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ON SUBGROUPS OF FINITE INDEX IN QUASI-FREE GROUPS

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Abstract

A group G is said to be a quasi-free group if G is a free product of a number of infinite cyclic groups and a certain number of cyclic groups of order 2. In this paper, we find the rank, structure, and a formula indicating the number of subgroups of finite index in a finitely generated quasi-free group.

1. Introduction

A group is said to be a *quasi-free group* if it is a free product of copies of infinite cyclic groups and of cyclic groups of order 2. So a group G is a quasi-free group if and only if G is a free product of a free group and copies of cyclic groups of order 2. In [6], Mahmood and Khanfar proved that a group G is a quasi-free group if and only if G is a fundamental group of a connected quasi-graph where an edge of the graph equals its inverse is

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allowed. Thus, if C_∞ stands for an infinite cyclic group and C_2 stands for a finite cyclic group of order 2, then a finitely generated quasi-free group G can be written as a free product $G = \underbrace{C_\infty * C_\infty * \cdots * C_\infty}_{p\text{-}factors} * \underbrace{C_2 * C_2 * \cdots * C_2}_{q\text{-}factors}$,

where one of the cases $p = 0$ or $q = 0$ is possible. In this case, G has the presentation $G = \langle y_1, ..., y_p, x_1, ..., x_q \mid x_1^2 = 1, ..., x_q^2 = 1 \rangle$, where all the symbols $y_1, ..., y_p, x_1, ..., x_q$ are distinct. This implies that every free group is a quasi-free group of a free product of a number of infinite cyclic groups and a zero number of cyclic groups of order 2. In view of Grushko-Neumann theorem [5, p. 192], the rank of the quasi-free group G introduced above is $r(G) = p + q$. The aim of this paper is to find the generators and the ranks of subgroups of finite index of finitely generated quasi-free groups by applying the theory of groups acting on trees with inversions introduced in [8], and the structures of subgroups of groups acting on trees with inversions introduced in [7], and then find the number of subgroups of finite index by using the methods of [1]. This paper is divided into 4 sections. In Section 2, we introduce basic concepts of groups acting on trees with inversions. In Section 3, we use the results of [7] and [8] to prove that a group G is a quasi-free group if and only if there exists a tree X such that G acts on X with inversions and the stabilizer of each vertex is trivial. Then we use the results of [7] to show that a subgroup of a quasi-free group is a quasi-free and find a formula of the rank of such subgroups. In Section 4, we generalize Theorem 5.2 [3] of Hall to obtain a formula for the number of subgroups of finite index of finitely generated quasi-free groups by using the methods of [1, Theorem 6.10] of Dey.

2. Basic Concepts of Groups Acting on Trees with Inversions

The theory of groups acting on trees without inversions, known Bass-Serre theory is introduced in [2] and [10] and with inversions is introduced in [8]. We begin a general background. A graph X consists of two disjoint sets $V(X)$ (the set of vertices of X) and $E(X)$ (the set of edges of X), with $V(X)$ non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$, and an involution $\eta : E(X) \to E(X)$ satisfying the conditions $\partial_0\eta = \partial_1$ and $\partial_1\eta = \partial_0$. For simplicity, if $e \in E(X)$, we write $\partial_0(e) = o(e)$, $\partial_1(e) = t(e)$ and $\eta(e) = \overline{e}$. This implies that $o(\overline{e}) = t(e)$, $t(\overline{e}) = o(e)$ and $\overline{\overline{e}} = e$. The case $\overline{e} = e$ is allowed. There are obvious definitions of trees, subtrees, morphisms of graphs and $Aut(X)$, the set of all automorphisms of the graph X which forms a group under the composition of morphisms of graphs. For more details, the interested readers are referred to [2, 8, 10]. We say that a group *G acts on a graph X* if there is a group homomorphism $\phi : G \to Aut(X)$. In this case, if $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. Thus, if $g \in G$, and $y \in E(X)$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$ and $g(\overline{y}) = \overline{g(y)}$. The case $g(y) = \overline{y}$ is allowed for some $g \in G$, and $y \in E(X)$.

