JP Journal of Algebra, Number Theory and Applications

Published Online: June 2016
http://dx.doi.org/10.17654/NT038040363
Volume 38, Number 4, 2016, Pages 363-372 **ISSN: 0972-5555**

FACTORIZATION OF FIBONACCI NUMBERS INTO PRODUCTS OF LUCAS NUMBERS AND RELATED RESULTS

Prapanpong Pongsriiam

Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom 73000
Thailand
e-mail: prapanpong@gmail.com
pongsriiam_p@silpakorn.edu

Abstract

We show that a Fibonacci number F_m can be written as a product of Lucas numbers if and only if $m = 2^{\ell}$ or $m = 3 \cdot 2^{\ell}$ for some $\ell \geq 0$. Similar results are also given.

1. Introduction and Preliminaries

Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and let $(L_n)_{n \geq 0}$ be the Lucas sequence given by the same recursive pattern but with the initial values $L_0 = 2$ and $L_1 = 1$.

Received: January 25, 2016; Accepted: April 21, 2016
2010 Mathematics Subject Classification: Primary 11B39; Secondary 11D99.
Keywords and phrases: Fibonacci number, Lucas number, Diophantine equation.
Communicated by Istvan Gaal

There have been some investigations on Diophantine equations involving Fibonacci and Lucas numbers, and on multiplicative partitions, where the number of representations of a positive integer as an unordered product is counted. In this article, we are interested in the factorization of Fibonacci numbers as a product of Lucas numbers and other closely related questions. More precisely, we explicitly solve the following Diophantine equations:

$$F_m^a = F_{n_1}F_{n_2}F_{n_3}\cdots F_{n_k}, \tag{1}$$

$$F_m^a = L_{n_1}L_{n_2}L_{n_3}\cdots L_{n_k}, \tag{2}$$

$$L_m^a = F_{n_1}F_{n_2}F_{n_3}\cdots F_{n_k}, \tag{3}$$

$$L_m^a = L_{n_1}L_{n_2}L_{n_3}\cdots L_{n_k}, \tag{4}$$

where $a \geq 1$, $m \geq 0$, $k \geq 1$ and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$.

Since our main tool in solving the above equations is the primitive divisor theorem of Carmichael [2], we first recall some facts about it. Let α and β be algebraic numbers such that $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers and $\alpha\beta^{-1}$ is not a root of unity. Let $(u_n)_{n\geq 0}$ be the sequence given by

$$u_0 = 0, \quad u_1 = 1 \quad \text{and} \quad u_n = (\alpha+\beta)u_{n-1} - (\alpha\beta)u_{n-2} \text{ for } n \geq 2.$$

Then we have Binet's formula for u_n given by

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ for } n \geq 0.$$

So, if $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then (u_n) is the Fibonacci sequence.

A prime p is said to be a *primitive divisor* of u_n if $p \mid u_n$ but p does not divide $u_1u_2\cdots u_{n-1}$. Then the primitive divisor theorem of Carmichael can be stated as follows:

Theorem 1.1 (Primitive divisor theorem of Carmichael [2])**.** *If* α *and* β *are real numbers and* $n \neq 1, 2, 6,$ *then* u_n *has a primitive divisor except when* $n = 12,$ $\alpha + \beta = 1$ *and* $\alpha\beta = -1.$

There is a long history about primitive divisors and the most remarkable results in this topic are given by Bilu et al. [1], by Stewart [6], and by Kunrui [3], but Theorem 1.1 is good enough in our situation.

We also need the concept of the order of appearance of a positive integer. Recall that for each positive integer *m*, the order of appearance of *m* in the Fibonacci sequence, denoted by $z(m)$, is the smallest positive integer *k* such that $m \mid F_k$. It is a well-known fact that if *p* is an odd prime and $z(p)$ is odd, then *p* does not divide any Lucas number. We will use this fact in the proof of main results without reference.

Finally, we remark that some variants of (1) to (4) are also considered by Szalay in [7] and by Pongsriiam in [4]. Other variants of (1) to (4) will appear in our next article [5].

2. Main Results

The reader will see that equation (2) has infinitely many solutions while (1), (3) and (4) have only a finite number of solutions. We begin this section by solving (2). The solutions to (1), (3) and (4) can be obtained similarly.

