ds$$

$$\le 2\varepsilon E_m(t) + (N+1)^2\frac{1}{\varepsilon}\| k_1\|^2_{L^2(0,T)}\int_0^t E_m(s)ds + \frac{1}{\varepsilon}\| k_2\|^2_{L^1(0,T)}, \tag{3.16}$$

$$J_2(t) \le 2(N+1)k_1(0)\int_0^t E_m(s)ds + 2k_2(0)\int_0^t \sqrt{E_m(s)}\,ds$$

$$\le [1 + 2(N+1)k_1(0)]\int_0^t E_m(s)ds + Tk_2^2(0), \tag{3.17}$$

$$J_3(t) \le 2(N+1)\int_0^t \sqrt{E_m(s)}\,ds\int_0^s k_1(s-\tau)\sqrt{E_m(\tau)}\,d\tau$$

$$+ 2\int_0^t \sqrt{E_m(s)}\,ds\int_0^s k_2(s-\tau)d\tau$$

$$\le (N+1)\left[\int_0^t E_m(s)ds + \int_0^t\left(\int_0^s k_1(s-\tau)\sqrt{E_m(\tau)}\,d\tau\right)^2 ds\right]$$

$$+ \int_0^t E_m(s)ds + T\| k_2\|^2_{L^1(0,T)}$$

$$\le \left[(N+1)(T\| k_1\|^2_{L^2(0,T)} + 1) + 1\right]\int_0^t E_m(s)ds + T\| k_1\|^2_{L^1(0,T)}. \tag{3.18}$$

Therefore, from (3.15)-(3.18), we get

$$I_6(t) \leq 2\varepsilon E_m(t) + d_T^1 \int_0^t E_m(s)ds + d_T^2, \tag{3.19}$$

in which

$$\begin{cases} d_T^1 = (N+1)\left(T + \dfrac{N+1}{\varepsilon}\right)\| k_1 \|_{L^2(0,T)}^2 + (N+1)(|k_1(0)|+1) + 2, \\ d_T^2 = \left(T + \dfrac{1}{\varepsilon}\right)\| k_2 \|_{L^1(0,T)}^2 + Tk_2^2(0). \end{cases} \tag{3.20}$$

Using the imbedding $H^1(0, 1) \hookrightarrow C^0([0, 1])$, there exists a constant $C > 0$ such that

$$E_m(0) + 2g(0)u_{0m}(0) + 2\hat{h}(u_{0m}(0)) + 2\int_0^1 \hat{f}(x, u_{0m}(x))dx \leq C. \tag{3.21}$$

We get from (3.8), (3.10)-(3.14), (3.19) and (3.21) that

$$E_m(t) \leq \varepsilon d_T^3 E_m(t) + \int_0^t d_T^4(s) E_m(s)ds + d_T^5, \tag{3.22}$$

where

$$\begin{cases} d_T^3 = 2\| f_1 \|_{L^1(0,1)} + 2h_1 + 3, \\ d_T^4(s) = 2\| F_1(s) \| + \| F_2(s) \| + | g'(s) | + 2| \hat{\lambda}(s) | \\ \qquad\qquad + 4T\left(1 + \dfrac{1}{\varepsilon}\right)(\| f_1 \|_{L^1(0,1)} + h_1) + d_T^1, \\ d_T^5 = \| F_2 \|_{L^1(0,T;L^2(0,1))} + 2\| f_2 \|_{L^1(0,1)} + \dfrac{1}{\varepsilon}\| g \|_{L^\infty(0,T)}^2 \\ \qquad + \| g' \|_{L^1(0,T)} + 4\left(1 + \dfrac{1}{\varepsilon}\right)(\| f_1 \|_{L^1(0,1)} + h_1)\| u_{0m} \|^2 \\ \qquad + C + d_T^2 + 2h_2. \end{cases} \tag{3.23}$$

Choosing $2\varepsilon d_T^3 \leq 1$, applying the Gronwall inequality, we get

$$E_m(t) \leq 2d_T^5 \exp(2\| d_T^4 \|_{L^1(0,T)}) \leq d_T, \tag{3.24}$$

where d_T is a positive constant depending on T.

Next, we shall require the following lemma:

Lemma 3.4. *There exists a positive constant* C_T *such that*

$$\sum_{i=0}^{N} \int_0^t |u'_m(\xi_i, s)|^2 ds \leq C_T, \quad \forall t \in [0, T]. \tag{3.25}$$

Proof of Lemma 3.4. We put

$$\begin{cases} k_{mi}(t) = \sum_{k=1}^{m} \cos(\mu_k \xi_i) \rho_k(t), \\ g_{mi}(t) = \sum_{k=1}^{m} \varphi_k(\xi_i)[a_{mk}\rho'_k(t) + b_{mk}\rho_k(t)] \\ \qquad - \sum_{k=1}^{m} \varphi_k(\xi_i) \int_0^t \rho_k(t-s)\lambda(s, \|u_m(s)\|^2)\omega'_{mk}(s) ds \\ \qquad - 2\sum_{k=1}^{m} \frac{\cos(\mu_k \xi_i)}{\varphi_k(0)} \int_0^t \rho_k(t-s)\langle F_m(\cdot, s), \varphi_k \rangle ds, \end{cases} \tag{3.26}$$

where $F_m(x, s) = f(x, u_m(s)) - F(x, s, \|u_m(s)\|^2)$.

