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A NOTE ON LIE-YAMAGUTI SUPERALGEBRAS

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Abstract

A Lie-Yamaguti algebra (or general Lie triple system or Lie triple algebra) is a tangent algebraic system of the reductive homogeneous space studied by Nomizu in [3]. The class of Lie-Yamaguti algebras contains Malcev algebras and Lie triple systems as special classes.

In this paper, we describe and give some properties of Lie-Yamaguti superalgebras. Also, we study a particular class of Lie-Yamaguti superalgebras.

1. Introduction

In what follows, the algebras and superalgebras considered will be defined over a field k of characteristic 0.

A Lie-Yamaguti algebra $(T, \cdot, [\ ,\ ,\ ])$ [7] is a vector space T equipped with a bilinear operation: $T \times T \to T$ and a trilinear operation $[\ ,\ ,\ ] : T \times T \times T \to T$ satisfying the following identities for any $x, y, z, v, w \in T$:

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(1) $xy = -yx;$

(2) $[x, y, z] = -[y, x, z];$

(3) $\sum_{cycl\ x, y, z} \{[x, y, z] + (xy)z\} = 0;$

(4) $\sum_{cycl\ x, y, z} [xy, z, w] = 0;$

(5) $[x, y, zw] = [x, y, z]w + z[x, y, w];$

(6) $[x, y, [z, v, w]] = [[x, y, z], v, w] + [z, [x, y, v], w] + [z, v, [x, y, w]].$

Lie-Yamaguti algebras are reduced to Lie algebras if we define $[x, y, z] = 0$ or $[x, y, z] = (xy)z$ while Lie-Yamaguti algebras with $xy = 0$ are Lie triple systems. In [2], Kinyon and Weinstein showed that if $(L, .)$ is a Leibniz algebra, that is, $x(yz) = (xy)z - y(xz),$ for any x, y, z in L, then $(L, [\ ,\ ], [\ ,\ ,\ ])$ with $[x, y] = \frac{1}{2}(xy - yx);$ $[x, y, z] = -\frac{1}{4}(xy)z$ is a Lie-Yamaguti algebra. Also, in [6], Yamaguti showed that if in a Malcev algebra $(M, \cdot)$ we set $[x, y, z] = x(yz) - y(xz) + (xy)z,$ then $(M, \cdot, [\ ,\ ,\ ])$ is a Lie-Yamaguti algebra. Then it is clear that Lie-Yamaguti algebras are a generalization of Lie algebras, Malcev algebras and Lie triple systems. In [5], Sagle gave the construction of remarkable Lie-Yamaguti algebras and in [1] examples of irreducible Lie-Yamaguti algebras are provided.

The purpose of this paper is twofold. We first introduce the concept and establish some properties of Lie-Yamaguti superalgebras and we study a particular class of Lie-Yamaguti superalgebras composed of Lie supertriple systems and Lie superalgebras. Second, we show that a Lie superalgebra with a reductive decomposition induces the structure of a Lie-Yamaguti superalgebra on the reductive complement to the $\mathbb{Z}_2$-graded subalgebra and we show that every Lie-Yamaguti superalgebra has an enveloping Lie superalgebra.

2. Lie-Yamaguti Superalgebras

Let us recall that a superalgebra over k is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$, where $A_i A_j \subset A_{i+j}$. The subspaces A_0 and A_1 are called the *even* and the *odd parts* of the superalgebra and so are called the *elements* from A_0 and A_1, respectively. Below all the elements are assumed to be homogeneous, that is, either even or odd and for a homogeneous element $x \in A_i$, $i = 0, 1$, the notation $\bar{x} = i$ is used and means the parity of x.

Let $G = alg(1, e_i, i = 1, 2, \ldots)$ be the *Grassmann algebra* over a countable set of generators e_i, with $e_i^2 = 0$, $e_i e_j = -e_j e_i$ for $i \neq j$. The elements $1, e_{i_1} e_{i_2} \cdots e_{i_r}$, $i_1 < i_2 < \cdots < i_r$ form a k basis of G. Denote by G_0 (respectively G_1) the span of the products of even length (respectively odd length) in the generators. The product of zero $e_i's$ is by convention equal to 1. Then $G = G_0 \oplus G_1$ is an associative and supercommutative superalgebra, that is, $g_1 g_2 = (-1)^{\bar{g}_1 \bar{g}_2} g_2 g_1$, where $g_1, g_2 \in G_0 \cup G_1$. Let $A = A_0 \oplus A_1$ be a superalgebra; the graded tensor product $G \otimes A$, where G is the Grassmann superalgebra, becomes a superalgebra with the product given by $(x \otimes g_1)(y \otimes g_2) = (-1)^{\bar{x}\bar{g}_2} xy \otimes g_1 g_2$, for homogeneous elements $g_1, g_2 \in G$, $x, y \in A$ and grading given by $(G \otimes A)_0 = G_0 \otimes A_0 \oplus G_1 \otimes A_1$, $(G \otimes A)_1 = G_0 \otimes A_1 \oplus G_1 \otimes A_0$. By the *Grassmann envelope* of the superalgebra $A = A_0 \oplus A_1$, we mean the subalgebra $G(A) = (G \otimes A)_0 = G_0 \otimes A_0 \oplus G_1 \otimes A_1$ of the tensor product $G \otimes A$.