Convention. If the group *G* acts on the graph *X* and $x \in X$ (*x* is a vertex or edge), then:

(1) The stabilizer of *x*, denoted G_x is the set $G_x = \{g \in G : g(x) = x\}$. It is clear that $G_x \leq G$, and if $x \in E(x)$ and $u \in \{o(x), t(x)\}$, then $G_{\overline{x}} = G_x$ and $G_x \leq G_u$.

(2) The orbit of *x* is the set $G(x) = \{g(x) : g \in G\} \subseteq X$. It is clear that *G* acts on the graph *X* without inversions if and only if $G(\overline{e}) \neq G(e)$ for any $e \in E(X)$. Otherwise, *G* acts on *X* with inversions if and only if $G(\overline{e}) = G(e)$ for some $e \in E(X)$.

Definition. Let *G* be a group acting on a tree *X* with inversions and let *T* and *Y* be two subtrees of *X* such that $T \subseteq Y$ and each edge of *Y* has at least one end in *T*. Assume that *T* and *Y* satisfy the following:

(i) *T* contains exactly one vertex from each vertex orbit.

(ii) Y contains exactly one edge y (say) from edge orbit if $G(y) \neq G(\bar{y})$ and exactly one pair $x, \bar{x}$ from each edge orbit if $G(x) = G(\bar{x})$. The pair $(T; Y)$ is called a *fundamental domain* for the action of G on X.

For the existence of fundamental domains, we refer the readers to [4].

For the rest of this section, G is a group acting on a tree X with inversions and $(T; Y)$ is the fundamental domain for the action of G on X. We have the following notation:

(1) For any vertex $v \in V(X)$, there exists a unique vertex denoted v^* of T and an element g (not necessarily unique) of G such that $g(v^*) = v$; that is, $G(v^*) = G(v)$. Moreover, if $v \in V(T)$, then $v^* = v$.

(2) For each edge $y \in E(Y)$, the value of y is denoted by $[y]$ and is defined to be an element of G satisfying the following:

(a) If $o(y) \in V(T)$, then $[y]((t(y))^*) = t(y)$, $[y] = 1$ in case $y \in E(T)$, and $[y](y) = \bar{y}$ if $G(y) = G(\bar{y})$.

(b) If $t(y) \in V(T)$, then $[y](o(y)) = (o(y))^*$, $[y] = [\bar{y}]^{-1}$ if $G(y) \neq G(\bar{y})$ and $[y] = [\bar{y}]$ if $G(y) = G(\bar{y})$.

(3) For each edge $y \in E(Y)$, let $+y$ be the edge $+y = y$ if $o(y) \in V(T)$ and $+y = [y](y)$ if $t(y) \in V(T)$. It is clear that $o(+y) = (o(y))^*$ and $G_{+y} \leq G_{(o(y))^*}$ and if $G(y) = G(\bar{y})$ or $y \in E(T)$, then $G_{+y} = G_y$.

In the next two theorems, G is a group acting on a tree X with inversions and $(T; Y)$ is a fundamental domain for the action of G on X. Furthermore, m, y and x stand for edges of $E(Y)$ such that $m \in E(T)$, $o(y) \in V(T)$, $t(y) \notin V(T)$, $G(y) \neq G(\bar{y})$, and $o(x) \in V(T)$, $t(x) \notin V(T)$ and $G(x) = G(\bar{x})$.

Theorem 2.1. *G has the presentation*

$$\langle gen(G_v),\ y,\ x \mid rel(G_v),\ G_m = G_{\overline{m}},\ y \cdot [y]^{-1} G_y [y] \cdot y^{-1} = G_y,$$

$$x \cdot G_x \cdot x^{-1} = G_x,\ x^2 = [x]^2 \rangle.$$

Proof. See [8, Th. 5.1].

