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A NOTE ON UNLIFTABLE IMMERSED SURFACES IN 3-SPACE

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Abstract

An immersed surface in 3-space is a surface with double curves and isolated triple points. In general, immersed surfaces in 3-space are not images of embedded surface in 4-space under the orthogonal projection. In this paper, we present a sufficient condition about a triple point for an immersed surface to become an unliftable surface into 4-space. Also we construct an orientable unliftable immersed sphere.

1. Introduction

Boy [1] constructed an immersed projective plane in 3-space; called the *Boy's surface*. This is an immersed surface with only one triple point and three double loops based at the triple point (see Figure 2 also [4]). It is known that the projective plane can be realized in 4-space. However, the Boy's surface is not the projected image of embedded projective plane in

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4-space. It is called an *unliftable immersed surface*. In [3], Carter and Saito gave sufficient and necessary conditions for a generic closed surface to be liftable into 4-space. In this paper, we discuss about an immersed surface with triple points. We will show that if there is a triple point having three-fold symmetry and there is a double curve passing through the triple point three times, then it is unliftable (Theorem 3.1).

2. Preliminaries

Let $\mathrm{proj} : \mathbf{R}^4 \to \mathbf{R}^3$ be the orthogonal projection defined by $\mathrm{proj}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. The function $h : \mathbf{R}^4 \to \mathbf{R}$ defined by $h(x_1, x_2, x_3, x_4) = x_4$ is called a *height function*.

Let F be a closed oriented surface. Let $f : F \to \mathbf{R}^3$ be an immersion with triple points. We define the *singular set* of f as follows:

$$S_f = \{x \in F \mid \#(f^{-1}(f(x))) > 1\},$$

where '#' means the cardinality. The singular set is a union of immersed circles. At a triple point p, the neighbourhood of p in $\mathbf{R}^3$ contains three small discs and their pre-images are disjoint discs in F (see Figure 1).

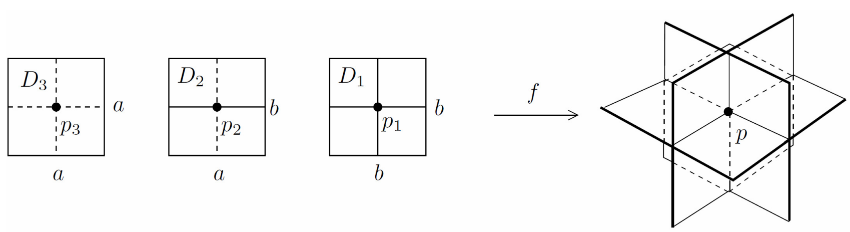

Figure 1. A triple point and its pre-images.

Let F be a closed orientable surface and let $f : F \to \mathbf{R}^3$ be an immersion. If there is an embedding $\tilde{f} : F \to \mathbf{R}^4$ such that $\mathrm{proj} \circ \tilde{f}(x) = f(x)$ for all $x \in F$, then $\tilde{f}$ is called a *lift* of f into $\mathbf{R}^4$ and f is said to be *liftable*. The image of the singular set S_f under f is called a *singularity set* in 3-space.

In [3], Carter and Saito proved that the liftability of orientable closed surface is determined by the singular set:

Lemma 2.1 (Carter and Saito [3]). *Let F be a closed surface and let* $f : F \to \mathbf{R}^3$ *be an immersion. Then f is liftable if and only if the crossing set satisfies the following two conditions*:

(1) *The components can be divided into two families and* (*called a-curves and b-curves*, *respectively*), *and*

(2) *The singular set is the union of two families* $\mathcal{S}_a = \{s_a^1, s_a^2, ..., s_a^k\}$, *called a-curves and* $\mathcal{S}_b = \{s_b^1, s_b^2, ..., s_b^k\}$, *called b-curves of immersed circles, such that* $f(s_a^i) = f(s_b^i)$ *for each* $i = 1, 2, ..., k$, *and for a triple point p with* $f^{-1}(p) = \{p_1, p_2, p_3\} \subset S_f$, *each point of* $f^{-1}(p)$ *is formed by a-curves or b-curves*; *the three combinations for the set* $f^{-1}(p)$ *is* $\{(b, b), (a, b), (a, a)\}$ (*see Figure* 1).