Let V be a homogeneous variety of algebras, that is, a class of algebras satisfying a certain set of homogeneous identities and all their partial linearizations [8]. A superalgebra $A = A_0 \oplus A_1$ is called a V-superalgebra, if its Grassmann envelope $G(A)$ belongs to V.

Thus, the concept of Lie-Yamaguti superalgebra introduced here has been modeled with the aim in mind that a superalgebra $T = T_0 \oplus T_1$ should be a Lie-Yamaguti superalgebra if its Grassmann envelope $G(T)$ is a Lie-Yamaguti algebra. We have the following definition.

Definition 2.1. A Lie-Yamaguti superalgebra is a $\mathbb{Z}_2$-graded vector space $T = T_0 \oplus T_1$, with a bilinear operation: $T \times T \to T$ and a trilinear operation $[\ ,\ ,\ ] : T \times T \times T \to T$ such that for any $x, y, z, v, w \in T_0 \cup T_1$, $i, j, k \in \mathbb{Z}_2$,

$(LYS1)$ $T_i T_j \subset T_{i+j}$; $i + j$ calculated modulo 2;

$(LYS2)$ $[T_i, T_j, T_k] \subset T_{i+j+k}$; $i + j + k$ calculated modulo 2;

$(LYS3)$ $xy = -(-1)^{\bar{x}\bar{y}} yx$;

$(LYS4)$ $[x, y, z] = -(-1)^{\bar{x}\bar{y}}[y, x, z]$;

$(LYS5)$ $\sum_{cycl\ x, y, z} (-1)^{\bar{x}\bar{z}}([x, y, z] + (xy)z) = 0$;

$(LYS6)$ $\sum_{cycl\ x, y, z} (-1)^{\bar{x}\bar{z}}[xy, z, w] = 0$;

$(LYS7)$ $[x, y, zw] = [x, y, z]w + (-1)^{\bar{z}(\bar{x}+\bar{y})} z[x, y, w]$;

$(LYS8)$ $[x, y, [z, v, w]] = [[x, y, z], v, w]$

$$+ (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, [x, y, v], w] + (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z, v, [x, y, w]].$$

Remark 2.2. 1. Obviously, T_0 is a Lie-Yamaguti algebra.

2. In [4] (see Definition II.1.4), the concept of Lie-Yamaguti superalgebras introduced as Lie triple superalgebras is defined as follows: a Lie triple superalgebra is a $\mathbb{Z}_2$-graded vector space $T = T_0 \oplus T_1$, with a bilinear operation: $T \times T \to T$ and a trilinear operation $[\ ,\ ,\ ] : T \times T \times T$

$\to T$ such that for any $x,\ y,\ z,\ v,\ w \in T_0 \cup T_1,\ i,\ j,\ k \in \mathbb{Z}_2,$

(i) $xy = -(-1)^{\bar{x}\bar{y}} yx;$

(ii) $[x,\ y,\ z] = -(-1)^{\bar{x}\bar{y}}[y,\ x,\ z];$

(iii) $[x,\ y,\ z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[y,\ z,\ x]$

$$+ (-1)^{\bar{z}(\bar{x}+\bar{y})}[z,\ x,\ y] + (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}+\bar{x}\bar{z}}\tilde{J}(x,\ y,\ z) = 0;$$

(iv) $[xy,\ z,\ v] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[yz,\ x,\ v] + (-1)^{\bar{y}(\bar{x}+\bar{z})}[zx,\ y,\ v] = 0;$

(v) $[x,\ y,\ [z,\ v,\ w]] = [[x,\ y,\ z],\ v,\ w]$

$$+ (-1)^{\bar{z}(\bar{x}+\bar{y})}[z,\ [x,\ y,\ v],\ w] + (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z,\ v,\ [x,\ y,\ w]],$$

where $\tilde{J}(x,\ y,\ z) = (xy)z - x(yz) - (-1)^{\bar{y}\bar{z}}(xz)y.$

By setting

$$\{x,\ y,\ z\} = (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}+\bar{x}\bar{z}}[x,\ y,\ z],$$

we can see by straightforward calculations that if $(T, ., [\ ,\ ,\ ])$ is a Lie-Yamaguti superalgebra in our sense, then $(T, ., \{\ ,\ ,\ \})$ is a Lie triple superalgebra in the sense of [4] and conversely.

