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CATEGORIFICATION VIA EQUIPPED GRAPHS

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Abstract

Indecomposable representations of the modular lattice $\mathcal{L}(A_n^0)$, where A_n^0 is an equipped Dynkin diagram, are used to give a categorification for Catalan numbers.

1. Introduction

The term *categorification* was coined by Fahr and Ringel to the process

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which consists of considering numbers in an integer sequence as invariants of objects of a given category in such a way that identities between numbers in the sequence can be viewed as functional relations between objects of the category [5, 6]. In such a work, the Gabriel's functor which identifies dimensions of representations of a given quiver Q and its corresponding universal covering $\overline{Q}$ plays a prominent role. In fact, Ringel and Fahr gave a categorification of Fibonacci numbers by using the Auslander-Reiten quiver of the 3-Kronecker quiver and its universal covering to obtain the following partitions for even index Fibonacci numbers:

$$f_{4t} = \sum_{r \text{ odd}} |T_r| a_t[r] = 3\sum_{m\geq 0} 2^{2m} a_t[2m+1],$$

$$f_{4t+2} = \sum_{r \text{ even}} |T_r| a_t[r] = a_t[0] + 3\sum_{m\geq 1} 2^{2m-1} a_t[2m], \tag{1}$$

where $|T_r|$ denotes a suitable distance between points of the Gabriel's universal covering of the 3-Kronecker quiver whereas symbols $a_t[r]$ denote suitable products of reflections in such a covering, in particular, sequences $a_t[0]$ and $a_t[1]$ are now listed in the On-Line Encyclopedia of Integer Sequences (OEIS) as A132262 and A110122, respectively [16]. We also recall that recently the first author et al. used the same techniques in order to obtain categorifications of sequences A002662, A016269 and A052558. To do that, they used the poset representation theory, semimaximal rings and Kronecker modules, respectively, [1-3].

In this paper, indecomposable representations of the modular lattice $\mathcal{L}(A_n^0)$ induced by the equipped Dynkin diagram A_n^0 in the sense of Gel'fand and Ponomarev are used to describe a categorification for Catalan numbers [8, 10].

This paper is organized as follows. Some of the basic definitions and notations regarding lattices are included in Section 2. In Section 3, we categorize Catalan numbers by counting the number of lattice paths in the distributive lattice $\mathcal{P}(A_n^0)$ of indecomposable representations of the modular lattice $\mathcal{L}(A_n^0)$.

2. Preliminaries

In this section, we introduce basic definitions and notations regarding posets and lattices as Stanley and the first author et al. describe in [4, 13, 14].

Let $\mathcal{P}$ be an ordered set and let $x, y \in \mathcal{P}$ we say x is *covered* by y if $x < y$ and $x \leq z < y$ implies $z = x$.

An *order ideal* of a poset $(\mathcal{P}, \leq)$ is a subset I of $\mathcal{P}$ such that if $x \in I$ and $y \leq x$, then $y \in I$. We let $J(\mathcal{P})$ denote the set of all order ideals of $\mathcal{P}$, ordered by inclusion. In particular, we define the order ideal or *down-set* of $a \in \mathcal{P}$ to be $a_\Delta = \{q \in \mathcal{P} : q \leq a\}$. Dually, $a^\nabla = \{q \in \mathcal{P} : a \leq q\}$ is the *filter* or *up-set* of a [11].

Note that, k-element antichains in $\mathcal{P}$ correspond to elements of $J(\mathcal{P})$ that cover exactly k-elements. If n is a positive integer we let $\mathbf{n}$ denote the n-element poset with the special property that any two elements are comparable [14].