Theorem 2.2. *If H is a subgroup of G, $v \in V(T)$ and $e \in E(Y)$, define the following*:

(a) D_v *is a double coset representative system for* $G \bmod(H, G_v)$;

(b) *for any element* $g \in G$, D_e^g *and* $D_{\overline{e}}^g$ *are any double coset representative systems for* $G_{o(e)} \bmod(G_{o(e)} \cap g^{-1}Hg,\ G_e)$ *and*

$$G_{(t(e))^*} \bmod(G_{(t(e))^*} \cap g^{-1}Hg,\ [e]^{-1} G_e [e]).$$

Then for any element $w \in G$, *there exist unique elements denoted* $\overline{w[e]} \in D_{(t(e))^*}$, $\overline{\overline{w[e]}} \in D_{\overline{e}}^{\overline{g}}$, *and an element denoted* $g_e \in G_e$ *such that* $\delta_{w,e} = w g_e [e] \overline{\overline{w[e]}}^{-1} \overline{w[e]}^{-1} \in H$. *Then H is generated by the elements of the following forms*:

(1) *the generators of* $H \cap aG_v a^{-1}$, *where* $a \in D_v$ *and* $v \in V(T)$;

(2) $\delta_{ab,m}$, *where* $a \in D_{o(m)}$ *and* $b \in D_m^a$ *such that* $ab \notin D_{t(m)}$;

(3) $\delta_{ab,y}$, *where* $a \in D_{o(y)}$ *and* $b \in D_y^a$ *such that* $ab[y] \notin D_{(t(y))*}$;

(4) $\delta_{ab,x}$, *where* $a \in D_{o(x)}$ *and* $b \in D_x^a$ *such that* $ab[x] \notin D_{o(x)}$ *and* $H \cap ab[x]G_x b^{-1}a^{-1} = \varnothing$;

(5) $ab[x]b^{-1}a^{-1}$, *where* $a \in D_{o(x)}$ *and* $b \in D_x^a$ *such that* $ab[x] \notin D_{o(x)}$, *and* $H \cap ab[x]G_x b^{-1}a^{-1} \neq \varnothing$.

Proof. See [7, Th. 3].

3. Quasi-free Groups and Groups Acting on Trees with Inversions

In this section, we find the relations between quasi-free groups and groups acting on trees with inversions. First, we start the following lemma:

Lemma 3.1. *A group G is a quasi-free group if and only if there exists a tree X such that G acts on X with inversions and the stabilizer of each vertex is trivial.*

Proof. Let the group G act on the tree X with inversions such that the stabilizer of each vertex $v \in V(X)$ of G is trivial. That is, $G_v = \{1\}$, where 1 is the identity element of G. Then G has a fundamental domain $(T; Y)$ for the action of G on X. By Theorem 2.1, G has the presentation

$$\langle gen(G_v),\ y,\ x \,|\, rel(G_v),\ G_m = G_{\overline{m}},\ y \cdot [y]^{-1} G_y [y] \cdot y^{-1} = G_y,$$