Proposition 2.3. *A superalgebra* $T = T_0 \oplus T_1$ *equipped with bilinear and trilinear products satisfying* $T_iT_j \subset T_{i+j}$ *and* $[T_i,\ T_j,\ T_k] \subset T_{i+j+k}$ *is a Lie-Yamaguti superalgebra if its Grassmann envelope* $G(T) = G_0 \otimes T_0 \oplus G_1 \otimes T_1$ *is a Lie-Yamaguti algebra under the following products*:

$$(x \otimes g_1)(y \otimes g_2) = (-1)^{\bar{x}\bar{y}} xy \otimes g_1g_2,$$

$$[x \otimes g_1,\ y \otimes g_2,\ z \otimes g_3] = (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}+\bar{x}\bar{z}}[x,\ y,\ z] \otimes g_1g_2g_3.$$

Proof. The proof is straightforward by using the fact that for any element $x \otimes g$ in $G(T)$, we have $\bar{x} = \bar{g}$. □

Examples. 1. It is clear that Lie superalgebras are Lie-Yamaguti superalgebras with $[x, y, z] = 0$.

2. If $xy = 0$ for any $x, y \in T_0 \cup T_1$, then $(LYS1)$, $(LYS3)$, $(LYS6)$, $(LYS7)$ are trivial and $(LYS2)$, $(LYS4)$, $(LYS5)$, $(LYS8)$ define a Lie supertriple system that is a $\mathbb{Z}_2$-graded vector space $T = T_0 \oplus T_1$, with trilinear composition $[\ ,\ ,\ ] : T \times T \times T \to T$ such that for any $x, y, z, v, w \in T_0 \cup T_1$, $i, j, k \in \mathbb{Z}_2$,

$(LSS1)$ $[T_i, T_j, T_k] \subset T_{i+j+k}$; $i + j + k$ calculated modulo 2;

$(LSS2)$ $[x, y, z] = -(-1)^{\bar{x}\bar{y}}[y, x, z]$;

$(LSS3)$ $\sum_{cycl\ x, y, z} (-1)^{\bar{x}\bar{z}}[x, y, z] = 0$;

$(LSS4)$ $[x, y, [z, v, w]] = [[x, y, z], v, w]$

$$+ (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, [x, y, v], w] + (-1)^{(\bar{z}+\bar{v})(x+\bar{y})}[z, v, [x, y, w]].$$

Let us recall that a superalgebra $M = M_0 \oplus M_1$ is said to be a *Malcev superalgebra* if the following identities are satisfied:

$$xy = -(-1)^{\bar{x}\bar{y}} yx;$$

$$-(-1)^{\bar{y}\bar{z}}(xz)(yt) = ((xy)z)t + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{t})}((yz)t)x$$

$$+ (-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{t})}((zt)x)y + (-1)^{\bar{t}(\bar{x}+\bar{y}+z)}((tx)y)z.$$

The next proposition is a superanalogue of the corresponding result for Malcev algebras [6].

Proposition 2.4. *Let* $(T, ., [\ ,\ ,\ ])$ *be a Lie-Yamaguti superalgebra such that* $[x, y, z] = x(yz) - (-1)^{\bar{x}\bar{y}} y(xz) + (xy)z$. *Then* $(T, \cdot)$ *is a Malcev superalgebra. Conversely, if we define on a Malcev superalgebra* $(T, \cdot)$ *a trilinear operation by*

$$[x,\ y,\ z] = x(yz) - (-1)^{\bar{x}\bar{y}} y(xz) + (xy)z,$$

then $(T, ., [\ ,\ ,\ ])$ *is a Lie-Yamaguti superalgebra*.

Proof. Let $(T, ., [\ ,\ ,\ ])$ be a Lie-Yamaguti superalgebra with a trilinear composition satisfying $[x,\ y,\ z] = x(yz) - (-1)^{\bar{x}\bar{y}} y(xz) + (xy)z.$ Then its Grassmann envelope $G(T) = G_0 \otimes T_0 \oplus G_1 \otimes T_1$ is a Lie-Yamaguti algebra such that

$$(x \otimes g_1)(y \otimes g_2) = (-1)^{\bar{x}\bar{y}} xy \otimes g_1 g_2,$$

$$[x \otimes g_1,\ y \otimes g_2,\ z \otimes g_3] = (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}+\bar{x}\bar{z}}[x,\ y,\ z] \otimes g_1 g_2 g_3.$$

We have:

$$\begin{aligned}
&(x \otimes g_1)((y \otimes g_2)(z \otimes g_3)) - (y \otimes g_2)((x \otimes g_1)(z \otimes g_3)) \\
&+ ((x \otimes g_1)(y \otimes g_2))(z \otimes g_3) \\
= {} & (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}+\overline{xz}}(x(yz)) \otimes g_1 g_2 g_3 - (-1)^{\overline{xy}+\bar{y}\bar{z}+\bar{x}\bar{z}} y(xz) \otimes g_2 g_1 g_3 \\
&+ (-1)^{\overline{xy}+\bar{y}\bar{z}+\bar{x}\bar{z}}(xy)z \otimes g_1 g_2 g_3 \\
= {} & (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}+\bar{x}\bar{z}}[x(yz) - (-1)^{\bar{x}\bar{y}} y(xz) + (xy)z] \otimes g_1 g_2 g_3 \\
= {} & (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}+\bar{x}\bar{z}}[x,\ y,\ z] \otimes g_1 g_2 g_3 = [x \otimes g_1,\ y \otimes g_2,\ z \otimes g_3].
\end{aligned}$$