A *lattice* is a poset L for which every pair of elements has a least upper bound and greatest lower bound. We say that a poset $\mathcal{P}$ *has a* $\hat{0}$ if there exists an element $\hat{0} \in \mathcal{P}$ such that $\hat{0} \leq x$ for all $x \in \mathcal{P}$. Similarly, $\mathcal{P}$ *has a* $\hat{1}$ if there exists $\hat{1} \in \mathcal{P}$ such that $x \leq \hat{1}$ for all $x \in \mathcal{P}$. Clearly all finite lattices have $\hat{0}$ and $\hat{1}$. Since the union and intersection of order ideals is again an order ideal, it follows from the well-known distributivity of set union and intersection over one another that $J(\mathcal{P})$ is indeed a distributive lattice [14].

A finite nonnegative *lattice path* in the plane (with unit steps to the right and down) is a sequence $L = (v_1, v_2, \ldots, v_k)$, where $v_i \in \mathbb{N}^2$ and $v_{i+1} - v_i = (1, 0)$ or $(0, -1)$ [14].

Given a finite poset $\mathcal{P}$ with $|\mathcal{P}| = n$ in [14] it is defined an *extension of* $\mathcal{P}$ *to a total order or linear extension of* $\mathcal{P}$ as an order-preserving bijection $\sigma : \mathcal{P} \to \mathbf{n}$. The number of extensions of $\mathcal{P}$ to a total order is denoted $e(\mathcal{P})$. Actually, $e(\mathcal{P})$ is also equal to the number of maximal chains of $J(\mathcal{P})$.

We may identify a maximal chain of $J(\mathcal{P})$ with a certain type of lattice path in Euclidean space as follows [14]. Let $C_1, \ldots, C_k$ be a partition of $\mathcal{P}$ into chains. Define a map $\delta : J(\mathcal{P}) \to \mathbb{N}^k$ by:

$$\delta(I) = (|I \cap C_1|, |I \cap C_2|, \ldots, |I \cap C_k|).$$

If we give to $\mathbb{N}^k$ the obvious product order, then δ is an injective lattice homomorphism that is cover-preserving (and therefore rank-preserving). Thus, in particular, $J(\mathcal{P})$ is isomorphic to a sublattice of $\mathbb{N}^k$. Given $\delta : J(\mathcal{P}) \to \mathbb{N}^k$, as above, define $\Gamma_\delta = \bigcup_T cx(\delta T)$, where cx denotes convex hull in $\mathbb{R}^k$, T ranges over all intervals of $J(\mathcal{P})$ that are isomorphic to boolean algebras. Thus Γ_δ is a compact polyhedral subset of $\mathbb{R}^k$. It is then clear that the number of maximal chains in $J(\mathcal{P})$ is equal to the number of lattice paths from the origin $(0, 0, \ldots, 0) = \delta(\hat{0})$ to $\delta(\hat{1})$, with unit steps in the direction of the coordinate axes. In other words, $e(\mathcal{P})$ is equal to the number of ways of writing

$$\delta(\hat{1}) = v_1 + v_2 + \cdots + v_n, \tag{2}$$

where each v_i is a unit coordinate vector in $\mathbb{R}^k$ and where $v_1 + v_2 + \cdots + v_i \in \Gamma_\delta$, for all i.

For example, let $\mathcal{M}_n = \mathbf{2} \times \mathbf{n}$, and $C_1 = \{(2, j) \mid j \in \mathbf{n}\}$, $C_2 = \{(1, j) \mid j \in \mathbf{n}\}$. Then $\delta(J(\mathcal{M}_n)) = \{(i, j) \in \mathbb{N}^2 \mid 0 \leq i \leq j \leq n\}$. Following figure represents the case $n = 3$ [4].

