Advances in Differential Equations and Control Processes
© 2016 Pushpa Publishing House, Allahabad, India
Published Online: July 2016
http://dx.doi.org/10.17654/DE017010001
Volume 17, Number 1, 2016, Pages 1-12 **ISSN: 0974-3243**

CONVERGENCE STUDY OF THE SBA METHOD ON THE VOLTERRA NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND KIND

Yaya Moussa[1], Bakari Abbo[1], Youssouf Pare[2] and Blaise Some[2]

[1]Département de Mathématique
Faculté des Sciences Exactes et Appliquées
Université de Ndamena
Tchad
e-mail: yayamoussayaya@yahoo.fr
bakariabbo@yahoo.fr

[2]Université Ouaga I Joseph Ki Zerbo
UFR/SEA, Burkina Faso
e-mail: youssouf.pare@univ-ouaga.bf
some@univ-ouaga.bf

Abstract

We establish the convergence of the SBA method for the Volterra nonlinear differential equations of second kind.

1. Introduction

In this paper, we discuss the convergence of the SBA method

Received: January 24, 2016; Accepted: April 19, 2016
2010 Mathematics Subject Classification: 45Axx, 45Bxx, 45Gxx, 47Jxx, 47H30, 34A12.
Keywords and phrases: numerical method SBA, Picard principle, Adomian method, successive approximation, Volterra nonlinear integral equation of second kind.
Communicated by K. K. Azad

(combination of the principle of Picard, Adomian method and the successive approximations) for the nonlinear integral equations of Volterra second kind of the form:

$$\varphi(x) = f(x) + \lambda\int_a^x K(x, t)g(\varphi(t))dt,$$

where g is given by $g(\varphi(t)) = l(\varphi(t)) + N(\varphi(t))$, $\lambda > 0$, $l(\varphi(t)) = \varphi(t)$, $a \le t \le x \le T < +\infty$ and N nonlinear. We get:

$$(E): \varphi(x) = f(x) + \lambda\int_a^x K(x, t)(\varphi(t))dt + \lambda\int_a^x K(x, t)N(\varphi(t))dt. \quad (1)$$

2. Convergence of the SBA Method

Let us consider the following equation (E):

$$\varphi(x) = f(x) + \lambda\int_a^x K(x, t)(\varphi(t))dt + \lambda\int_a^x K(x, t)N(\varphi(t))dt. \quad (2)$$

The approach equation associated to (E) is: for $k \ge 1$,

$$(E_{app}): \varphi^k(x) = f^k(x) + \lambda\int_a^x K(x, t)\varphi^k(t)dt + \lambda\int_a^x K(x, t)N(\varphi^{k-1}(t))dt \quad (3)$$

and so the SBA algorithm associated to (E_{app}) is:

$$\begin{cases} \varphi_0^k(x) = f(x) + \lambda\int_a^x K(x, t)N(\varphi^{k-1}(t))dt, \ k = 1, 2, \ldots \\ \varphi_n^k(x) = \lambda\int_a^x K(x, t)\varphi_{n-1}^k(t)dt, \ n = 1, 2, \ldots, \end{cases} \quad (4)$$

where for $k \ge 1$, $f^k(x) = f(x)$.

Theorem 1. *Consider* $f, \varphi \in C([a, T])$ *and* $K \in C([a, T] \times [a, T])$. *Then the following Volterra nonlinear integral equation of second kind is given by*:

$$(E): \varphi(x) = f(x) + \lambda\int_a^x K(x, t)(\varphi(t))dt + \lambda\int_a^x K(x, t)N(\varphi(t))dt,$$

where $\lambda > 0$, *is approached by*: *for* $k \geq 1$,

$$(E_{app}): \varphi^k(x) = f(x) + \lambda\int_a^x K(x, t)\varphi^k(t)dt + \lambda\int_a^x K(x, t)N(\varphi^{k-1}(t))dt$$

and the SBA algorithm associated to (E_{app}) *is given by*

$$\begin{cases} \varphi_0^k(x) = f(x) + \lambda\int_a^x K(x, t)N(\varphi^{k-1}(t))dt, \ k = 1, 2, \ldots \\ \varphi_{n+1}^k(x) = \lambda\int_a^x K(x, t)\varphi_n^k(t)dt, \ n = 0, 1, 2, \ldots. \end{cases}$$