Thus, $(G(T), ., [\ ,\ ,\ ])$ is a Lie-Yamaguti algebra such that

$$\begin{aligned}
&[x \otimes g_1,\ y \otimes g_2,\ z \otimes g_3] \\
= {} & (x \otimes g_1)((y \otimes g_2)(z \otimes g_3)) \\
&- (y \otimes g_2)((x \otimes g_1)(z \otimes g_3)) + ((x \otimes g_1)(y \otimes g_2))(z \otimes g_3).
\end{aligned}$$

Having in mind that any Lie-Yamaguti algebra $(T, ., [\ ,\ ,\ ])$ with $[x,\ y,\ z]$ $= x(yz) - y(xz) + (xy)z$ is a Malcev algebra [6], we deduce that $(G(T), .)$ is a Malcev algebra and then $(T, .)$ is a Malcev superalgebra.

The converse is proved in the same way. $\square$

Definition 2.5. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. A graded subspace $H = H_0 \oplus H_1$ of T is a graded Lie-Yamaguti subalgebra of T if $H_i H_j \subset H_{i+j}$ and $[H_i, H_j, H_k] \subset H_{i+j+k}$ for any $i, j, k \in \mathbb{Z}_2$.

Definition 2.6. Let $T = T_0 \oplus T_1$ and $T' = T'_0 \oplus T'_1$ be Lie-Yamaguti superalgebras. A linear map $f : T \to T'$ is said to be of *degree* α if $f(T_i) \subset T'_{\alpha+i}$ for all $\alpha, i \in \mathbb{Z}_2$.

Definition 2.7. Let $T = T_0 \oplus T_1$ and $T' = T'_0 \oplus T'_1$ be Lie-Yamaguti superalgebras. A linear map $f : T \to T'$ is called a *homomorphism* of Lie-Yamaguti superalgebras if

1. f preserves the grading, that is, $f(T_i) \subset T'_i$, $i \in \mathbb{Z}_2$;

2. $f(xy) = f(x)f(y)$;

3. $f([x, y, z]) = [f(x), f(y), f(z)]$ for any $x, y, z \in T_0 \cup T_1$.

Let us recall that if $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space, then we set $End_\alpha(V) = \{f \in End(V) / f(V_i) \subset V_{\alpha+i}\}$, we obtain an associative superalgebra $End(V) = End_0(V) \oplus End_1(V)$; $End_\alpha(V)$ consists of the linear mappings of V into itself which are homogeneous of degree α. The bracket $[f, g] = fg - (-1)^{\bar{f}\bar{g}} gf$ makes $End(V)$ into a Lie superalgebra.

Definition 2.8. Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. Then $D \in End_\alpha(T)$ is a superderivation of T if for any $x, y, z \in T_0 \cup T_1$,

1. $D(xy) = D(x)y + (-1)^{\alpha\bar{x}} xD(y)$;

2. $D([x, y, z]) = [D(x), y, z] + (-1)^{\alpha\bar{x}}[x, D(y), z]$

$$+ (-1)^{\alpha(\bar{x}+\bar{y})}[x, y, D(z)].$$

Let $D_\alpha(T)$ consist of the superderivation of degree α and $D(T) = D_0(T) \oplus D_1(T)$. It is easy to check that $D(T)$ is a graded subalgebra of $End(T)$ called the *Lie superalgebra* of superderivation of *T*.

Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. For any $x, y \in T_0 \cup T_1$, denote by $D_{x,y}$ the endomorphism of *T* defined by $D_{x,y}(z) = [x, y, z]$ for any $z \in T$. We have for any $x, y \in T_0 \cup T_1$, $\alpha \in \mathbb{Z}_2$, $D_{x,y}(T_\alpha) \subset T_{\alpha+\bar{x}+\bar{y}}$, that is, $D_{x,y}$ is a linear map of degree $\bar{x} + \bar{y}$. The axioms (*LYS*4), (*LYS*6), (*LYS*7) and (*LYS*8) can be written as:

(i) $D_{x,y} = -(-1)^{\bar{x}\bar{y}} D_{y,x}$;

(ii) $\sum_{cycl\ x,y,z} (-1)^{\bar{x}\bar{z}} D_{xy,z} = 0$;

(iii) $D_{x,y}(zw) = D_{x,y}(z)w + (-1)^{\bar{z}(\bar{x}+\bar{y})} z D_{x,y}(w)$;

(iv) $D_{x,y}([z, v, w]) = [D_{x,y}(z), v, w]$

$$+ (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, D_{x,y}(v), w] + (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z, v, D_{x,y}(w)]$$

for any $x, y, z, v, w \in T_0 \cup T_1$. By (iii) and (iv), we have that $D_{x,y}$ is a superderivation of *T* called an *inner superderivation* of *T*. From (iv), we also have that for any $x, y, z, v, w \in T_0 \cup T_1$,

(iv)' $[D_{x,y}, D_{z,v}] = D_{[x,y,z],v} + (-1)^{\bar{z}(\bar{x}+\bar{y})} D_{z,[x,y,v]}$.