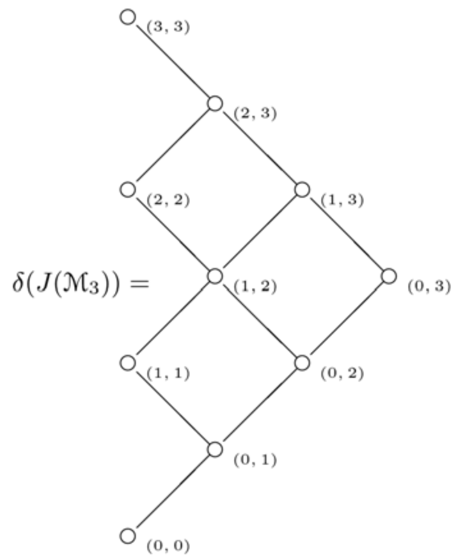


Figure 1

In fact, for each $n \geq 1$, $\delta(J(\mathcal{M}_n))$ is a poset endowed with the order $\preceq$ such that:

$$(i,\ j) \preceq (i',\ j') \text{ if and only if } i \leq i' \text{ and } j \leq j',$$
$$\text{for all } (i,\ j), (i',\ j') \in \delta(J(\mathcal{M}_n)). \tag{3}$$

Hence $e(\mathcal{M}_n)$ is equal to the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$ and $(0, 1)$, which never rise above the main diagonal $x = y$ of the $(x,\ y)$-plane. It can be shown that

$$e(\mathcal{M}_n) = \frac{1}{n+1}\binom{2n}{n} = C_n. \tag{4}$$

These numbers are called *Catalan numbers* [7, 12, 14].

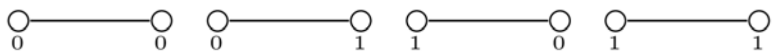
3. Categorification of Catalan Numbers

In this section, we introduce equipped graphs and the category of representations of the modular lattice induced by such graphs as Gel'fand and Ponomarev described in [8] and [10]. In particular, it is described a generalization of the Gabriel's theorem which establishes a bijection between positive roots of a given quiver Ω and the indecomposable representations of the category of representations $\mathrm{rep}_k\Omega$ over a given field k. Further, in this section, we also describe how indecomposable representations of modular lattices induced by equipped graphs allow to describe a categorification of Catalan numbers.

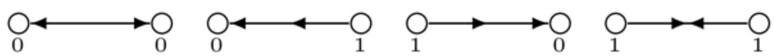
3.1. Equipped graphs

If $\Gamma = (\Gamma_0, \Gamma_1, \mu)$ is a graph without orientation, where Γ_0 is the set of vertices, Γ_1 is the set of edges and μ is a function that assigns to each edge w a set $\mu(w)$ of vertices, $\mu(w) = \{i, j\}$, $i, j \in \Gamma_0$ [10].

An equipped graph Γ^{ν} is a graph without loops and orientation endowed with a collection of functions $\{\nu_w\}$ such that for each $w \in \Gamma_1$, $\nu_w : \mu(w) \to \{0, 1\} \subset \mathbb{N}$. Hence in Γ^{ν} there are four types of equipped edges:



Sometimes these edges are presented in the following way:



Thus each oriented graph can be associated to an equipped graph by assigning to an oriented edge of the form $\circ\!\!\longrightarrow\!\!\circ$ a corresponding equipped edge starting in 1 and ending in 0.

Each equipped graph Γ^{ν} induces a modular lattice $\mathcal{L}(\Gamma^{\nu})$ which is defined by generators $\{v_i, h_w\}$ with $i \in \Gamma_0$ and $w \in \Gamma_1$ and the relations:

(a_Γ) $v_i\left(\sum_{j \neq i} v_j\right) = 0$ for each vertex $i \in \Gamma_0$,

(b_Γ) if $\mu(w) = \{i, j\}$, then $h_w \subseteq v_i + v_j$ for each $w \in \Gamma_1$,

(c_Γ) if $\nu_w(i) = 0$, then $h_w v_i = 0$,

(d_Γ) if $\nu_w(i) = 1$, then $h_w + v_j = v_i + v_j$.