By (3.3), $(3.5)_1$ and (3.7), thus $u_m(\xi_i, t)$ is rewritten as follows:

$$u_m(\xi_i, t) = g_{mi}(t) - 2\int_0^t k_{mi}(t-s) v_m(s) ds. \tag{3.27}$$

In connection with $g_{mi}(t)$, we have the following lemma:

Lemma 3.5. *There exists a positive constant* $\hat{C}_T$ *such that*

$$\sum_{i=1}^{N} \int_0^t |g'_{mi}(s)|^2 ds \leq \hat{C}_T, \quad \forall t \in [0, T]. \tag{3.28}$$

Proof of Lemma 3.5. We define

$$g'_{mi}(t) = -a_{mi}(t) + b_{mi}(t) - c_{mi}(t) - d_{mi}(t) - e_{mi}(t), \tag{3.29}$$

with

$$\begin{cases} a_{mi}(t) = \sum_{k=1}^{m} \varphi_k(0)\mu_k a_{mk} \cos(\mu_k \xi_i) \sin(\mu_k t), \\ b_{mi}(t) = \sum_{k=1}^{m} \varphi_k(0) b_{mk} \cos(\mu_k \xi_i) \cos(\mu_k t), \\ c_{mi}(t) = \sum_{k=1}^{m} \varphi_k(0) \cos(\mu_k \xi_i) \int_0^t \rho_k'(t-s) \lambda(s, \| u_m(s) \|^2) \omega_{mk}'(s) ds, \\ d_{mi}(t) = 2 \sum_{k=1}^{m} \frac{\cos(\mu_k \xi_i)}{\varphi_k(0)} \rho_k(t) \langle F_m(\cdot, 0), \varphi_k \rangle, \\ e_{mi}(t) = 2 \sum_{k=1}^{m} \frac{\cos(\mu_k \xi_i)}{\varphi_k(0)} \int_0^t \rho_k(t-s) \langle F_m'(\cdot, s), \varphi_k \rangle ds. \end{cases} \tag{3.30}$$

Moreover, using the following inequality:

$$(a+b+c+d+e)^2 \leq 5(a^2+b^2+c^2+d^2+e^2), \quad \forall a, b, c, d, e \in \mathbb{R}. \tag{3.31}$$

Hence,

$$\int_0^t | g_{mi}'(s) |^2 ds \leq 5 \sum_{J \in \{a, b, c, d, e\}} \int_0^t | J_{mi}(s) |^2 ds = \sum_{j=1}^{5} K_j(t). \tag{3.32}$$

We will estimate each term on the right-hand side of this inequality.

Estimating $K_1(t) = 5\int_0^t | a_{mi}(s) |^2 ds.$ From $(3.30)_1$ and (3.32), we easily see that

$$K_1(t) \leq \frac{5}{2} \int_{\xi_i}^{t+\xi_i} \left[\sum_{k=1}^{m} \varphi_k(0)\mu_k a_{mk} \sin(\mu_k s) \right]^2 ds$$

$$+ \frac{5}{2} \int_{-\xi_i}^{t-\xi_i} \left[\sum_{k=1}^{m} \varphi_k(0)\mu_k a_{mk} \sin(\mu_k s) \right]^2 ds. \tag{3.33}$$

Next, we need the following lemma. The proof of it is simple, we omit the details.

Lemma 3.6. *Let* $a, b \in \mathbb{R}$, $a < b$ *and* $c_k \in \mathbb{R}$, $k = \overline{1, m}$. *Then*

$$\int_a^b \left[\sum_{k=1}^m c_k \sin(\mu_k s)\right]^2 ds \le 2(\max\{|a|, |b|\} + 2)\int_0^1 \left[\sum_{k=1}^m c_k \sin(\mu_k s)\right]^2 ds. \tag{3.34}$$

By Lemma 3.6, we get from $(3.5)_2$, (3.33) and (A_1) that

$$K_1(t) \le 10(T + \xi_i + 2)\| u_{0m} \|^2_{H^1(0,1)} \le C_T, \tag{3.35}$$

where C_T always indicates a constant depending on T.

Estimating $K_2(t) = 5\int_0^t | b_{mi}(s)|^2 ds.$ Similarly, we get

$$K_2(t) \le 10(T + \xi_i + 2)\| u_{1m} \|^2 \le C_T. \tag{3.36}$$

Estimating $K_3(t) = 5\int_0^t | c_{mi}(s)|^2 ds.$ We deduce from $(3.30)_3$ that

$$\begin{aligned} | c_{mi}(t)| &= \Bigg| \frac{1}{2}\int_0^t \lambda(s, \| u_m(s)\|^2)\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(t - s + \xi_i))\omega'_{mk}(s)ds \\ &\quad + \frac{1}{2}\int_0^t \lambda(s, \| u_m(s)\|^2)\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(t - s - \xi_i))\omega'_{mk}(s)ds \Bigg| \\ &\le \frac{1}{2}\int_0^t \hat{\lambda}(s)\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(t - s + \xi_i))\omega'_{mk}(s)\right| ds \\ &\quad + \frac{1}{2}\int_0^t \hat{\lambda}(s)\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(t - s - \xi_i))\omega'_{mk}(s)\right| ds. \end{aligned} \tag{3.37}$$

Thus,

$$|c_{mi}(t)|^2 \le \frac{1}{2}\|\hat{\lambda}\|_{L^1(0,T)}\int_0^t \hat{\lambda}(s)\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(t-s+\xi_i))\omega'_{mk}(s)\right|^2 ds$$

$$+\frac{1}{2}\|\hat{\lambda}\|_{L^1(0,T)}\int_0^t \hat{\lambda}(s)\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(t-s-\xi_i))\omega'_{mk}(s)\right|^2 ds. \tag{3.38}$$