$$x \cdot G_x \cdot x^{-1} = G_x,\ x^2 = [x]^2 \rangle,$$

where m, y and x are edges of $E(Y)$ such that $m \in E(T)$, $o(y) \in V(T)$, $t(y) \notin V(T)$, $G(y) \neq G(\overline{y})$, and $o(x) \in V(T)$, $t(x) \notin V(T)$, $G(x) = G(\overline{x})$. The condition that the stabilizer of each vertex is trivial implies that $G_m = G_v = G_y = \{1\}$ and G has the presentation $G = \langle y, x \,|\, x^2 = 1 \rangle$. This implies that G is a free product of infinite cyclic groups generated by the edges $y \in E(Y)$ and cyclic groups of order 2 generated by the edges $x \in E(Y)$. Now assume that G is a quasi-free group. We need to find a tree X such that G acts on X with inversions and the stabilizer of each vertex $v \in V(X)$ of X is $G_v = \{1\}$. Then G is the free product of infinite cyclic groups generated by t_i, $i \in I$, and cyclic groups of order 2 generated by t_j, $j \in J$. Then G has the presentation $G = \langle t_i, t_j \,|\, t_j^2 = 1 \rangle$ for $i \in I$ and $j \in J$. Let X be the graph where the set of vertices is $V(X) = \{g : g \in G\} = G$, and the set of edges is $E(X) = \{(g, t_i), (g, t_i^{-1}), (g, t_j)\}$, where $g \in G$, $i \in I$ and $j \in J$. For the edges (g, t_i), (g, t_i^{-1}) and (g, t_j), $i \in I$, $j \in J$, define $o(g, t_i) =$

$o(g, t_i^{-1}) = o(g, t_j) = g$, $t(g, t_i) = gt_i$, $t(g, t_i^{-1}) = gt_i^{-1}$, and $t(g, t_j) = gt_j$, and $\overline{(g, t_i)} = (gt_i, t_i^{-1})$, $\overline{(g, t_i^{-1})} = (gt_i^{-1}, t_i)$, and $\overline{(g, t_j)} = (gt_j, t_j^{-1}) = (gt_j, t_j)$ because $t_j^{-1} = t_j$ on which t_j has order 2. G acts on X as follows. Let $f \in G$. Then for the vertex $g \in G$ and the edges (g, t_i), (g, t_i^{-1}) and (g, t_j) of X, define $f(g) = fg$, $f(g, t_i) = (fg, t_i)$, $f(g, t_i^{-1}) = (fg, t_i^{-1})$, and $f(g, t_j) = (fg, t_j)$. The action of G on X is with inversions because the element $t_j \in G$ maps the edge $(1, t_j)$ to its inverse $\overline{(1, t_j)}$; that is, $t_j(1, t_j) = (t_j, t_j) = \overline{(1, t_j)}$. The stabilizer of the vertex $v \in g$ is $G_v = \{1\}$, the stabilizers of the edges (g, t_i), (g, t_i^{-1}) and (g, t_j) are $\{1\}$. Each element $g \in G$, $g \neq 1$ can be written uniquely as $g = g_1 g_2 \ldots g_n$, where $g_s \in \{t_i, t_j : i \in I,\ j \in J\}$, $s = 1, ..., n$. Then it is clear that

$$P_g : (1; g_1), (g_1; g_2), (g_1 g_2; g_3), ..., (g_1 g_2 \ldots g_{n-1}; g_n)$$

is a reduced path in X joining the vertices 1 and g. Then for any non-identity elements a, b of G, $P_a^{-1} P_b$ is a path in X joining the vertices a and b. This implies that X is a connected graph. By the normal theorem for free product of groups, we can show that any two vertices of the graph are joined by exactly one reduced path. Consequently, X is a tree. It is clear that T and Y are subtrees of X, where $T = \{1\}$, $V(Y) = \{1, t_i, t_j : i \in I,\ j \in J\}$ and $E(Y) = \{(1, t_i), (1, t_i^{-1}), (1, t_j), (t_j, t_j)\}$. So $(T; Y)$ is a fundamental domain for the action of G on X. The structure of $E(Y)$ implies that if $e \in E(Y)$, then the value of e is $[e] \in \{t_i, t_i^{-1}, t_j : i \in I,\ j \in J\}$. This completes the proof.

Theorem 3.1. *Let G be a quasi-free group of free product of infinite cyclic groups generated by t_i, $i \in I$ and finite cyclic groups of order* 2 *generated by t_j, $j \in J$.*

Let $A = \{t_i, t_j : i \in I,\ j \in J\}$. *Let H be a subgroup of G and B be a right transversal for H in G. Then for each elements* $a \in A$ *and* $b \in B$, *there exists a unique element denoted* $\overline{ba} \in B$ *such that* $ba(\overline{ba})^{-1} \in H$. *Furthermore, H is a quasi-free group generated by the set* $\{ba(\overline{ba})^{-1} : a \in A, b \in B\}$.