If, for $k = 1, 2, \ldots,$ *then there exists* $\varphi^{k-1} \in C([a, T])$ *such that* $N(\varphi^{k-1}) = 0,$ *and if the SBA algorithm associated to* (E_{app}) *converges at the step* $k = 1,$ *then the solution* $\varphi(x)$ *of the equation* (E) *is unique, and* $\varphi(x) = \lim_{k\to+\infty} \varphi^k(x).$

Proof. When $k = 1$, $N(\varphi^0(t)) = 0$ and the SBA algorithm is given by

$$\begin{cases} \varphi_0^1(x) = f(x), \\ \varphi_n^1(x) = \lambda\int_a^x K(x, t)\varphi_{n-1}^1(t)dt, \ n = 1, 2, \ldots, \end{cases}$$

we have $f \in C([0, T])$ and $K \in C([a, T] \times [a, T]) \Rightarrow \exists m > 0, M > 0$ such as $\forall x \in [a, T]$ and $\forall (x, t) \in [a, T] \times [a, T], |f(x)| \leq m, |K(x, t)| \leq M.$

Then we have:

$$\begin{cases} |\varphi_0^1(x)| = |f(x)| \leq m, \\ |\varphi_n^1(x)| \leq \lambda\int_a^x |K(x, t)||\varphi_{n-1}^1(t)|dt, \ n = 1, 2, \ldots \end{cases}$$

and we get

$$\begin{cases} |\varphi_0^1(x)| \le m \\ |\varphi_1^1(x)| \le \lambda m M(x-a) \\ |\varphi_2^1(x)| \le \dfrac{\lambda^2 m^2 M^2 (x-a)^2}{2!} \\ \dots \\ |\varphi_n^1(x)| \le \dfrac{\lambda^n m^n M^n (x-a)^n}{n!} \end{cases}$$

$$\Rightarrow \sum_{n=0}^{+\infty} |\varphi_n^1(x)| \le \sum_{n=0}^{+\infty} \frac{\lambda^n m^n M^n (x-a)^n}{n!} = \exp(\lambda m M(x-a))$$

which shows that $\left(\sum_{n=0}^{+\infty} \varphi_n^1(x) \right)$ is absolutely convergent.

We suppose for the step $k = p \ge 1$, $N(\varphi^p(x)) = 0$ and we get at the step $k = p+1$:

$$\begin{cases} |\varphi_0^{p+1}(x)| \le m \\ |\varphi_1^{p+1}(x)| \le \lambda m M(x-a) \\ |\varphi_2^{p+1}(x)| \le \dfrac{\lambda^2 m^2 M^2 (x-a)^2}{2!} \\ \dots \\ |\varphi_n^{p+1}(x)| \le \dfrac{\lambda^n m^n M^n (x-a)^n}{n!} \end{cases}$$

$$\Rightarrow \sum_{n=0}^{+\infty} |\varphi_n^{p+1}(x)| \le \sum_{n=0}^{+\infty} \frac{\lambda^n m^n M^n (x-a)^n}{n!} = \exp(\lambda m M(x-a))$$

which shows that $\left(\sum_{n=0}^{+\infty} \varphi_n^{p+1}(x) \right)$ is absolutely convergent, and hence $\varphi(x) = \lim_{k \to +\infty} \varphi^k(x)$.