By the Fubini theorem and Lemma 3.6, we deduce from (3.24) and (3.38) that

$$K_3(t) \le \frac{5}{2}\|\hat{\lambda}\|_{L^1(0,T)}\int_0^t\int_\tau^t \hat{\lambda}(\tau)\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(s-\tau+\xi_i))\omega'_{mk}(\tau)\right|^2 ds d\tau$$

$$+\frac{5}{2}\|\hat{\lambda}\|_{L^1(0,T)}\int_0^t\int_\tau^t \hat{\lambda}(\tau)\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k(s-\tau-\xi_i))\omega'_{mk}(\tau)\right|^2 ds d\tau$$

$$= \frac{5}{2}\|\hat{\lambda}\|_{L^1(0,T)}\int_0^t \hat{\lambda}(\tau)\left(\int_{\xi_i}^{t-\tau+\xi_i}\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k z)\omega'_{mk}(\tau)\right|^2 dz\right) d\tau$$

$$= \frac{5}{2}\|\hat{\lambda}\|_{L^1(0,T)}\int_0^t \hat{\lambda}(\tau)\left(\int_{-\xi_i}^{t-\tau-\xi_i}\left|\sum_{k=1}^m \varphi_k(0)\cos(\mu_k z)\omega'_{mk}(\tau)\right|^2 dz\right) d\tau$$

$$\le 20(T+\xi_i+1)\|\hat{\lambda}\|_{L^1(0,T)}\int_0^t \hat{\lambda}(\tau)\|u'_m(\tau)\|^2 d\tau \le C_T. \tag{3.39}$$

Estimating $K_4(t) = 5\int_0^t |d_{mi}(s)|^2 ds$. By $(3.30)_4$, we have

$$|d_{mi}(s)| \le \frac{1}{2}\int_0^1 \left|\sum_{k=1}^m \frac{\sin(\mu_k(s+\xi_i+x)) + \sin(\mu_k(s-\xi_i+x))}{\mu_k}\right|$$

$$\times |F_m(x, 0)| dx$$

$$+ \frac{1}{2}\int_0^1 \left| \sum_{k=1}^{m} \frac{\sin(\mu_k(s+\xi_i - x)) + \sin(\mu_k(s-\xi_i - x))}{\mu_k} \right|$$

$$\times |F_m(x, 0)| dx. \tag{3.40}$$

Now using (3.40) and Lemma 2.3, we arrive at

$$|d_{mi}(s)| \leq 2\left(1+\frac{4}{\pi}\right)(\| f(\cdot, u_{0m})\|_{L^1(0,1)} + \| F(\cdot, 0, \| u_{0m}\|^2)\|_{L^1(0,1)}). \tag{3.41}$$

Hence, we obtain from $(3.5)_2$, (3.41) and (A_1), (A_4), (A_5) that

$$K_4(t) \leq 40T\left(1+\frac{4}{\pi}\right)^2 (\| f(\cdot, u_{0m})\|^2_{L^1(0,1)} + \| F(\cdot, 0, \| u_{0m}\|^2)\|^2_{L^1(0,1)}) \leq C_T. \tag{3.42}$$

Estimating $K_5(t) = 5\int_0^t |e_{mi}(s)|^2 ds.$ Proving in the same way as in (3.42), we have

$$K_5(t) \leq 10T^2\left(1+\frac{4}{\pi}\right)^2 \int_0^t \| F_m'(\cdot, s)\|^2_{L^1(0,1)} ds. \tag{3.43}$$

On the other hand, we remark that

$$\begin{aligned} |F_m'(x, s)| &= | D_2 f(x, u_m(s))u_m'(s) - D_2F(x, s, \| u_m(s)\|^2) \\ &\quad - 2D_3F(x, s, \| u_m(s)\|^2)\langle u_m'(s), u_m(s)\rangle | \\ &\leq | D_2 f(x, u_m(s))|| u_m'(s)| + | D_2F(x, s, \| u_m(s)\|^2)| \\ &\quad + 2| D_3F(x, s, \| u_m(s)\|^2)| \| u_m'(s)\| \| u_m(s)\| \\ &\leq C_T(| u_m'(s)| + 1). \end{aligned} \tag{3.44}$$

Applying the Cauchy-Schwarz inequality, we get

$$K_5(t) \le 20T^2\left(1 + \frac{4}{\pi}\right)^2 C_T^2 \int_0^t (\| u_m'(s)\|^2 + 1)ds \le C_T. \tag{3.45}$$

Combining (3.32), (3.35), (3.36), (3.39), (3.42) and (3.45), we have Lemma 3.5. □

Remark 3.7. Lemma 3 in [5] is a special case of Lemma 3.5 with $F = h = k = 0$.