Let $D(T, T)$ be the vector space spanned by all $D_{x,y}(x, y \in T)$.

We can define naturally a $\mathbb{Z}_2$-gradation by setting $D(T, T) = D_0(T, T) \oplus D_1(T, T)$, where $D_\alpha(T, T)$ consists of the superderivation $D_{x,y}$ of degree α. It is clear from (iv)' that $D(T, T)$ is a $\mathbb{Z}_2$-graded Lie subalgebra of $D(T)$ called the *Lie superalgebra* of all inner superderivations of *T*.

Let $(T, ., [\ ,\ ,\ ])$ be a Lie-Yamaguti superalgebra; T can be written as a pair $T = (T_B, T_D)$, where $T_B = (T, .)$ is the bilinear system and $T_D = (T, ., [\ ,\ ,\ ])$ the trilinear system. Thus, the relations (iii) and (iv) mean that every inner superderivation of $(T, ., [\ ,\ ,\ ])$ is a superderivation of both of the bilinear system T_B and the trilinear system T_D, that is, $D(T, T) \subset DerT_B \cap DerT_D$, where $DerT_B$ (respectively $DerT_D$) denotes the Lie superalgebra of all superderivations of T_B (respectively T_D).

Now, we consider a particular class of Lie-Yamaguti superalgebras composed of Lie supertriple systems and Lie superalgebras.

The following result is obvious:

Proposition 2.9. *Let* $(T, ., [\ ,\ ,\ ])$ *be a Lie-Yamaguti superalgebra. Then the bilinear system* $T_B = (T, .)$ *forms a Lie superalgebra if and only if the trilinear system* $T_D = (T, ., [\ ,\ ,\ ])$ *forms a Lie supertriple system. A pair of a Lie superalgebra* T_B *and a Lie supertriple system* T_D *on the same underlying vector space T forms a Lie-Yamaguti superalgebra if and only if the following relations are verified*:

(i) $\sum_{cycl\ x, y, z} (-1)^{\bar{x}\bar{z}} [xy, z, v] = 0;$

(ii) $D(T, T) \subset DerT_B.$

Proposition 2.10. *Let* $T = (T_B, T_D)$ *be a Lie-Yamaguti superalgebra. Suppose that* T_B *forms a Lie superalgebra* (*or* T_D *forms a Lie supertriple system*). *Then the following assertions are equivalent for any* $x, y, z, v \in T$:

(i) $B_x \in DerT_D$, *where* $B_x : T \to T,\ y \mapsto xy;$

(ii) $[xy, z, v] = [x, y, z]v.$

Proof. Suppose that for any x in T, $B_x \in DerT_D$. Notice that $B_x(T_i) = xT_i \subset T_{i+\bar{x}}$, that is, B_x is homogeneous of degree $\bar{x}$.

By (LYS6), it follows that for any $x, y, z, v \in T$,

$$(-1)^{\bar{x}\bar{z}}[xy, z, v] + (-1)^{\bar{x}\bar{y}}[yz, x, v] + (-1)^{\bar{y}\bar{z}}[zx, y, v] = 0,$$

that is, $[xy, z, v] = -(-1)^{\bar{x}(\bar{y}+\bar{z})}[yz, x, v] - (-1)^{(\bar{x}+\bar{y})\bar{z}}[zx, y, v]$.

But, for any z in T, $B_z \in DerT_D$ gives

$$z[x, y, v] = [zx, y, v] + (-1)^{\bar{z}\bar{x}}[x, zy, v] + (-1)^{(\bar{x}+\bar{y})\bar{z}}[x, y, zv],$$

that is,

$$\begin{aligned}[zx, y, v] &= z[x, y, v] - (-1)^{\bar{z}\bar{x}}[x, zy, v] - (-1)^{(\bar{x}+\bar{y})\bar{z}}[x, y, zv] \\ &= z[x, y, v] - (-1)^{(\bar{x}+\bar{z})\bar{y}}[yz, x, v] - (-1)^{(\bar{x}+\bar{y})\bar{z}}[x, y, zv].\end{aligned}$$