Proof. Lemma 3.1 implies that there exists a tree X such that G acts on X with inversions and the stabilizer of each vertex $v \in E(X)$ under the action of G on X is trivial. That is, $G_v = \{1\}$. This implies that H acts on X and the stabilizer of each vertex $v \in E(X)$ under the action of H on X is trivial because $H_v = H \cap G_v = H \cap \{1\} = \{1\}$. Again, Lemma 3.1 implies that H is a quasi-free group. Since for each edge $x \in E(X)$ of X, $G_x \leq G_u$, where $u \in \{o(x), t(x)\}$, and G_u is trivial, this implies that the stabilizer G_x is trivial. Then the elements D_e^g and $D_{\bar{e}}^g$ of Theorem 2.2 are trivial. So, for any element w of G, we have $\overline{\overline{w[e]}} = 1$. Furthermore, the double cosets become right cosets. Since $T = \{1\}$ is the tree of representatives for the action of G on X, where T has no edges, H is generated by the elements of forms (2) and (3) of Theorem 2.2. As Y is a transversal for the action of G on X, where $V(Y) = \{1, t_i, t_j : i \in I,\ j \in J\}$,

$$E(Y) = \{(1, t_i), (1, t_i^{-1}), (1, t_j), (t_j, t_j)\}$$

and for each $e \in E(Y)$, we have $[e] \in \{t_i, t_i^{-1}, t_j : i \in I,\ j \in J\}$. Then H is generated by $\{ba(\overline{ba})^{-1} : a \in A, b \in B\}$. This completes the proof.

Corollary 3.1. *Let H be a non-decomposable subgroup of the quasi-free group*

$$G = \underbrace{C_\infty * C_\infty * \cdots * C_\infty}_{p\text{-factors}} * \underbrace{C_2 * C_2 * \cdots * C_2}_{q\text{-factors}}.$$

Then either $H \cong C_\infty$ *or* $H \cong C_2$.

4. The Number of Subgroups of Index n in Quasi-free Groups

Given a finitely generated quasi-free group

$$G = \underbrace{C_\infty * C_\infty * \cdots * C_\infty}_{p\text{-}factors} * \underbrace{C_2 * C_2 * \cdots * C_2}_{q\text{-}factors}$$

of rank $r = p + q$. In this section, we obtain a formula that calculates the number of subgroups of G of index n. In the symmetric group S_n, a transposition is a cycle of length 2, and any element of S_n of order 2 is a product of disjoint transpositions. It is clear that S_n has $n(n-1)/2$ transpositions. Furthermore, in [9, p. 133], Rotman showed that the number D_k of sets of k disjoint transpositions in S_n is given by the formula

$$D_k = \frac{1}{k!}\frac{1}{2^k} n(n-1)(n-2)\cdots(n-2k+1).$$

This implies that there are $\sum_{k=0}^{\left[\frac{n}{2}\right]} D_k$ elements of order 2 in S_n, where $[x]$ is the integer function of the real number x. By taking $D_0 = 1$, we see that in S_n, the number of elements E_n of order 2 including the identity element of S_n is given by the formula $E_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} D_k$.

The following lemma is needed to prove the main result of this paper.

Lemma 4.1. (1) *If d_∞^n denotes the number of homomorphisms from the infinite cyclic group C_∞ to the symmetric group S_n, then $d_\infty^n = n!$.*

(2) *If d_2^n denotes the number of homomorphisms from the cyclic group C_2 to the symmetric group S_n, then $d_2^n = E_n$.*

Proof. (1) Let c be a generator of C_∞ and g be any element of S_n, and $\phi : \{c\} \to S_n$ be the mapping given by $\phi(c) = g$. C_∞ being a free group of base $\{c\}$ implies that there exists a unique homomorphism $\widetilde{\phi} : C_\infty \to S_n$ given by the formula $\widetilde{\phi}(c^m) = g^m$, g being an arbitrary element of S_n and the order of S_n is $|S_n| = n!$ which implies that $d_\infty^n = n!$.