We now return to the proof of Lemma 3.4. Noting that

$$v_m'(t) = h'(u_m(\xi_1, t))u_m'(\xi_1, t) + g'(t) + k(0, u_m(\xi_1, t), \ldots, u_m(\xi_N, t))$$
$$+ \int_0^t D_1 k(t-s, u_m(\xi_1, s), \ldots, u_m(\xi_N, s))ds. \tag{3.46}$$

On account of (3.24) and $(A_2), (A_6), (A_7)$, we get

$$| v_m'(t)| \le C_T(| u_m'(\xi_1, t)| + 1) + | g'(t)|. \tag{3.47}$$

By Lemma 2.3 and the imbedding $H^1(0, 1) \hookrightarrow C^0([0, 1])$, we get from (3.27) that

$$| u_m'(\xi_i, t)| \le | g_{mi}'(t)| + 2\left(1 + \frac{4}{\pi}\right)| v_m(0)| + 2\left(1 + \frac{4}{\pi}\right)\int_0^t | v_m'(s)| ds$$
$$\le | g_{mi}'(t)| + C_T\left(1 + \int_0^t | v_m'(s)| ds\right). \tag{3.48}$$

Using Lemma 3.5, we deduce from (3.47) and (3.48) that

$$\int_0^t | u_m'(\xi_i, s)|^2 ds \le 3\int_0^t | g_{mi}'(t)|^2 ds + 3C_T^2 t + 3C_T^2 \int_0^t \tau d\tau \int_0^\tau | v_m'(s)|^2 ds$$
$$\le C_T + C_T \int_0^t d\tau \int_0^\tau | u_m'(\xi_1, s)|^2 ds, \quad i = \overline{1, N}. \tag{3.49}$$

Applying the Gronwall inequality, we obtain Lemma 3.4. □

Step 3. Limiting process. With the help of (3.24) and Lemma 3.4, we can extract a subsequence of sequence $\{u_m\}$, still labeled by the same notation such that

$$\begin{cases} u_m \to u & \text{weakly}^* \text{ in } L^\infty(0, T; H^1(0, 1)), \\ u'_m \to u' & \text{weakly}^* \text{ in } L^\infty(0, T; L^2(0, 1)), \\ u_m(\xi_i, \cdot) \to u(\xi_i, \cdot) & \text{weakly in } H^1(0, T), \ i = \overline{1, N}. \end{cases} \tag{3.50}$$

By the compactness of the imbedding $H^1(0, T) \hookrightarrow C^0([0, T])$ and the lemma of Lions [8], then (3.50) leads to

$$\begin{cases} u_m \to u & \text{strongly in } L^2((0, 1) \times (0, T)) \\ & \qquad \text{and a.e. in } (0, 1) \times (0, T), \\ u_m(\xi_i, \cdot) \to u(\xi_i, \cdot) & \text{strongly in } C^0([0, T]), \ i = \overline{1, N}. \end{cases} \tag{3.51}$$

Owing to (A_3)-(A_5) and (A_7), thus there exist constants f_T, F_T, k_T and λ_T such that

$$\begin{cases} |\lambda(t, \|u_m(t)\|^2) - \lambda(t, \|u(t)\|^2)| \le \lambda_T \|u_m(t) - u(t)\|, \\ |F(x, t, \|u_m(t)\|^2) - F(x, t, \|u(t)\|^2)| \le F_T \|u_m(t) - u(t)\|, \\ |f(x, u_m(x, t)) - f(x, u(x, t))| \le f_T |u_m(x, t) - u(x, t)|, \\ |k(t, u_m(\xi_1, s), \dots, u_m(\xi_N, s)) - k(t, u(\xi_1, s), \dots, u(\xi_N, s))| \\ \quad \le k_T \sum_{i=1}^{N} |u_m(\xi_i, s) - u(\xi_i, s)|. \end{cases} \tag{3.52}$$

Hence, it follows from (3.51) and (3.52) that

$$\begin{cases} f(x, u_m) \to f(x, u) & \text{strongly in } L^2((0,1) \times (0, T)), \\ F(x, t, \|u_m(t)\|^2) \to F(x, t, \|u(t)\|^2) & \text{strongly in } L^2((0,1) \times (0, T)), \\ v_m \to v & \text{strongly in } L^2(0, T), \\ \lambda(t, \|u_m(t)\|^2) \to \lambda(t, \|u(t)\|^2) & \text{strongly in } L^2(0, T), \end{cases} \tag{3.53}$$

where

$$v(t) = \int_0^t k(t - s, u(\xi_1, s), \dots, u(\xi_N, s))ds + h(u(\xi_1, t)) + g(t). \tag{3.54}$$

Therefore, passing to the limit in (3.4) by $(3.50)_{1,2}$ and (3.53), we have that u satisfies the following equations:

$$\begin{cases} \frac{d}{dt}\langle u'(t), \varphi\rangle + \langle u_x(t), \varphi_x\rangle + v(t)\varphi(0) + \langle f(\cdot, u(t)), \varphi\rangle \\ \qquad + \lambda(t, \| u(t)\|^2)\langle u'(t), \varphi\rangle = \langle F(\cdot, t, \| u(t)\|^2), \varphi\rangle, \forall \varphi \in H, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \end{cases} \tag{3.55}$$

The existence of weak solutions is proved.