Then

$$\begin{aligned}[xy, z, v] &= -(-1)^{\bar{x}(\bar{y}+\bar{z})}[yz, x, v] - (-1)^{(\bar{x}+\bar{y})\bar{z}}[zx, y, v] \\ &= -(-1)^{(\bar{x}+\bar{y})\bar{z}}z[x, y, v] + [x, y, zv] \\ &= [x, y, z]v \text{ by } (LYS7).\end{aligned}$$

Now, let us consider that $[xy, z, v] = [x, y, z]v$. Then, by the equality (LYS7), we have

$$[y, z, xv] = [y, z, x]v + (-1)^{\bar{x}(\bar{y}+\bar{z})}x[y, z, v]$$

and

$$x[y, z, v] = (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, xv] - (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, x]v,$$

that is,

$$x[y, z, v] = (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, xv] - (-1)^{\bar{x}(\bar{y}+\bar{z})}[yz, x, v]$$

by using the assumption.

It follows that

$$B_x[y, z, v] = x[y, z, v] = (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, xv] - (-1)^{\bar{x}(\bar{y}+\bar{z})}[yz, x, v].$$

But (*LYS*6) gives

$$(-1)^{\bar{x}\bar{y}}[yz, x, v] = -(-1)^{\bar{y}\bar{z}}[zx, y, v] - (-1)^{\bar{x}\bar{z}}[xy, z, v].$$

Then

$$B_x[y, z, v]$$

$$= (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, xv] - (-1)^{\bar{x}(\bar{y}+\bar{z})}[yz, x, v]$$

$$= (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, xv] - (-1)^{\bar{x}\bar{z}}(-(-1)^{\bar{y}\bar{z}}[zx, y, v] - (-1)^{\bar{x}\bar{z}}[xy, z, v])$$

$$= [xy, z, v] + (-1)^{\bar{x}\bar{y}}[y, xz, v] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, xv].$$

Thus, $B_x \in DerT_D$. □

Theorem 2.11. *Let* $T = (T_B, T_D)$ *be a Lie-Yamaguti superalgebra. Suppose that* T_B *forms a Lie superalgebra* (*or* T_D *forms a Lie supertriple system*). *Then the following assertions are equivalent*:

1. *For any fixed* $\alpha \in k$, *the trilinear system* T_D^{α} *given by*

   $[x, y, z]^{\alpha} = [x, y, z] + \alpha(xy)z$ *forms a Lie supertriple system*.

2. $B_{xy} \in DerT_D$.

Proof. Suppose that the trilinear system T_D^{α} forms a Lie supertriple system. Let us show that $B_{xy} \in DerT_D$.

As T_D^{α} is a Lie supertriple system, then by definition, we have

$$[x, y, [z, v, w]^1]^1 = [[x, y, z]^1, v, w]^1 + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, [x, y, v]^1, w]^1$$
$$+ (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z, v, [x, y, w]^1]^1.$$

As $[x, y, z]^1 = [x, y, z] + (xy)z,$ we have

$$[x, y, [z, v, w]^1]^1 = [x, y, [z, v, w]^1] + (xy)[z, v, w]^1$$
$$= [x, y, [z, v, w] + (zv)w] + (xy)([z, v, w] + (zv)w),$$

$$[[x, y, z]^1, v, w]^1 = [[x, y, z]^1, v, w] + ([x, y, z]^1 v)w$$
$$= [[x, y, z] + (xy)z, v, w] + (([x, y, z] + (xy)z)v)w,$$

$$[z, [x, y, v]^1, w]^1 = [z, [x, y, v]^1, w] + (z[x, y, v]^1)w$$
$$= [z, [x, y, v] + (xy)v, w] + (z([x, y, v] + (xy)v))w,$$

$$[z, v, [x, y, w]^1]^1 = [z, v, [x, y, w]^1] + (zv)[x, y, w]^1$$
$$= [z, v, [x, y, w] + (xy)w] + (zv)([x, y, w] + (xy)w).$$

Hence, we have

$$[x, y, [z, v, w] + (zv)w] + (xy)([z, v, w] + (zv)w)$$
$$= [[x, y, z] + (xy)z, v, w] + (([x, y, z] + (xy)z)v)w$$
$$+ (-1)^{\bar{z}(\bar{x}+\bar{y})}([z, [x, y, v] + (xy)v, w] + (z([x, y, v] + (xy)v))w)$$
$$+ (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}([z, v, [x, y, w] + (xy)w] + (zv)([x, y, w] + (xy)w)).$$

Using the fact that T_B is a Lie superalgebra, we also have due to the Jacobi's superidentity:

$$(xy)((zv)w) = (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}(zv)((xy)w) + (-1)^{\bar{z}(\bar{x}+\bar{y})}(z((xy)v))w$$
$$+ (((xy)z)v)w.$$

In consequence, we have:

$$[x, y, (zv)w] + (xy)[z, v, w]$$
$$= [(xy)z, v, w] + ([x, y, z]v + (-1)^{\bar{z}(\bar{x}+\bar{y})}z[x, y, v])w + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, (xy)v, w]$$

$$+ (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z, v, (xy)w] + (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}(zv)[x, y, w]$$

$$= [(xy)z, v, w] + [x, y, zv]w + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, (xy)v, w]$$

$$+ (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z, v, (xy)w] + (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}(zv)[x, y, w],$$

that is,

$$(xy)[z, v, w] = [(xy)z, v, w] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, (xy)v, w]$$

$$+ (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z, v, (xy)w]$$

and $B_{xy} \in DerT_D$.