(2) Any homomorphism from C_2 to S_n takes every element of C_2 to the identity element of S_n, or takes the identity to the identity, and takes the generator of C_2 to any element of S_n of order 2. Consequently, there are E_n homomorphisms from C_2 to S_n.

This completes the proof.

Convention. Let $d_\infty^0 = d_2^0 = 1$.

The main result of this paper is the following theorem:

Theorem 4.1. *The number of subgroups N_n of finite index n of the finitely generated quasi-free group*

$$G = \underbrace{C_\infty * C_\infty * \cdots * C_\infty}_{p\text{-factors}} * \underbrace{C_2 * C_2 * \cdots * C_2}_{q\text{-factors}}$$

is given by the formula $N_1 = 1$ if $n = 1$, and if $n > 1$, then

$$N_n = \frac{(n!)^p (E_n)^q}{(n-1)!} - \sum_{i=1}^{n-1} \frac{[(n!)^p (E_n)^q]^{n-1} N_i}{(n-1)!}.$$

Proof. In [1, Theorem 6.10], Dey proved that if $A = \prod^* A_j \ (j \in J)$ is a free product of the groups A_j, where J is the finite set $\{1, 2, ..., k\}$, and $d_j^n \ (n > 0)$ is the number of homomorphisms of A_j into the symmetric group S_n on n symbols and $d_j^0 = 1$ then the number N_n of the subgroups of A of index n is given by $N_1 = 1$ and if $n > 1$, then

$$N_n = \frac{\prod_{j=1}^{k} d_j^n}{(n-1)!} - \sum_{i=1}^{n-1} \left\{ \frac{1}{(n-i)!} \right\} \prod_{j=1}^{k} d_j^{n-i} N_i.$$

In our case, the product $\prod_{j=1}^{k} d_j^n$ is replaced by the product

$$\underbrace{d_\infty^n \times d_\infty^n \times \cdots \times d_\infty^n}_{p\text{-}factors} \times \underbrace{d_2^n \times d_2^n \times \cdots \times d_2^n}_{q\text{-}factors} = (d_\infty^n)^p (d_2^n)^q = (n!)^p (E_n)^q.$$

From above, we see that the number of subgroups N_n of finite index n of the group G is given by the formula

$$N_n = \frac{\prod_{j=1}^{k} d_j^n}{(n-1)!} - \sum_{i=1}^{n-1} \left\{ \frac{1}{(n-i)!} \right\} \prod_{j=1}^{k} d_j^{n-i} N_i.$$

This completes the proof.

We have the following corollaries:

Corollary 1. *If G is the finitely generated free group*

$$G = \underbrace{C_\infty * C_\infty * \cdots * C_\infty}_{p\text{-}factors}$$

of rank p, then the number of subgroups N_n *of finite index n of G is given by the formula* $N_1 = 1$ *if* $n = 1$, *and if* $n > 1$, *then*

$$N_n = \frac{(n!)^p}{(n-1)!} - \sum_{i=1}^{n-1} \frac{[(n!)^p]^{n-i} N_i}{(n-i)!}$$

which is the formula of Theorem 5.2 *of* [3].

Corollary 2. *If G is the quasi-free group*

$$G = \underbrace{C_2 * C_2 * \cdots * C_2}_{q\text{-}factors}$$

of rank q, *then the number of subgroups of finite index n of G is given by the formula* $N_1 = 1$ *if* $n = 1$, *and if* $n > 1$, *then*

$$N_n = \frac{(E_n)^q}{(n-1)!} - \sum_{i=1}^{n-1} \frac{[(E_n)^q]^{n-1} N_i}{(n-1)!}.$$

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