Step 4. Uniqueness of the weak solutions. Let u_1 and u_2 be two weak solutions of the problem (1.1)-(1.4). Then $u = u_1 - u_2$ is a solution of the following problem:

$$\begin{cases} u_{tt} - u_{xx} + f(x, u_1) - f(x, u_2) + \lambda(t, \| u_1\|^2)u_1' - \lambda(t, \| u_2\|^2)u_2' \\ \qquad = F(x, t, \| u_1\|^2) - F(x, t, \| u_2\|^2), \\ u(1, t) = 0, u_x(0, t) = v(t), u(x, 0) = u_t(x, 0) = 0, \end{cases} \tag{3.56}$$

where

$$v(t) = \int_0^t \sum_{i=1}^{2} (-1)^{i-1} k(t-s, u_i(\xi_i, s), \ldots, u_i(\xi_N, s))ds$$

$$+ \sum_{i=1}^{2} (-1)^{i-1} h(u_i(\xi_1, t)). \tag{3.57}$$

With $u_0 = u_1 = 0$,

$$\chi(x, t) = \sum_{i=1}^{2} (-1)^{i-1}[F(x, t, \| u_i\|^2) - f(x, u_i) - \lambda(t, \| u_i\|)u_i'],$$

we deduce from Lemma 2.4 that

$$E(t) = -2\Bigg(\int_0^t \langle f(\cdot, u_1(s)) - f(\cdot, u_2(s)), u'(s)\rangle ds$$

$$- \int_0^t \langle F(\cdot, s, \| u_1(s)\|^2) - F(\cdot, s, \| u_2(s)\|^2), u'(s)\rangle ds\Bigg)$$

$$- 2\int_0^t u'(0, s)\sum_{i=1}^{2}(-1)^{i-1}h(u_i(0, s))ds$$

$$- 2\int_0^t \langle \lambda(s, \| u_1(s)\|^2)u_1'(s) - \lambda(s, \| u_2(s)\|^2)u_2'(s), u'(s)\rangle ds$$

$$- 2u(0, t)\int_0^t \sum_{i=1}^{2}(-1)^{i-1}k(t-s, u_i(\xi_1, s), \ldots, u_i(\xi_N, s))ds$$

$$+ 2\int_0^t u(0, s)\sum_{i=1}^{2}(-1)^{i-1}k(0, u_i(\xi_1, s), \ldots, u_i(\xi_N, s))ds$$

$$+ 2\int_0^t u(0, r)dr\int_0^r \sum_{i=1}^{2}(-1)^{i-1}D_1k(r-s, u_i(\xi_1, s), \ldots, u_i(\xi_N, s))ds$$

$$= J_1(t) + J_2(t) + \cdots + J_6(t), \tag{3.58}$$

in which $E(t) = \| u'(t)\|^2 + \| u_x(t)\|^2$.

Next, we can estimate the integrals in the right-hand side of (3.58) as follows:

First term $J_1(t)$. Using (A_4) and (A_5), hence it follows from Lemma 2.1 that

$$J_1(t) \le (f_M + 4MF_M)\int_0^t E(s)ds, \tag{3.59}$$

where

$$\begin{cases} f_M = \| D_2 f \|_{C^0([0,1]\times[-M, M])}, \\ F_M = \| D_3 F \|_{C^0([0,1]\times[0,T]\times[0, M^2])}, \\ M = \sum_{i=1}^{2}(\| u_i \|_{L^\infty(0,T;H)} + \| u_i' \|_{L^\infty(0,T;L^2(0,1))}). \end{cases} \tag{3.60}$$

Second term $J_2(t)$. Setting $h_M = \| h \|_{C^2([-M, M])}$. Integrating by parts, then

$$J_2(t) = -u^2(0, t)\int_0^1 h'(u_2(0, t) + \theta u(0, t))d\theta$$

$$+ \int_0^t u^2(0, s)ds\int_0^1 h''(u_2(0, s) + \theta u(0, s))(u_2'(0, s) + \theta u'(0, s))d\theta$$

$$\leq h_M\left[u^2(0, t) + \int_0^t (|u_1'(0, s)| + |u_2'(0, s)|)u^2(0, s)ds\right]. \tag{3.61}$$

On the other hand, by Lemma 2.2, we get

$$u^2(0, t) \leq \varepsilon E(t) + T\left(1 + \frac{1}{\varepsilon}\right)\int_0^t E(s)ds, \quad \forall \varepsilon > 0. \tag{3.62}$$

We deduce from (3.61) and (3.62) that

$$J_2(t) \leq \varepsilon h_M E(t) + \int_0^t \hat{h}(s)E(s)ds, \tag{3.63}$$

in which

$$\hat{h}(s) = \varepsilon h_M(|u_1'(0, s)| + |u_2'(0, s)|) + \left(1 + \frac{1}{\varepsilon}\right)T(2\sqrt{TC_T} + 1)h_M. \tag{3.64}$$

Third term $J_3(t)$. With $\lambda_M = \| D_2\lambda \|_{C^0([0, T]\times[0, M^2])}$, by (A_3), this yields

$$J_3(t) = -2\int_0^t \lambda(s, \| u_1(s) \|^2)\| u'(s) \|^2 ds$$

$$- 2\int_0^t [\lambda(s, \| u_1(s) \|^2) - \lambda(s, \| u_2(s) \|^2)]\langle u_2'(s), u'(s)\rangle ds$$

$$\leq 2\int_0^t \hat{\lambda}(s)\| u'(s) \|^2 ds + 2M\lambda_M\int_0^t E(s)ds$$

$$\leq 2\int_0^t (\hat{\lambda}(s) + M\lambda_M)E(s)ds. \tag{3.65}$$

In addition, from (A_7), we obtain the estimates for the other terms as follows:

$$J_4(t) \le 2k_M \sqrt{E(t)} \int_0^t N\sqrt{E(s)}\,ds \le \frac{1}{4}E(t) + 4N^2Tk_M^2 \int_0^t E(s)ds,$$

$$J_5(t) \le 2\int_0^t \sqrt{E(s)}\,k_M N\sqrt{E(s)}\,ds = 2Nk_M \int_0^t E(s)ds,$$

$$J_6(t) \le 2\int_0^t \sqrt{E(r)}\,dr \int_0^r Nk_M \sqrt{E(s)}\,ds \le N(T^2+1)k_M \int_0^t E(s)ds, \quad (3.66)$$

where

$$k_M = \sum_{i=2}^{N+1} \left(\| D_{i,1}g \|_{C^0([0,T]\times[-M,M]^N)} + \| D_i g \|_{C^0([0,T]\times[-M,M]^N)}\right). \quad (3.67)$$