Let us suppose that the equation (E) admits two distinct solutions $\varphi(x)$ and $\phi(x)$. Taking $\delta(x) = \varphi(x) - \phi(x)$, and applying the SBA algorithm with the preceding hypotheses, we have:

$$\begin{cases} \delta_0^k(x) = 0;\ k = 1,\ 2,\ \ldots \\ \delta_n^k(x) = \lambda \int_a^x K(x,\ t)\delta_{n-1}^k(t)dt;\ n = 1,\ 2,\ \ldots \end{cases} \tag{5}$$

the solution of which at each step k is $\delta^k(x) = 0$. Then $\delta(x) = \lim_{k\to+\infty} \delta^k(x) = 0$. Hence, for $t \in [a,\ x]$, $\delta(t) = \varphi(t) - \phi(t) = 0 \Rightarrow \varphi(t) = \phi(t)$. Thus, the solution of the equation (E) is unique. □

3. Numerical Examples

3.1. Example 1

Let us consider the following Volterra nonlinear equation of the second kind, which is the canonical form of Adomian:

$$\varphi(x) = 1 - \int_0^x (x - t)\varphi(t)dt + \int_0^x (x - t)[\varphi^5(t) - \varphi^4(t)\cos t]dt. \tag{6}$$

The SBA algorithm for this equation is the following:

$$\begin{cases} \varphi_0^k(x) = 1 + \int_0^x (x - t)N(\varphi^{k-1}(t))dt;\ k \geq 1, \\ \varphi_{n+1}^k(x) = -\int_0^x (x - t)\varphi_n^k(t)dt;\ n \geq 0, \end{cases}$$

where

$$N(\varphi(t)) = \varphi^5(t) - \varphi^4(t)\cos t.$$

When $k = 1$, applying the principle of Picard, for $\varphi^0(x) = 0$, we have $N(\varphi^0(x)) = 0$ and the approached solution is:

$$\begin{cases} \varphi_0^1(x) = 1 \\ \varphi_1^1(x) = -\dfrac{x^2}{2!} \\ \varphi_2^1(x) = \dfrac{x^4}{4!} \\ \dots \\ \varphi_n^1(x) = (-1)^n \dfrac{x^{2n}}{(2n)!};\ n \geq 0 \end{cases}$$

which converges to $\varphi^1(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Hence, $\varphi^1(x) = \cos x$.

When $k = 2$, we have $N(\varphi^1(x)) = 0$, and the approached solution is:

$$\begin{cases} \varphi_0^2(x) = 1 \\ \varphi_1^2(x) = -\dfrac{x^2}{2!} \\ \varphi_2^2(x) = \dfrac{x^4}{4!} \\ \dots \\ \varphi_n^2(x) = (-1)^n \dfrac{x^{2n}}{(2n)!};\ n \geq 0 \end{cases}$$

which converges to $\varphi^2(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Hence $\varphi^2(x) = \cos x$.

In the recursive way, $\varphi^1(x) = \varphi^2(x) = \cdots = \varphi^k(x) = \cos x$. Therefore, we obtain the exact solution of equation (6) as

$$\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = \cos x.$$

3.2. Example 2

Let us consider the following Volterra nonlinear equation of the second kind, which is the canonical form of Adomian:

$$\varphi(x) = x - \frac{1}{3}x^4 + \int_0^x xt\varphi(t)\,dt + \frac{1}{2}\int_0^x xt(t\varphi^3(t) - \varphi^4(t))\,dt. \tag{7}$$

The SBA algorithm for this equation is the following:

$$\begin{cases} \varphi_0^k(x) = x + \dfrac{1}{2}\displaystyle\int_0^x xtN(\varphi^{k-1}(t))dt;\ k \geq 1, \\ \varphi_1^k(x) = -\dfrac{1}{3}x^4 + \displaystyle\int_0^x xt\,\varphi_0^k(t)\,dt, \\ \varphi_{n+1}^k(x) = \displaystyle\int_0^x xt\,\varphi_n^k(t)\,dt;\ n \geq 1, \end{cases}$$

where $N(\varphi(t)) = t\varphi^3(t) - \varphi^4(t)$.