Theorem 2.1. *A Fibonacci number* F_m *can be written as a product of Lucas numbers if and only if* $m = 2^{\ell}$ *or* $m = 3 \cdot 2^{\ell}$ *for some* $\ell \geq 0$. *Furthermore, there is a unique representation of* $F_{2^{\ell}}$ $(\textit{for } \ell \geq 2)$, *and exactly five representations of* $F_{3 \cdot 2^{\ell}}$ $(\textit{for } \ell \geq 2)$ *as a nontrivial unordered product of Lucas numbers*:

$$F_{2^{\ell}} = L_{2^{\ell-1}} L_{2^{\ell-2}} \cdots L_2 \text{ for } \ell \geq 2, \tag{5}$$

$$F_{3 \cdot 2^{\ell}} = L_{3 \cdot 2^{\ell-1}} L_{3 \cdot 2^{\ell-2}} \cdots L_{12} A \text{ for } \ell \geq 2, \text{ where} \tag{6}$$

$$A = F_{12} = L_2L_2L_0L_0L_0L_0 = L_3L_2L_2L_0L_0 = L_6L_0L_0L_0$$

$$= L_6L_3L_0 = L_3L_3L_2L_2. \tag{7}$$

Here nontrivial product means that there is no $L_1 = 1$ *in the representation*.

Remark 2.2. $F_1 = F_2 = 1 = L_1$ is a representation of F_1 and F_2 as a product of Lucas numbers. But when we would like to count the number of nontrivial representations, we consider only those in which every factor is larger than 1. This is why we restrict our attention to the case $\ell \geq 2$ in (5). In addition, if $\ell = 0$ or $\ell = 1$ in (6), then we have $F_3 = L_0$ and $F_6 = L_3L_0 = L_0^3$. Moreover, if $\ell = 2$, then the product $L_{3\cdot 2^{\ell-1}}L_{3\cdot 2^{\ell-2}} \cdots L_{12}$ appearing in (6) is empty. In this case, (6) becomes $F_{3\cdot 2^{\ell}} = A = F_{12}$ and its representation as a product of Lucas numbers is given by (7).

Proof of Theorem 2.1. We first eliminate the following two cases:

Case 1. m is odd and $m \geq 5$. By Theorem 1.1, there exists an odd primitive prime divisor p of F_m so that $z(p) = m$. Therefore, p does not divide any Lucas number. Since $p \mid F_m$, we see that F_m is not a product of Lucas numbers.

Case 2. $m = 2^{\ell}a$, $\ell \geq 1$, $a \geq 5$ and a is odd. Since $a \mid m$, $F_a \mid F_m$. From the argument in Case 1, there exists a prime p such that $p \mid F_a$ but p does not divide any Lucas numbers. Since $p \mid F_a$ and $F_a \mid F_m$, we have $p \mid F_m$. Therefore, F_m is not a product of Lucas numbers.

From Case 1 and Case 2, we only need to consider

$$m = 2^{\ell} \quad \text{or} \quad m = 3 \cdot 2^{\ell} \text{ for some } \ell \geq 0.$$

We have $F_1 = L_1$, $F_3 = L_0$, $F_6 = L_3L_0$, and by the well-known identity $F_{2n} = L_nF_n$, we also obtain

$$F_{2^\ell} = L_{2^{\ell-1}} L_{2^{\ell-2}} \cdots L_2 \text{ for every } \ell \geq 2, \text{ and} \tag{8}$$

$$F_{3\cdot 2^\ell} = L_{3\cdot 2^{\ell-1}} L_{3\cdot 2^{\ell-2}} \cdots L_6 L_3 L_0 \text{ for every } \ell \geq 2. \tag{9}$$

This proves the first part. For the second part, it is easy to check that F_4 and F_8 have a unique representation as a product of Lucas numbers given by (8) and $F_{12} = 2^4 \cdot 3^2 = A$ has exactly five representations given in (7). Next, we show that the representation of F_{2^ℓ} in (8) is unique for every $\ell \geq 4$. Consider the equation

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} = F_{2^\ell} = L_2 L_4 L_8 \cdots L_{2^{\ell-1}}, \tag{10}$$

where $\ell \geq 4$, $n_1 \leq n_2 \leq \cdots \leq n_k$, and $n_j \neq 1$ for any j. By the identity $F_{2n} = L_n F_n$, (10) can be written as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-1}} F_{2n_k} F_{2^{\ell-1}} = L_2 L_4 L_8 \cdots L_{2^{\ell-2}} F_{2^\ell} F_{n_k}. \tag{11}$$