The converse is obtained by straightforward calculations. □

3. Lie Enveloping Superalgebra

Let $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ be two $\mathbb{Z}_2$-graded vector spaces; $A \oplus B$ has a natural $\mathbb{Z}_2$-gradation defined by $(A \oplus B)_\alpha = A_\alpha \oplus B_\alpha$, $\alpha = 0, 1$. Let $(L, [,])$ be a Lie superalgebra with a *reductive decomposition* $L = h \oplus m$, that is, $h = h_0 \oplus h_1$ is a $\mathbb{Z}_2$-graded subalgebra of L and $m = m_0 \oplus m_1$ a $\mathbb{Z}_2$-graded subspace of L with $[m, h] \subset m$. On m, define a bilinear map : $m \times m \to m$ and a trilinear map $[\ ,\ ,\ ] : m \times m \times m \to m$ by:

$$xy = \pi_m([x, y]);$$

$$[x, y, z] = [\pi_h([x, y]), z],$$

for any $x, y, z \in m$, where π_m and π_h denote the projections on m and h with respect to the reductive decomposition. We have the following:

Theorem 3.1. $(m, ., [\ ,\ ,\ ])$ *is a Lie-Yamaguti superalgebra*.

Proof. Obviously, we have $m_i m_j \subset m_{i+j}$ and $[m_i, m_j, m_k] \subset m_{i+j+k}$ for any $i, j, k \in \mathbb{Z}_2$. For any $x, y \in m$, $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$ gives $xy +$

$\pi_h([x, y]) = -(-1)^{\bar{x}\bar{y}}(yx + \pi_h([y, x]))$, that is, $xy = -(-1)^{\bar{x}\bar{y}} yx$ and $\pi_h([x, y]) = -(-1)^{\bar{x}\bar{y}}(yx + \pi_h([y, x]))$. This gives

$$[x, y, z] = [\pi_h([x, y]), z] = -(-1)^{\bar{x}\bar{y}}[\pi_h([y, x]), z] = -(-1)^{\bar{x}\bar{y}}[y, x, z]$$

for any $z \in m$.

For any $x, y, z \in m$, the relation

$$(-1)^{\bar{x}\bar{z}}[[x, y], z] + (-1)^{\bar{x}\bar{y}}[[y, z], x] + (-1)^{\bar{y}\bar{z}}[[z, x], y] = 0$$

implies

$$(-1)^{\bar{x}\bar{z}}[\pi_m([x, y]) + \pi_h([x, y]), z] + (-1)^{\bar{x}\bar{y}}[\pi_m([y, z]) + \pi_h([y, z]), x]$$

$$+ (-1)^{\bar{y}\bar{z}}[\pi_m([z, x]) + \pi_h([z, x]), y] = 0$$

and

$$(-1)^{\bar{x}\bar{z}}[xy, z] + (-1)^{\bar{x}\bar{z}}[\pi_h([x, y]), z] + (-1)^{\bar{x}\bar{y}}[yz, x] + (-1)^{\bar{x}\bar{y}}[\pi_h([y, z]), x]$$

$$+ (-1)^{\bar{y}\bar{z}}[zx, y] + (-1)^{\bar{y}\bar{z}}[\pi_h([z, x]), y] = 0.$$

We obtain

$$(-1)^{\bar{x}\bar{z}}\pi_m([xy, z]) + (-1)^{\bar{x}\bar{z}}[x, y, z] + (-1)^{\bar{x}\bar{y}}\pi_m([yz, x]) + (-1)^{\bar{x}\bar{y}}[y, z, x]$$

$$+ (-1)^{\bar{y}\bar{z}}\pi_m([zx, y]) + (-1)^{\overline{yz}}[z, x, y] = 0$$

and

$$(-1)^{\bar{x}\bar{z}}\pi_h([xy, z]) + (-1)^{\bar{x}\bar{y}}\pi_h([yz, x]) + (-1)^{\bar{y}\bar{z}}\pi_h([zx, y]) = 0,$$

that is,

$$(-1)^{\bar{x}\bar{z}}[x, y, z] + (-1)^{\bar{x}\bar{y}}[y, z, x] + (-1)^{\bar{y}\bar{z}}[z, x, y]$$

$$+ (-1)^{\bar{x}\bar{z}}(xy)z + (-1)^{\bar{x}\bar{y}}(yz)x + (-1)^{\overline{zy}}(zx)y = 0$$

and for any $v \in m$,

$$(-1)^{\bar{x}\bar{z}}[xy,\ z,\ v] + (-1)^{\bar{x}\bar{y}}[yz,\ x,\ v] + (-1)^{\bar{y}\bar{z}}[zx,\ y,\ v] = 0.$$