When $k = 1$, applying the principle of Picard, for $\varphi^0(x) = 0$, we have $N(\varphi^0(x)) = 0$ and the approached solution is:

$$\begin{cases} \varphi_0^1(x) = x \\ \varphi_1^1(x) = 0 \\ \varphi_2^1(x) = 0 \\ \dots \\ \varphi_n^1(x) = 0;\ n \geq 1 \end{cases}$$

which converges to $\varphi^1(x) = \sum_{n=0}^{+\infty} \varphi_n^1(x)$. Hence, $\varphi^1(x) = x$.

When $k = 2$, we have $N(\varphi^1(x)) = 0$, and the approached solution is:

$$\begin{cases} \varphi_0^2(x) = x \\ \varphi_1^2(x) = 0 \\ \varphi_2^2(x) = 0 \\ \dots \\ \varphi_n^2(x) = 0;\ n \geq 1 \end{cases}$$

which converges to $\varphi^2(x) = \sum_{n=0}^{+\infty} \varphi_n^2(x)$. Hence, $\varphi^2(x) = x$.

In the recursive way, $\varphi^1(x) = \varphi^2(x) = \cdots = \varphi^k(x) = x.$ Therefore, we obtain the exact solution of equation (7) as

$$\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = x.$$

3.3. Example 3

Let us consider the following Volterra nonlinear equation of the second kind, which is the canonical form of Adomian:

$$\varphi(x) = e^{2x} - e^x(e^x - 1) + \int_0^x e^{x-t}\varphi(t)dt - \frac{3}{4}\int_0^x e^{x-t}(e^{3x}\sqrt{\varphi(t)} - \varphi^2(t))dt. \quad (8)$$

The SBA algorithm for this equation is the following:

$$\begin{cases} \varphi_0^k(x) = e^{2x} + \int_0^x e^{x-t} N(\varphi^{k-1}(t))dt;\ k \geq 1, \\ \varphi_1^k(x) = -e^x(e^x - 1) + \int_0^x e^{x-t}\varphi_0^k(t)dt, \\ \varphi_{n+1}^k(x) = \int_0^x e^{x-t}\varphi_n^k(t)dt;\ n \geq 1, \end{cases}$$

where

$$N(\varphi(t)) = e^{3x}\sqrt{\varphi(t)} - \varphi^2(t).$$

When $k = 1,$ applying the principle of Picard, for $\varphi^0(x) = 0,$ we have $N(\varphi^0(x)) = 0$ and the approached solution is:

$$\begin{cases} \varphi_0^1(x) = e^{2x} \\ \varphi_1^1(x) = 0 \\ \varphi_2^1(x) = 0 \\ \cdots \\ \varphi_n^1(x) = 0;\ n \geq 1 \end{cases}$$

which converges to

$$\varphi^1(x) = \sum_{n=0}^{+\infty} \varphi_n^1(x) \Rightarrow \varphi^1(x) = e^{2x}.$$

When $k = 2$, we have $N(\varphi^1(x)) = 0$ and the approached solution is:

$$\begin{cases} \varphi_0^2(x) = e^{2x} \\ \varphi_1^2(x) = 0 \\ \varphi_2^2(x) = 0 \\ \dots \\ \varphi_n^2(x) = 0;\ n \geq 1 \end{cases}$$

which converges to

$$\varphi^2(x) = \sum_{n=0}^{+\infty} \varphi_n^2(x) \Rightarrow \varphi^2(x) = e^{2x}.$$

In the recursive way, $\varphi^1(x) = \varphi^2(x) = \cdots = \varphi^k(x) = e^{2x}$. Therefore, we obtain the exact solution of equation (8) as

$$\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = e^{2x}.$$

3.4. Example 4

Let us consider the following Volterra nonlinear equation of the second kind, which is the canonical form of Adomian:

$$\varphi(x) = e^{2x} - e^x(e^x - 1) + \int_0^x e^{x-t}\varphi(t)\,dt$$

$$- \frac{3}{4}\int_0^x e^{x-t}(e^{3t}\sqrt{\varphi(t)} - \varphi^2(t))\,dt. \qquad (9)$$

The SBA algorithm for this equation is the following:

$$\begin{cases} \varphi_0^k(x) = e^{2x} - \dfrac{3}{4}\displaystyle\int_0^x e^{x-t} N(\varphi^{k-1}(t))\,dt;\ k \geq 1, \\ \varphi_1^k(x) = -e^x(e^x - 1) + \displaystyle\int_0^x e^{x-t}\varphi_0^k(t)\,dt, \\ \varphi_{n+1}^k(x) = \displaystyle\int_0^x e^{x-t}\varphi_n^k(t)\,dt;\ n \geq 1, \end{cases}$$

where $N(\varphi(t)) = e^{3t}\sqrt{\varphi(t)} - \varphi^2(t)$.