Proposition 2.1. *Let F be a closed oriented surface and let* $f : F \to \mathbf{R}^3$ *be an immersion with at least one triple point p*. *Let* $P = \{p_1, p_2, p_3\}$ *be the set of pre-images of a triple point p*. *Suppose that one of immersed circles c in* S_f *contains all points of P*. *Then f is unliftable into* 4-*space*.

Proof. Suppose that *f* is liftable. Then by Lemma 2.1 S_f is a union of at least two immersed circles c_a and c_b such that $f(c_a) = f(c_b)$. In general, $c_a \cup c_b$ can be viewed as a graph with vertices p_1, p_2 and p_3. Each p_i is a four-regular vertex and $P \subset c = c_a$ (or c_b). By Lemma 2.1, these points p_1, p_2 and p_3 have colours $\{(a, a), (a, b), (b, b)\}$ but this contradicts that c_a contains all three points. Therefore, *f* is unliftable. □

3. Pre-images of Triple points

Let $f : F \to \mathbf{R}^3$ be an immersion with a triple point p and let

$P = f^{-1}(p) = \{p_1, p_2, p_3\}$ be the set of pre-images of a triple point p. Suppose that an immersed circle c in S_f contains some of elements of P; that is, $p \in f(c)$. From Proposition 2.1, if the map f is liftable, then $\#(c \cap P) \leq 2$.

Let $N(S_f)$ be a small neighbourhood of S_f in F and denote $f(N(S_f))$ by M_f. Let $\varphi : \mathbf{R}^3 \to \mathbf{R}^3$ be an orientation preserving homeomorphism such that $\varphi(M_f) = M_f$. If the minimal period n of φ; that is, $\varphi^n(x) = x$ for all $x \in M_f$, is called the *order* of φ. If $n = 3$, and $\varphi(p) = p$ for some triple point, then we say that M_f has a *three-fold symmetry* around p.

The map φ induces a map $\tilde{\varphi} : F \to F$. We define $\tilde{\varphi}' : N(S_f)\backslash S_f \to N(S_f)$ by $\tilde{\varphi}'(x) = f^{-1}\varphi f(x)$. This map is an embedding and it naturally extends to a homeomorphism $\tilde{\varphi} : F \to F$.

Lemma 3.1. *If* φ *has order* 3, *then* $\tilde{\varphi}$ *also has order* 3.

Proof. From the definition, it is trivial. □

Theorem 3.1. *Let* $f : F \to \mathbf{R}^3$ *be an immersion with at least one triple point p. Suppose that the neighbourhood of singularity set* M_f *has three-fold symmetry around a triple point p. If there exists c in* S_f *such that* $f(c)$ *passes the triple point p three times, then f is unliftable.*

Proof. Suppose that f is liftable. Then from Lemma 2.1, there are double decker curves c_a and c_b such that $f(c_a) = f(c_b)$.

Let $P = \{p_1, p_2, p_3\}$ be the pre-image of the triple point p. Without loss of generality, we can assume that $h(p_1) < h(p_2) < h(p_3)$. There are three disjoint closed discs D_1, D_2 and D_3 in F such that $f(D_1 \cup D_2 \cup D_3) \subset M_f$

forms a closed neighbourhood of the triple point p. We assume that $p_i \subset D_i$ for $i = 1, 2, 3.$ Each D_i contains two line segments $d_i \cup e_i$:

$$D_i \cap (c_a \cup c_b) = d_i \cup e_i,$$
$$p_i = d_i \cap e_i. \quad (1)$$

By Lemma 2.1, D_1 contains only part of b-curves; D_2 contains both a and b-curves; and D_3 contains only a-curves.