Choosing $4\varepsilon h_M \le 1$, hence it follows from (3.58), (3.59), (3.63), (3.65) and (3.66) that

$$E(t) \le \int_0^t d_M(s)E(s)ds, \quad (3.68)$$

where

$$d_M(s) = \hat{h}(s) + 2\hat{\lambda}(s) + f_M + 4M(F_M + \lambda_M) + 4N^2Tk_M^2 + N(T^2+3)k_M. \quad (3.69)$$

Using the Gronwall inequality, we obtain $E(t) = 0$, i.e., $u_1 = u_2$.

This completes the proof of Theorem 3.2. □

Remark 3.8. Theorem 3.2 still holds if (A_7) is replaced by the following condition:

(A_7') $f \in C^2([0,T]\times\mathbb{R}^N)$, there exist positive functions $k_1, k_2 \in W^{1,1}(0,T)$ and $k_3, k_4 \in L^1(0,T)$ such that

$$\begin{cases} |k(t, u_1, ..., u_N)| \le k_1(t)\sum_{i=1}^{N}|u_i| + k_2(t), \\ |D_1 k(t, u_1, ..., u_N)| \le k_3(t)\sum_{i=1}^{N}|u_i| + k_4(t). \end{cases}$$

4. Stability of the Weak Solutions

Let $(u_0, u_1) \in H^1(0, 1) \times L^2(0, 1)$ be fixed functions. In addition, we assume that h_1 and h_2 are fixed constants, $f_1, f_2 \in L^2(0, 1), \hat{\lambda} \in L^1(0, T)$, $k_1, k_2 \in W^{1,1}(0, T)$ and $F_1, F_2 \in L^1(0, T; L^2(0, 1))$ are fixed functions satisfying (A_3)-(A_7) (independent of f, F, h, k and λ). Using Theorem 3.2, the problem (1.1)-(1.4) has a unique weak solution u depending on f, F, g, h, k and λ. We denote

$$u = u(f, F, g, h, k, \lambda), \tag{4.1}$$

where f, F, g, h, k, λ satisfy (A_2)-(A_7).

Then the stability of the solutions of the problems (1.1)-(1.4) is given by:

Theorem 4.1. *Let (A_1)-(A_7) hold. Then the solutions of the problems* (1.1)-(1.4) *are stable with respect to the data (f, F, g, h, k, λ) in the following sense*:

If $(f^j, F^j, g^j, h^j, k^j, \lambda^j)$ and (f, F, g, h, k, λ) satisfy (A_2)-(A_7) and

$$\begin{cases} g^j \to g \text{ strongly in } W^{1,1}(0, T), \\ (f^j, F^j, k^j, \partial k^j/\partial t, h^j, \lambda^j) \to (f, F, k, \partial k/\partial t, h, \lambda) \text{ strongly in} \end{cases}$$

$$C^0([0, 1] \times [-M, M]) \times C^0([0, 1] \times [0, T] \times [0, M^2])$$

$$\times [C^0([0, T] \times [-M, M]^N)]^2 \times C^1([-M, M]) \times C^0([0, T] \times [0, M^2])$$

$$\text{as } j \to \infty, \quad \forall M > 0. \tag{4.2}$$

Then

$$(u^j,\ u_t^j,\ u^j(\xi_1, \cdot), \ldots, u^j(\xi_N, \cdot)) \to (u,\ u_t,\ u(\xi_1, \cdot), \ldots, u(\xi_N, \cdot)) \tag{4.3}$$

strongly in $L^\infty(0, T; H^1(0,1)) \times L^\infty(0, T; L^2(0,1)) \times [C^0([0, T])]^N$ *as* $j \to \infty$, *where*

$$u^j = u(f^j,\ F^j,\ g^j,\ h^j,\ k^j,\ \lambda^j),\ u = u(f,\ F,\ g,\ h,\ k,\ \lambda).$$

Proof of Theorem 4.1. Firstly, we can assume that

$$\| g^j \|_{W^{1,1}(0,T)} + \| g \|_{W^{1,1}(0,T)} \leq g_*, \tag{4.4}$$

where g_* is a fixed positive constant.

On the other hand, by the proof of Theorem 3.2, we easily show that

$$\| u_t(t) \|^2 + \| u_x(t) \|^2 + \sum_{i=1}^N \int_0^t | u_t(\xi_i, s) |^2 ds \leq C_T, \tag{4.5}$$

$$\| u_t^j(t) \|^2 + \| u_x^j(t) \|^2 + \sum_{i=1}^N \int_0^t | u_t^j(\xi_i, s) |^2 ds \leq C_T, \tag{4.6}$$

where C_T is a constant depending on $u_0,\ u_1,\ f_1,\ f_2,\ F_1,\ F_2,\ k_1,\ k_2,\ \hat{\lambda},\ g_*,\ h_1,\ h_2,\ T.$ We set

$$\hat{\mu}_j = \mu^j - \mu, \quad \mu \in \{f,\ F,\ g,\ h,\ k,\ \lambda\}. \tag{4.7}$$