Also, due to the Jacobi's superidentity, we have, for any $x,\ y,\ z,\ v \in m$,

$$(-1)^{(\bar{x}+\bar{y})\bar{v}}[\pi_h([x,\ y]),\ [z,\ v]] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z,\ [v,\ \pi_h([x,\ y])]]$$

$$+ (-1)^{\bar{v}\bar{z}}[v,\ [\pi_h([x,\ y]),\ z]] = 0.$$

Hence,

$$(-1)^{(\bar{x}+\bar{y})\bar{v}}[\pi_h([x,\ y]),\ \pi_m[z,\ v]] + (-1)^{\bar{z}(\bar{x}+\bar{y})}\pi_m[z,\ [v,\ \pi_h([x,\ y])]]$$

$$+ (-1)^{\bar{v}\bar{z}}\pi_m[v,\ [\pi_h([x,\ y]),\ z]] = 0$$

and

$$(-1)^{(\bar{x}+\bar{y})\bar{v}}[x,\ y,\ zv] - (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}z[x,\ y,\ v] + (-1)^{\bar{v}\bar{z}}v[x,\ y,\ z] = 0,$$

that is, $[x,\ y,\ zv] = [x,\ y,\ z]v + (-1)^{\bar{z}(\bar{x}+\bar{y})}z[x,\ y,\ v]$.

Again, due to the Jacobi's superidentity, we have:

$$(-1)^{(\bar{x}+\bar{y})\bar{w}}[\pi_h([x,\ y]),\ [\pi_h([z,\ v]),\ w]]$$

$$+ (-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{v})}[\pi_h([z,\ v]),\ [w,\ \pi_h([x,\ y])]]$$

$$+ (-1)^{\bar{w}(\bar{z}+v)}[w,\ [\pi_h([x,\ y]),\ \pi_h([z,\ v])]] = 0.$$

But

$$[\pi_h([x,\ y]),\ [z,\ v]] = -(-1)^{\bar{z}(\bar{x}+\bar{y})}[z,\ [\pi_h([x,\ y]),\ v]] + [[\pi_h([x,\ y]),\ z],\ v],$$

that is,

$$[\pi_h([x,\ y]),\ \pi_h([z,\ v])] = -(-1)^{\bar{z}(\bar{x}+\bar{y})}\pi_h[z,\ [x,\ y,\ v]] + \pi_h[[x,\ y,\ z],\ v].$$

Thus,

$$[\pi_h([x, y]), [z, v, w]] - (-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{v})}[z, v, [x, y, w]]$$
$$- (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, [x, y, v], w] - [[x, y, z], v, w] = 0$$

and

$$[x, y, [z, v, w]] = [[x, y, z], v, w] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, [x, y, v], w]$$
$$+ (-1)^{(\bar{z}+\bar{v})(\bar{x}+\bar{y})}[z, v, [x, y, w]].$$

It follows that $(m, ., [\ ,\ ,\ ])$ is a Lie-Yamaguti superalgebra. □

Now, let $(T, ., [\ ,\ ,\ ])$ be a Lie-Yamaguti superalgebra. Set $L_i = T_i + D_i(T, T)$, $i = 0, 1$ and define a new bracket operation in $L = L_0 \oplus L_1$ as follows: for any $x, y \in T_0 \cup T_1$, $D_1, D_2 \in D_0(T, T) \cup D_1(T, T)$,

$$[x, y] = xy + D(x, y);$$

$$[D, x] = -(-1)^{\bar{x}\bar{D}}[x, D] = D(x);$$

$$[D_1, D_2] = D_1 D_2 - (-1)^{\bar{D}_1 \bar{D}_2} D_2 D_1. \qquad (*)$$

Theorem 3.2. *Let $T = T_0 \oplus T_1$ be a Lie-Yamaguti superalgebra. Then $L = L_0 \oplus L_1 = T \oplus D(T, T)$ is a reductive decomposition and the operations in T coincide with those given by $(*)$.*

Proof. The bracket $[,]$ is bilinear by definition and $XY = -(-1)^{\bar{X}\bar{Y}}YX$ for any $X, Y \in L$ by (LTS3) and (LTS4). The Jacobi's superidentity follows from $(LTS5\text{-}LTS8)$; $[D(T, T), T] \subset T$ is obvious from $(*)$, so the decomposition is reductive. Also, we have $\pi_T([x, y]) = xy$ and

$$[\pi_{D(T,T)}([x, y]), z] = [x, y, z]$$

for any $x, y, z \in T$ which proves the remaining assertion. □

Note that L is called the standard enveloping lie superalgebra of T.

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