Since $f(c)$ passes the triple point p three times, there is a loop l_1 based at p_1 such that $l_1 \cap D_1$ is a pair of arcs a_1 and b_1 and $a_1 \subset d_1$ and $b_1 \subset e_1$. Since $\tilde{\varphi}$ is cyclic, $\tilde{\varphi}(D_1) = D_2$, $\tilde{\varphi}(D_2) = D_3$ and $\tilde{\varphi}(D_3) = D_1$. Thus, $\tilde{\varphi}(l_1) = l_2$ is also a loop at p_2 with arcs $\tilde{\varphi}(a_1) = a_2$, $\tilde{\varphi}(b_1) = b_2$ and $a_2 \subset d_2$, $b_2 \subset e_2$. We may assume that $d_2 \subset c_a$ and $e_2 \subset c_b$ or $d_2 \subset c_b$ and $e_2 \subset c_a$. Either case is a contradiction. Therefore, f is unliftable. □

4. Examples

Example 4.1. It is known that Boy's surface (Figure 2) is unliftable (see [4, 5]).

By Theorem 3.1, we can justify this: Boy's surface has only one triple point and one double curve and it has three-fold symmetry. Therefore, from Theorem 3.1 it is unliftable.

We can construct an orientable unliftable immersed sphere (see also [5, 6]). First provide the following immersed disc depicted in Figure 3.

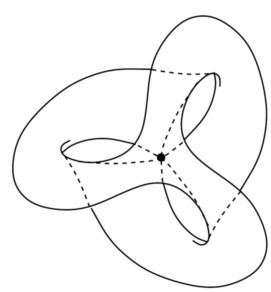

Figure 2. Boy's surface.

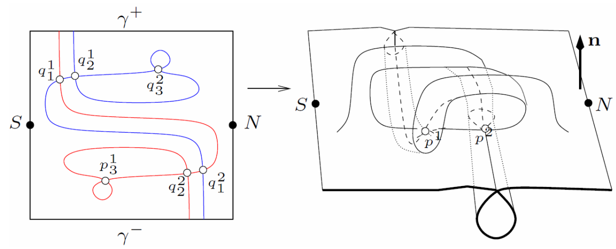

Figure 3. *Z*-disc.

We call this immersed disc a *Z-disc*. Take three copies of the *Z*-disc and paste these boundaries together so that the points *N* and *S* are pasting together and it has only one double curve (see Figure 4). The resulting diagram will be an immersed sphere.

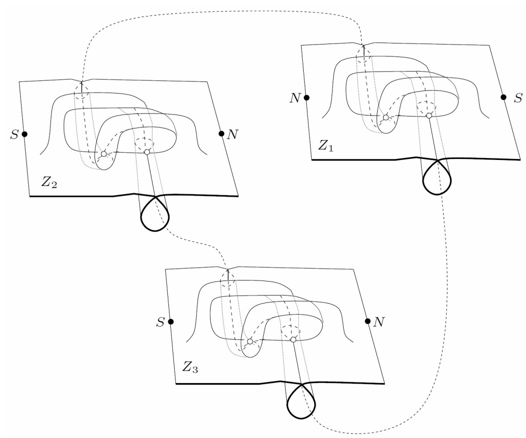

Figure 4. Three copies of *Z*-disc.

In a neighbourhood of the point *N*, we have three double curves meeting together. Merge them to make a pair of triple points *p* and *q* (see Figure 5).

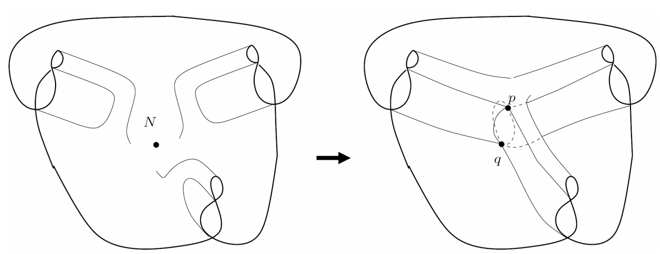

Figure 5. Intersecting three double curves to create triple points *p* and *q*.

The resulting immersed sphere has a three-fold symmetry around the triple point p and it has only one double curve and it passes through p three times. Therefore, by Theorem 3.1, it is unliftable.

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