When $k = 1$, applying the principle of Picard, for $\varphi^0(x) = 0$, we have $N(\varphi^0(x)) = 0$ and the approached solution is:

$$\begin{cases} \varphi_0^1(x) = 1 \\ \varphi_1^1(x) = 0 \\ \varphi_2^1(x) = 0 \\ \ldots \\ \varphi_n^1(x) = 0;\ n \geq 1 \end{cases}$$

which converges to

$$\varphi^1(x) = \sum_{n=0}^{+\infty} \varphi_n^1(x) \Rightarrow \varphi^1(x) = 1.$$

When $k = 2$, we have $N(\varphi^1(x)) = 0$ and the approached solution is:

$$\begin{cases} \varphi_0^2(x) = 1 \\ \varphi_1^2(x) = 0 \\ \varphi_2^2(x) = 0 \\ \ldots \\ \varphi_n^2(x) = 0;\ n \geq 1 \end{cases}$$

which converges to

$$\varphi^2(x) = \sum_{n=0}^{+\infty} \varphi_n^2(x) \Rightarrow \varphi^2(x) = 1.$$

In the recursive way, $\varphi^1(x) = \varphi^2(x) = \cdots = \varphi^k(x) = 1.$ Therefore, we obtain the exact solution of equation (9) as

$$\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = 1.$$

References

[1] K. Abbaoui and Y. Cherruault, The decomposition method applied to the Cauchy problem, Kybernetes 28(1) (1999), 68-74.

[2] S. Kheilifa and Y. Cherruault, The decomposition method for solving first order partial differential equations, Kybernetes 31(6) (2002), 844-871.

[3] Bakari Abbo, Ngarkodje Ngarasta, Benjamin Mampassi, Blaise Some and Longin Some, A new approach of the Adomian algorithm for solving nonlinear ordinary or partial differential equations, Far East J. Appl. Math. 23(3) (2006), 299-312.

[4] Bakari Abbo, Nouvel algorithme numérique de résolution des équations différentielles ordinaires (EDO) et des équations aux dérivées partielles non linéaires, Thèse de Doctorat unique, Université de Ouagadougou, Janvier, 2007.

[5] Pare Youssouf, Saley Bisso, Blaise Some, So Ousseini and Bakari Abbo, A numerical method for solving Cauchy evolution problem of partial differential equations (PDEs) for several variables, Inter. J. Numer. Method Appl. 1(1) (2009), 87-100.

[6] Pare Youssouf, Yaro Rasmane, Elysee Gouba and Blaise Some, Solving a system of nonlinear equations second kind of Volterra by the SBA method, Far East. J. Appl. Math. 71(1) (2012), 43-54.

[7] Bakari Abbo, Abba Danna, Pare Youssouf and Blaise Some, Extension of the new iterative approach of Adomian algorithm to partial differential equations (PDE) strongly nonlinear with initial and boundary conditions, Far East J. Math. Sci. (FJMS) 75(2) (2013), 245-255.

[8] Youssouf Pare, Françis Bassono and Blaise Some, A new technique for numerical resolution of few non linear integral equations of Fredholm by SBA method, Far East J. Appl. Math. 70(1) (2012), 21-33.

[9] Youssouf Pare, Abbo Bakari, Rasmane Yaro and Blaise Some, Solving first kind Abel integral equations using the SBA numerical method, Nonlinear Analysis and Differential Equations 1(3) (2013), 115-128.