JP Journal of Geometry and Topology

Published Online: June 2016
http://dx.doi.org/10.17654/GT019010001
Volume 19, Number 1, 2016, Pages 1-18 **ISSN: 0972-415X**

ON SUBMANIFOLDS OF PARA-SASAKIAN MANIFOLDS

Bilal Eftal Acet[1], **Selcen Yüksel Perktaş**[1] **and Erol Kılıç**[2]

[1]Department of Mathematics
Art and Science Faculty
Adiyaman University
02040 Adiyaman, Turkey
e-mail: eacet@adiyaman.edu.tr
sperktas@adiyaman.edu.tr

[2]Department of Mathematics
Art and Science Faculty
İnönü University
44280 Malatya, Turkey
e-mail: ekilic@inonu.edu.tr

Abstract

Studying in submanifolds of para-Sasakian manifolds, we obtain that (1) semi-parallel and 2-semi-parallel invariant submanifolds are totally geodesic, (2) necessary and sufficient conditions for the integrability of distributions and (3) some characterizations for submanifolds to be semi-invariant.

Received: October 2, 2015; Revised: December 14, 2015; Accepted: January 3, 2016
2010 Mathematics Subject Classification: 53C15, 53C25.
Keywords and phrases: para-Sasakian manifolds, invariant submanifolds, semi-invariant submanifolds.
Communicated by Yasuo Matsushita

1. Introduction

In [1], on a semi-Riemannian manifold M^{2n+1}, Kaneyuki and Konzai introduced a structure which is called the *almost paracontact structure* and characterized the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. Recently, Zamkovoy [2] studied paracontact metric manifolds and some subclasses which are known as para-Sasakian manifolds and then the study of paracontact geometry was continued by several papers [3-7] which are contained role of paracontact geometry about semi-Riemannian geometry, mathematical physics and relationships with the para-Kähler manifolds.

Assume that M and $\hat{M}$ are Riemannian manifolds, $f : M \to \hat{M}$ is an isometric immersion, h is the second fundamental form and $\widetilde{\nabla}$ is the Van der Waerden-Bortolotti connection of M. If the following condition

$$\widetilde{R}(X, Y) \cdot h = (\widetilde{\nabla}_X \widetilde{\nabla}_Y - \widetilde{\nabla}_Y \widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]})h = 0, \tag{1.1}$$

is provided, then an immersion is called *semi-parallel* [8]. Several papers on semi-parallel immersions have appeared (see [9-11]). In [12], Arslan et al. introduced the following condition

$$\widetilde{R}(X, Y) \cdot \widetilde{\nabla} h = 0, \tag{1.2}$$

for all $X, Y \in \Gamma(M)$. If the condition (1.2) satisfying, then submanifold is said to be 2-semi-parallel.

Also, in [13], Bejancu and Papaghiuc examined semi-invariant submanifolds and then the study of semi-invariant submanifold was continued by several papers [14-18]. In our paper, we give some geometric results about invariant and semi-invariant submanifolds of a para-Sasakian manifold.

This manuscript is arranged as follows. There are some basic definitions and equations for submanifolds and almost paracontact manifolds in Section 2. In Section 3, we show that semi-parallel and 2-semi-parallel invariant submanifolds of a para-Sasakian manifold are totally geodesic. In Section 4, some characterizations of semi-invariant submanifold of a para-Sasakian manifold are given. Finally, totally paracontact umbilical and totally paracontact geodesic submanifolds are introduced.

2. Preliminaries

Let $(\hat{M}^{n+m}, \hat{g})$ be a semi-Riemannian manifold and (M^n, g) be a submanifold of $\hat{M}$. Let $\hat{\nabla}$ and ∇ be the Levi-Civita connection of $\hat{M}$ and M, respectively. Then the Gauss and Weingarten formulas are given by

$$\hat{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\hat{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.2}$$

where $X, Y \in \Gamma(M)$ and N is a normal vector field on M.

The covariant derivative of second fundamental form h is given by

$$(\hat{\nabla}_X h)(Y, Z) = \nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.3}$$

for $X, Y, Z \in \Gamma(TM)$. Then $\hat{\nabla} h$ is called the *third fundamental form* of M. If $\hat{\nabla} h = 0$, then M is said to have *parallel second fundamental form* [19].

In view of (1.1), we have

$$\begin{aligned}(\hat{R}(X, Y) \cdot h)(U, V) = R^{\perp}(X, Y)h(U, V) - h(R(X, Y)U, V)\\ - h(U, R(X, Y)V),\end{aligned} \tag{2.4}$$

for $X, Y, U, V \in \Gamma(TM)$. Furthermore, we get

$$\begin{aligned}&(\hat{R}(X, Y) \cdot \hat{\nabla} h)(Z, W, U)\\ &= R^{\perp}(X, Y)(\hat{\nabla} h)(Z, W, U) - \hat{\nabla} h(R(X, Y)Z, W, U)\\ &\quad - \hat{\nabla} h(Z, R(X, Y)W, U) - \hat{\nabla} h(Z, W, R(X, Y)U),\end{aligned} \tag{2.5}$$

for $X, Y, Z, W, U \in \Gamma(TM)$, where $(\hat{\nabla} h)(Z, W, U) = (\hat{\nabla} h_Z)(W, U)$ [12].

A paracontact manifold $\hat{M}^{2n+1}$ is smooth manifold equipped with a 1-form η, a characteristic vector field ξ and a tensor field φ of type (1, 1) such that [1]:

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.6}$$

If we set $D = \ker \eta = \{X \in \Gamma(T\hat{M}) : \eta(X) = 0\}$, then φ induces an almost paracomplex structure on the codimension 1 distribution defined by D [1].