In accordance with the definition, the modular lattice $\mathcal{L}(A_n^{\nu})$ induced by an equipped Dynkin diagram A_n^{ν} is generated by $\{v_i, h_j \mid 1 \leq i \leq n, 1 \leq j \leq$

$n-1\}$ with relations of the form:

(a_{A_n}) $v_i\left(\sum_{j\neq i} v_j\right)=0,$ for all i,

(b_{A_n}) $h_j \subseteq v_j + v_{j+1},$

(c_{A_n}) $h_j v_s = 0,$ if $\nu_j(s)=0,$ where $s \in \{j,\ j+1\},$

(d_{A_n}) $h_j + v_t = v_s + v_t$ if $\nu_j(s)=1$ and $\{s,\ t\} \in \{j,\ j+1\}.$

For instance, if $x = wv_1 + wv_2 = (v_1 + wv_2)(v_2 + wv_1)$ and $y = (v_1 + w)(v_2 + w) = (v_1 + w)v_2 + (v_2 + w)v_1,$ then

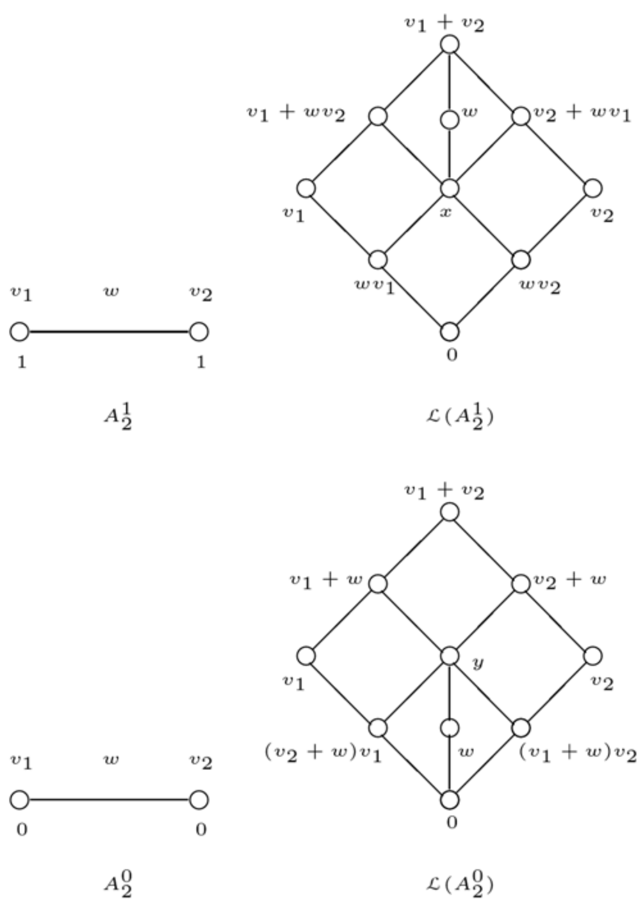


Figure 2

3.2. The category of representations of a modular lattice

Ponomarev [10] defined the category of representations of a lattice $\mathcal{L}$ as follows:

If $(\mathcal{L}, \leq)$ is a modular lattice with maximal and minimal points $\hat{1}$ and $\hat{0}$, respectively, V is a finite-dimensional vector space and $\mathcal{L}(V)$ is the lattice of all vector subspaces of V, then a *representation* ρ of $\mathcal{L}$ in V is a lattice homomorphism $\rho : \mathcal{L} \to \mathcal{L}(V)$ such that $\rho(\hat{0}) = 0$ and $\rho(\hat{1}) = V$, ρ associates to any pair of points x, y subspaces $\rho(x)$ and $\rho(y)$ in such a way that $\rho(xy) = \rho(x)\rho(y)$ and $\rho(x+y) = \rho(x) + \rho(y)$.