Then $v^j = u^j - u$ satisfies the following problem:

$$\begin{cases} v_{tt}^j - v_{xx}^j = \chi^j,\ 0 < x < 1,\ 0 < t < T, \\ v^j(x, 0) = v_t^j(x, 0) = 0,\ v^j(1, t) = 0,\ v_t^j(0, t) = w^j(t), \end{cases} \tag{4.8}$$

where

$$\begin{cases} \chi^j(x,t) = -\hat{f}_j(x, u^j(t)) - [f(x, u^j(t)) - f(x, u(t))] \\ \qquad - [\lambda^j(t, \| u^j(t) \|^2) u_t^j(t) - \lambda(t, \| u(t) \|^2) u_t(t)] \\ \qquad + \hat{F}_j(x, t, \| u^j(t) \|^2) + [F(x, t, \| u^j(t) \|^2) - F(x, t, \| u(t) \|^2)], \\ w^j(t) = \hat{g}_j(t) + \hat{h}_j(u^{(j)}(\xi_1, t)) + [h(u^{(j)}(\xi_1, t)) - h(u(\xi_1, t))] \\ \qquad + \int_0^t \hat{k}_j(t-s, u^j(\xi_1, s), \dots, u^j(\xi_N, s)) ds \\ \qquad + \int_0^t k(t-s, u^j(\xi_1, s), \dots, u^j(\xi_N, s)) ds \\ \qquad - \int_0^t k(t-s, u(\xi_1, s), \dots, u(\xi_N, s)) ds. \end{cases} \tag{4.9}$$

Applying Lemma 2.4 with $u_0 = u_1 = 0,$ we obtain

$$E_j(t) = -2\int_0^t \langle \lambda^j(s, \| u^j(s) \|^2) u_t^j(s) - \lambda(s, \| u(s) \|^2) u_t(s), v_t^j(s) \rangle ds$$

$$+ 2\int_0^t \langle F(\cdot, s, \| u^j(s) \|^2) - F(\cdot, s, \| u(s) \|^2)$$

$$- f(\cdot, u^j) + f(\cdot, u), v_t^j(s) \rangle ds$$

$$+ 2\int_0^t \langle \hat{F}_j(\cdot, s, \| u^j(s) \|^2) - \hat{f}_j(\cdot, u^j), v_t^j(s) \rangle ds$$

$$- 2\int_0^t \hat{g}_j(s) v_t^j(0, s) ds - 2\int_0^t \hat{h}_j(u^j(\xi_0, s)) v_t^j(0, s) ds$$

$$- 2\int_0^t [h(u^j(\xi_1, t)) - h(u(\xi_1, t))] v_t^j(0, s) ds$$

$$- 2\int_0^t v_t^j(0, r) dr \int_0^r \hat{k}_j(r-s, u^j(\xi_1, s), \dots, u^j(\xi_N, s)) ds$$

$$- 2\Big[\int_0^t v_t^j(0, r) dr \int_0^r k(r-s, u^j(\xi_1, s), \dots, u^j(\xi_N, s)) ds$$

$$- \int_0^t v_t^j(0, r) dr \int_0^r k(r - s, u(\xi_1, s), ..., u(\xi_N, s)) ds \Big]$$

$$= K_1(t) + K_2(t) + \cdots + K_8(t), \tag{4.10}$$

where

$$E_j(t) = \| v_t^j(t) \|^2 + \| v_x^j(t) \|^2. \tag{4.11}$$

Now we can estimate the integrals in the right-hand side of (4.10).

Estimating $K_1(t)$: Set $\lambda_M = \| D_2 \lambda \|_{C^0([0, T] \times [0, M^2])}$, $M = 2\sqrt{C_T}$, then using (4.5), (4.6) and the Cauchy-Schwarz inequality, we obtain

$$K_1(t) = -2 \int_0^t \lambda(s, \| u(s) \|^2) \| v_t^j(s) \|^2 ds$$

$$- 2 \int_0^t \hat{\lambda}_j(s, \| u^j(s) \|^2) \langle u_t^j(s), v_t^j(s) \rangle ds$$

$$- 2 \int_0^t [\lambda(s, \| u^j(s) \|^2) - \lambda(s, \| u(s) \|^2)] \langle u_t^j(s), v_t^j(s) \rangle ds$$

$$\leq TC_T \| \hat{\lambda}_j \|^2_{C^0([0, T] \times [0, M^2])}$$

$$+ \int_0^t (2\hat{\lambda}(s) + M\lambda_M + 1) E_j(s) ds. \tag{4.12}$$

Estimating $K_2(t)$: Due to (A_4) and (A_5), it yields

$$K_2(t) \leq (f_M + F_M) \int_0^t E_j(s) ds, \tag{4.13}$$

with $f_M = \| D_2 f \|_{C^0([0, 1] \times [-M, M])}$, $F_M = \| D_2 f \|_{C^0([0, 1] \times [0, T] \times [0, M^2])}$.