If $2^\ell > 2n_k$, then by Theorem 1.1, there exists a prime p dividing F_{2^ℓ} but p does not divide any term on the left hand side of (11), which is not the case. Similarly, $2n_k > 2^\ell$ leads to a contradiction. Therefore, $2n_k = 2^\ell$ and (10) is reduced to

$$L_{n_1} L_{n_2} \cdots L_{n_{k-1}} = L_2 L_4 L_8 \cdots L_{2^{\ell-2}}$$

which is in the same form as (10). So we can repeat the same process to obtain $n_{k-1} = 2^{\ell-2}$, $n_{k-2} = 2^{\ell-3}$, ..., and (10) is reduced to

$$L_{n_1} L_{n_2} \cdots L_{n_j} = L_2 L_4.$$

From this, it is easy to check that $j = 2$, $n_1 = 2$ and $n_2 = 4$. Hence, $k = \ell - 1$, $n_1 = 2$, $n_2 = 4$, ..., and $n_k = 2^{\ell-1}$. This proves the uniqueness of (8). Similarly, we consider from (9), the equation

$$L_{n_1}L_{n_2}L_{n_3}\cdots L_{n_k} = F_{3\cdot 2^\ell} = L_0L_3L_6\cdots L_{3\cdot 2^{\ell-2}}L_{3\cdot 2^{\ell-1}}, \tag{12}$$

with $\ell \geq 3$, $n_1 \leq n_2 \leq \cdots \leq n_k$, and $n_j \neq 1$ for any j. Applying Theorem 1.1 and the same argument given above, we obtain $n_k = 3 \cdot 2^{\ell-1}$, $n_{k-1} = 3 \cdot 2^{\ell-2}$, ... and (12) is reduced to

$$L_{n_1}L_{n_2}L_{n_3}\cdots L_{n_j} = L_0L_3L_6.$$

Note that $L_0L_3L_6 = 2^4 \cdot 3^2 = A$, which can be written as an unordered product of Lucas numbers in 5 ways. This completes the proof. □

Every Fibonacci number F_n is obviously a product of Fibonacci numbers $F_n = F_n$ but it is more interesting to consider the product that has more than one nontrivial factor.

Theorem 2.3. *A Fibonacci number* F_m *can be written as a nontrivial product of at least two Fibonacci numbers if and only if* $m = 6$ *or* 12. *Here nontrivial product means that there is no* $F_0 = 0$, $F_1 = F_2 = 1$ *in the factor.*

Proof. Here the factor of F_m is always smaller than F_m. So, if $m \geq 13$ or $m = 3, 4, 5, 7, 8, 9, 10, 11,$ then F_m has a primitive divisor by Theorem 1.1. Therefore, F_m cannot be written as a product of smaller Fibonacci numbers. So we only need to consider $m = 6, 12$. We have $F_6 = F_3^3$ and $F_{12} = F_3^4F_4^2$. This completes the proof. □

Theorem 2.4. *A Lucas number* L_m *can be written as a nontrivial product of at least two Lucas numbers if and only if* $m = 3$ *or* $m = 6$. *Here nontrivial means that there is no* $L_1 = 1$ *in the factor.*

Proof. Since $L_mF_m = F_{2m}$, we see that if F_{2m} has a primitive divisor p, L_m also has p as a primitive divisor. So, if $m \geq 7$ or $m = 2, 4, 5,$ then L_m has a primitive divisor. By the same argument used in Theorem 2.3, we only need to consider $m = 3, 6$. We have $L_3 = L_0^2$ and $L_6 = L_0 L_2^2$. This completes the proof. □

Theorem 2.5. *A Lucas number* L_m *can be written as a nontrivial product of Fibonacci numbers if and only if* $m = 0, 2, 3, 6$. *Here nontrivial product means that there is no* $F_0 = 0$, $F_1 = F_2 = 1$ *in the factor*.

Proof. Suppose $m \geq 7$ and L_m is a nontrivial product of Fibonacci numbers. Since $F_{2m} = F_m L_m$, we see that F_{2m} is a nontrivial product of at least two Fibonacci numbers. By Theorem 2.3, we have $2m = 6$ or $2m = 12$, a contradiction. Therefore, we only need to check the result for $m \leq 6$. This can be easily done, so the proof is complete. □

Our argument can be applied to obtain more general results as follows.

Theorem 2.6. *Let* $m \geq 0$ *and* $n \geq 2$. *Then* F_m^n *can be written as a product of Lucas numbers if and only if* $m = 2^\ell$ *or* $m = 3 \cdot 2^\ell$ *for some* $\ell \geq 0$. *Furthermore, all representations of* $F_{2^\ell}^n$ $(\ell \geq 2)$ *and of* $F_{3 \cdot 2^\ell}^n$ $(\ell \geq 2)$ *can be obtained directly from those given in Theorem* 2.1.

Remark 2.7. There can be more than five representations of $F_{3 \cdot 2^\ell}^n$ as a product of Lucas numbers but they only come from the factor $A^n = (2^4 \cdot 3^2)^n$. For example, if $n = 2$, then there are 13 representations of A^n, and therefore $F_{3 \cdot 2^\ell}^n$ can be written as a product of Lucas numbers in 13 different ways.