If ρ_1 and ρ_2 are representations of the same lattice $\mathcal{L}$ in spaces V_1 and V_2 over the same field k, then a *morphism* $\psi : \rho_1 \to \rho_2$ is a linear map $\psi : V_1 \to V_2$ such that $\psi\rho_1(x) \subseteq \rho_2(x)$ for every $x \in \mathcal{L}$. Henceforth, we let $\mathrm{rep}_k\mathcal{L}$ denote the category of representations of the lattice $\mathcal{L}$ over a given field k. Actually, a representation ρ of a lattice $\mathcal{L}$ over a field k in a k-vector space V can be interpreted as a system of subspaces of the form:

$$\rho(\mathcal{L}) = (V; V_x \mid x \in \mathcal{L}),$$

where $V_{\hat{0}} = 0$, $V_{\hat{1}} = V$, $V_{x+y} = V_x + V_y$, $V_{xy} = V_x \cap V_y$ and $V_x \subseteq V_y$ if $x \leq y$ in $\mathcal{L}$. Note that $\mathrm{rep}_k\mathcal{L}$ is a Krull-Schmidt category.

Ponomarev proved the following results [9, 10]:

Theorem 1. *The category* $\mathrm{rep}_k\Gamma^{\vee}$ *of representations of an equipped graph* $\Gamma^{\vee}$ *is equivalent to the category* $\mathrm{rep}_k\mathcal{L}(\Gamma^{\vee})$ *of representations of the modular lattice* $\mathcal{L}(\Gamma^{\vee})$.

Theorem 2. *The category* $\mathrm{rep}_k\mathcal{L}(\Gamma^{\vee})$ *of representations of the modular lattice* $\mathcal{L}(\Gamma^{\vee})$ *of a connected graph* Γ *is of finite representation type, i.e., has a finite number of mutually non-isomorphic indecomposable* *representations if and only if* Γ *is one of the Dynkin diagrams* A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 *or* E_8.

The following result generalizes the Gabriel's theorem [10].

Theorem 3. *There is one-to-one correspondence between the indecomposable representations of a lattice* $\mathcal{L}(A_n^{\nu})$ *and the positive roots of the Dynkin diagram* A_n.

Each positive root α can be written in the form $\alpha = \sum_{i \leq j \leq k} \alpha_j$, where $1 \leq i \leq k \leq n$ and α_j is a simple root. In [10], Ponomarev denoted such roots as α_{ik} and $\tau_\alpha = \tau_{ik}$ the corresponding indecomposable representation.

The representation $\tau_{i,k}$ is a representation in a vector space $T_{i,k} \simeq k^{k-i+1}$ over a field *k*. $T_{i,k}$ has a basis $\{e_j \mid i \leq j \leq k\}$ such that:

$$\tau_{i,k}(v_j) = \begin{cases} ke_j, & \text{if } i \leq j \leq k, \\ 0, & \text{if } j < i \text{ or } k < j, \end{cases}$$

$$\tau_{i,k}(w_j) = \begin{cases} k(e_j + e_{j+1}), & \text{if } i \leq j \leq k, \\ 0, & \text{if } k < j, \\ 0, & \text{if } j < i - 1, \end{cases}$$

$$\tau_{i,k}(w_{i-1}) = \begin{cases} ke_i, & \text{if } \nu_{i-1}(i) = 1, \\ 0, & \text{if } \nu_{i-1}(i) = 0, \end{cases}$$

$$\tau_i(w_k) = \begin{cases} ke_k, & \text{if } \nu_k(k) = 1, \\ 0, & \text{if } \nu_{i-1}(i) = 0. \end{cases}$$

The classes of isomorphisms of indecomposable representations of $\mathcal{L}(A_n^{\nu})$ constitute a poset $\mathcal{P}(A_n^{\nu}, \trianglelefteq)$ defined in such a way that:

$\tau_\beta \trianglelefteq \tau_\alpha$ if and only if there is a sequence $\tau_1, \tau_2, ..., \tau_m \in \mathcal{P}(A_n^{\nu})$ such that $\tau_1 \simeq \tau_\alpha \tau_m \simeq \tau_\beta$ and $\mathrm{Hom}(\tau_j, \tau_{j+1}) \neq 0$ for all $1 \leq j \leq m$.