Estimating $K_3(t)$: We easily show that

$$K_3(t) \leq 2T(\| \hat{f}_j \|^2_{C^0([0, 1] \times [-M, M])} + \| \hat{F}_j \|^2_{C^0([0, 1] \times [0, T] \times [0, M^2])})$$

$$+ \int_0^t E_j(s) ds. \tag{4.14}$$

Estimating $K_4(t)$: Since $\|v\|_{C^0([0,T])} \le k_T \|v\|_{W^{1,1}(0,T)}$, $\forall v \in W^{1,1}(0,T)$, we get

$$K_4(t) \le \varepsilon E_j(t) + \frac{k_T^2}{\varepsilon} \| \hat{g}_j \|^2_{W^{1,1}(0,T)} + \| \hat{g}_j \|_{W^{1,1}(0,T)}$$

$$+ \int_0^t | \hat{g}'_j(s) | E_j(s) ds. \tag{4.15}$$

Estimating $K_5(t)$: By using (A_6) and (4.6), we get

$$K_5(t) \le \varepsilon E(t) + \int_0^t E_j(s) ds + \left(C_T + \frac{1}{\varepsilon}\right) \| \hat{h}_j \|^2_{C^1([-M,M])}. \tag{4.16}$$

Estimating $K_6(t)$: Similarly as in (3.63), we also have

$$K_6(t) \le \varepsilon E_j(t) \| h \|_{C^2([-M,M])} + \int_0^t M_j(s) E(s) ds, \tag{4.17}$$

where

$$M_j(s) = \left[\varepsilon(| u_t^j(0, s) | + | u_t(0, s) |) + 2T^{\frac{3}{2}}(M+1)\left(1 + \frac{1}{\varepsilon}\right) \right]$$

$$\times \| h \|_{C^2([-M,M])}. \tag{4.18}$$

Estimating $K_7(t)$: By using (4.6) and (A_7), then

$$K_7(t) \le \varepsilon E_j(t) + \int_0^t E_j(s) ds + \frac{T}{\varepsilon}(T + \varepsilon) \| \hat{k}_j \|^2_{C^0([0,T]\times[-M,M]^N)}$$

$$+ T^2 \| D_1 \hat{k}_j \|^2_{C^0([0,T]\times[-M,M]^N)}. \tag{4.19}$$

Estimating $K_8(t)$: From (A_7), it is easy to show that

$$K_8(t) \le \varepsilon E_j(t) + k_M \int_0^t E_j(s) ds, \tag{4.20}$$

where

$$k_M = N(3N + T)\sum_{i=2}^{N+1}(\| D_{i,1}g \|_{C^0([0,T]\times[-M,M]^N)} + \| D_i g \|_{C^0([0,T]\times[-M,M]^N)}). \tag{4.21}$$

Combining (4.10)-(4.17), (4.19) and (4.20), then choosing $\varepsilon M_T^1 = \frac{1}{2}$, we get

$$\begin{aligned} E_j(t) \leq 2M_T^2 \exp\| M_T^3 \|(\| \hat{f}_j \|^2_{C^0([0,1]\times[-M,M])} &+ \| \hat{F}_j \|^2_{C^0([0,1]\times[0,T]\times[0,M^2])} + \| \hat{g}_j \|^2_{W^{1,1}(0,T)} + \| \hat{g}_j \|_{W^{1,1}(0,T)} \\ &+ \| \hat{h}_j \|^2_{C^1([-M,M])} + \| \hat{k}_j \|^2_{C^0([0,T]\times[-M,M]^N)} \\ &+ \| D_1\hat{k}_j \|^2_{C^0([0,T]\times[-M,M]^N)} + \| \hat{\lambda}_j \|^2_{C^0([0,T]\times[0,M^2])}), \end{aligned} \tag{4.22}$$

where

$$\begin{cases} M_T^1 = \| h \|_{C^2([-M,M])} + 4, \\ M_T^2 = \left(1 + \dfrac{1}{\varepsilon}\right)(2T^2 + k_T^2 + 1) + 3(T+1)(C_T + 1), \\ M_T^3(s) = M_j(s) + | \hat{g}'_j(s) | + 2\hat{\lambda}(s) + f_M + F_M + k_M + M\lambda_M + 3. \end{cases} \tag{4.23}$$

Consequently,

$$\begin{aligned} &\| v_t^j \|^2_{L^\infty(0,T;L^2(0,1))} + \| v^j \|^2_{L^\infty(0,T;H^1(0,1))} + \sum_{i=1}^{N} \| v^j(\xi_i, \cdot) \|^2_{C^0([0,T])} \\ &\leq M_T(\| \hat{f}_j \|^2_{C^0([0,1]\times[-M,M])} + \| \hat{F}_j \|^2_{C^0([0,1]\times[0,T]\times[0,M^2])} \\ &\quad + \| \hat{g}_j \|^2_{W^{1,1}(0,T)} + \| \hat{g}_j \|_{W^{1,1}(0,T)} \end{aligned}$$

$$+\| \hat{h}_j \|^2_{C^1([-M,M])} + \| \hat{k}_j \|^2_{C^0([0,T]\times[-M,M]^N)}$$

$$+\| D_1\hat{k}_j \|^2_{C^0([0,T]\times[-M,M]^N)} + \| \hat{\lambda}_j \|^2_{C^0([0,T]\times[0,M^2])}), \tag{4.24}$$

where M_T is a constant depending on u_0, u_1, f_1, f_2, F_1, F_2, k_1, k_2, $\hat{\lambda}$, g_*, h_1, h_2, T. Theorem 4.1 is completely proved. □

Remark 4.2. Theorem 13 in [16] is a special case of Theorem 4.1 with $F = \lambda = 0$ and $k(t, u_1, ..., u_N) = k_1(t)k_2(u_1, ..., u_N)$, where k_1, k_2 are given functions.

Acknowledgement

The author thanks the anonymous referees for their constructive criticism and valuable suggestions.

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