Proof of Theorem 2.6. The proof is similar to that of Theorem 2.1. We first eliminate those *m* which are not of the form 2^ℓ or $3 \cdot 2^\ell$. For those *m* of the form 2^ℓ or $3 \cdot 2^\ell$, we can use the representations of F_m to obtain those of F_m^n. The details are omitted. □

Theorem 2.8. *Let* $m \geq 0$ *and* $n \geq 2$. *A number* F_m^n *can be written as*

$$F_m^n = F_{m_1} F_{m_2} \cdots F_{m_k}, \tag{13}$$

where $k \geq 2$, $3 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, *and* $m_i \neq m_j$ *for some* $i, j \in \{1, 2, ..., k\}$ *if and only if* $m = 6, 12$ *or* $(m = 3$ *and* $n \geq 4)$.

Proof. Assume that (13) holds. If $m \geq 13$, then the repeat application of Theorem 1.1 leads to $m_k = m, m_{k-1} = m, ..., m_1 = m$, respectively, which contradicts the assumption that $m_i \neq m_j$ for some i, j. If $m = 0, 1, 2$, then the left hand side of (13) is smaller than the right hand side, which is not possible. If $m = 4$, 5, 7, 8, 9, 10 or 11, then the same argument based on the primitive divisor can still be applied as follows. Let p be a primitive divisor of F_m. Then $p \mid F_{m_i}$ for some i. So $m_i \geq m$. If $m_i > m$, then there is a prime divisor q of F_{m_i} which does not divide F_m, a contradiction. So $m_i = m$. Repeating this argument, we see that $m_j = m$ for every j, a contradiction. So we only need to check $m = 3$, 6, 12. For $m = 3$ and $n \geq 4$, we have $F_m^n = 2^n = F_3^{n-3} F_6$. The remaining cases can be checked easily as well. This completes the proof. $\square$

Theorem 2.9. *Let* $m \geq 0$ *and* $n \geq 2$. *A number* L_m^n *can be written as*

$$L_m^n = L_{m_1} L_{m_2} \cdots L_{m_k},$$

where $k \geq 2$, $0 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, $m_j \neq 1$ *for any j, and* $m_i \neq m_j$ *for some i, j if and only if* $m = 3, 6$ *or* $(m = 0$ *and* $n \geq 3)$.

Proof. We omit the proof as it is very similar to that of Theorem 2.4 and Theorem 2.8. $\square$

Theorem 2.10. *For* $m \geq 0$ *and* $n \geq 2$, *a number* L_m^n *can be written as*

$$L_m^n = F_{m_1} F_{m_2} \cdots F_{m_k} \tag{14}$$

with $k \geq 2$, $3 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, *and* $m_i \neq m_j$ *for some i, j if and only if* $m = 3, 6$ *or* $(m = 0$ *and* $n \geq 4)$.

Proof. We first consider the case $m > 0$. By (14) and the identity $F_{2m} = L_m F_m$, we have

$$F_{2m}^n = F_{m_1} F_{m_2} F_{m_3} \cdots F_{m_k} F_m^n.$$

By Theorem 2.8, we obtain $2m = 6, 12$. So $m = 3, 6$. We have

$$L_3^n = 2^{2n} = F_3^{2n-3} F_6,$$

$$L_6^n = 2^n 3^{2n} = F_3^n F_4^{2n}.$$

For $m = 0$ and $n \geq 4$, we have $L_0^n = 2^n = F_3^{n-3} F_6$. The case $m = 0$, $n < 4$ can be checked easily. This completes the proof. □

Acknowledgments

The author receives financial support from Faculty of Science, Silpakorn University, Thailand, Grant number SRF-JRG-2558-03. The author would also like to thank Professor Szalay for sending him his article [7], which gives him an idea and leads to the publication of this article.

References

[1] Y. Bilu, G. Hanrot and P. M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers, J. Reine Angew. Math. 539 (2001), 75-122.

[2] R. D. Carmichael, On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$, Ann. of Math. (2) 15 (1913), 30-48.

[3] Y. Kunrui, *p*-adic logarithmic forms and a problem of Erdös, Acta Math. 211 (2013), 315-382.

[4] P. Pongsriiam, Fibonacci and Lucas numbers associated with Brocard-Ramanujan equation, preprint.

[5] P. Pongsriiam, Fibonacci and Lucas Numbers associated with Brocard-Ramanujan equation II, in preparation.

[6] C. L. Stewart, On divisors of Lucas and Lehmer numbers, Acta Math. 211 (2013), 291-314.

[7] L. Szalay, Diophantine equations with binary recurrences associated to the Brocard-Ramanujan problem, Port. Math. 69 (2012), 213-220.