Remark 4. The ordering $\trianglelefteq$ can be defined on $\mathcal{L}(A_n^0)$ in such a way that $\tau_{i_1, j_1} \trianglelefteq \tau_{i_2, j_2}$ if and only if $i_1 \leq i_2 \leq j_2 \leq j_1$ [9].

3.3. Categorification

The following result associates Catalan numbers to the distributive lattice $\mathcal{P}(A_n^0)$.

Theorem 5. *If C_n is the nth Catalan number, then C_{n-1} is the number of lattice paths from* $(1, 1)$ *to* (n, n) *in* $(\mathcal{P}(A_n^0), \trianglelefteq)$.

Proof. Since the poset $(\mathcal{P}(A_n^0), \trianglelefteq)$ can be defined in such a way that if $\alpha = (i_1, j_1)$ and $\beta = (i_2, j_2)$, then $\tau_\beta \trianglelefteq \tau_\alpha$ if and only if $i_2 \leq i_1 \leq j_1 \leq j_2$ if and only if $\beta \leq \alpha$. If we let $h(1, n)$ denote the number of lattice paths from $(1, 1)$ to (n, n), then it is possible to define a bijection $\sigma : \mathcal{P}(A_n^0) \to \delta(J(\mathbf{2} \times \mathbf{n} - 1))$ such that $\sigma(i, j) = (i - 1, j - 1)$. Thus, there exists a bijection between the set of lattice paths of $(\mathcal{P}(A_n^0), \trianglelefteq)$ from $(1, 1)$ to (n, n) and the set of lattice paths in $(\delta(J(\mathbf{2} \times \mathbf{n} - 1)), \preceq)$ from $(0, 0)$ to $(n - 1, n - 1)$. Therefore, $h(1, n) = e(\mathbf{2} \times \mathbf{n} - 1)$ and with this identity we are done. □

We also have the following result.

Corollary 6. *The number of lattice paths from $(1, n)$ to (i, j) in $(\mathcal{P}(A_n^0), \trianglelefteq)$, $1 \leq i \leq j \leq n$ is* $h(1, i, j) = \binom{n + i - j - 1}{i - 1}$.

Proof. The map $\gamma : (\mathcal{P}(A_n^0), \trianglelefteq) \to (\underline{\mathbf{n} \times \mathbf{n}}, \preceq)$ defined in such a way that $\gamma((i, j)) = (i - 1, n - j)$ is a poset isomorphism, where $\underline{\mathbf{n} \times \mathbf{n}} = \{(i, j) \in \mathbf{n}^2 \mid 0 \leq i \leq j \leq n\}$. Then the result follows since for any $I \in J(\mathbf{n} \times \mathbf{n})$ the number of saturated chains between 0 and I is given by the identity $e(I) = e(I_1) + \cdots + e(I_r)$, where the terms I_i, $1 \leq i \leq r$ are the elements of

$J(\mathbf{n} \times \mathbf{n})$ which I covers, it holds that the segment $[0, I]$ has the form $(\mathbf{i}+1) \times (\mathbf{j}+1)$ and $e(I) = \binom{i+j}{j}$ which is the number of lattice paths from $(0, 0)$ to (i, j) in $\mathbf{n} \times \mathbf{n}$. □

Remark 7. On $\mathcal{P}(A_n^0)$ we recall some ideas regarding the subject given by Stanley in [15]. In particular, C_k is the total number of elements in $J(S(\mathbf{k}-1))$, where $S(\mathcal{P})$ denotes the set of segments (or intervals) of $\mathcal{P}$ ordered by inclusion. Moreover, the number of maximal chains in $J(S(\mathbf{k}))$ is given by the formula:

$$\frac{\binom{k+1}{2}!}{(2k-1)(2k-3)^2(2k-5)^3 \cdots 3^{k-1}1^k}. \tag{5}$$

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