Moreover, if the manifold $\hat{M}$ is equipped with a semi-Riemannian metric $\hat{g}$ of signature $(n+1, n)$ which is called *compatible metric* satisfying [2]

$$\hat{g}(\varphi X, \varphi Y) = -\hat{g}(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \Gamma(T\hat{M}), \tag{2.7}$$

then we say that $\hat{M}$ is an *almost paracontact metric manifold* with an *almost paracontact metric structure* $(\varphi, \xi, \eta, \hat{g})$.

From the definition, one can see that [2]

$$\eta(X) = \hat{g}(X, \xi). \tag{2.8}$$

The fundamental 2-form of $\hat{M}$ is defined by

$$\Phi(X, Y) = \hat{g}(X, \varphi Y). \tag{2.9}$$

Definition 2.1. If $\hat{g}(X, \varphi Y) = d\eta(X, Y)$ (where $d\eta(X, Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X, Y])\}$), then η is a paracontact form and the almost paracontact metric manifold $(\hat{M}, \varphi, \xi, \eta, \hat{g})$ is said to be *paracontact metric manifold*.

An almost paracontact metric structure $(\varphi, \xi, \eta, \hat{g})$ is a para-Sasakian manifold if and only if [2]

$$(\hat{\nabla}_X \varphi)Y = -\hat{g}(X, Y)\xi + \eta(Y)X, \tag{2.10}$$

where $X, Y \in \Gamma(T\hat{M})$ and $\hat{\nabla}$ is a Levi-Civita connection on $\hat{M}$.

From (2.10), we arrive at

$$\hat{\nabla}_X \xi = -\varphi X. \tag{2.11}$$

Also, the following relation holds in a para-Sasakian manifold [2]:

$$\hat{R}(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.12}$$

for $X, Y \in \Gamma(T\hat{M})$.

3. Invariant Submanifolds

Let M be a submanifold of a para-Sasakian manifold $(\hat{M}^{2n+1}, \varphi, \xi, \eta, g)$. If the structure vector field $\xi \in \Gamma(TM)$ and φX is tangent to M, for any $X \in \Gamma(TM)$ at every point of M, that is, $\varphi T_x(M) \subset T_x(M)$, for all $x \in M$, then M is called *invariant submanifold* of $\hat{M}$. Since $\xi \in \Gamma(TM)$, we write

$$-\varphi X = \hat{\nabla}_X \xi = \nabla_X \xi + h(X, \xi), \quad X \in \Gamma(TM).$$

In that case, we obtain $h(X, \xi) = 0$.

Proposition 3.1. *If M is an invariant submanifold of a para-Sasakian manifold $\hat{M}$, then the following relations hold*:

(i) $h(X, \xi) = 0,\ A_N \xi = 0,$ (3.1)

(ii) $h(X, \varphi Y) = h(\varphi X, Y) = \varphi h(X, Y),$ (3.2)

(iii) $\varphi A_N X = -A_N \varphi X = A_{\varphi N} X.$ (3.3)

Proposition 3.2. *Let M be an invariant submanifold of a para-Sasakian manifold $\hat{M}$. If the second fundamental form h of M is parallel*, *then M is totally geodesic*.

Proof. Assume that h is parallel. Thus, for any $X, Y \in \Gamma(TM)$, we have

$$0 = (\nabla_X h)(Y, \xi) = \nabla_X h(Y, \xi) - h(\nabla_X Y, \xi) - h(Y, \nabla_X \xi).$$

From (2.11) and (3.1), we get

$$h(\varphi X, Y) = 0.$$

By use of (3.2), we conclude

$$h(X, Y) = 0,$$

which completes the proof. □

Theorem 3.1. *Let M be an invariant submanifold of a para-Sasakian manifold* $\hat{M}$. *If M is totally umbilical, then M is totally geodesic.*

Proof. From (3.1), we obtain

$$h(\xi, \xi) = 0.$$

If M is totally umbilical, then we get

$$h(\xi, \xi) = 0 = \mu g(\xi, \xi),$$

which implies $\mu = 0.$ Hence, we obtain

$$h(X, Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

This completes the proof. □

Theorem 3.2. *Let M be an invariant submanifold of a para-Sasakian manifold* $\hat{M}$. *Then M is semi-parallel if and only if M is totally geodesic.*

Proof. Let M be a semi-parallel invariant submanifold of $\hat{M}$. Then from (2.4), we get

$$R^{\perp}(X, Y)h(V, Z) - h(R(X, Y)V, Z) - h(V, R(X, Y)Z) = 0, \tag{3.4}$$

for any $X, Y, V, Z \in \Gamma(TM)$.

Putting $X = Z = \xi$ in (3.4) and using (3.1), we have

$$h(V, R(\xi, Y)\xi) = 0.$$

From (2.12) and (3.1), we get

$$h(V, Y) = 0,$$

which implies that *M* is totally geodesic.

The proof sufficiency part is clear. □

Theorem 3.3. *An invariant submanifold M of a para-Sasakian manifold* $\hat{M}$ *has parallel second fundamental form if and only if M is totally geodesic.*

Proof. By use of (2.3), for any $X, Y, W \in \Gamma(TM)$, we get

$$\nabla_X^{\perp}(h(Y, W)) - h(\nabla_X Y, W) - h(Y, \nabla_X W) = 0. \tag{3.5}$$

Taking $W = \xi$ in (3.5), we find

$$\nabla_X^{\perp}(h(Y, \xi)) - h(\nabla_X Y, \xi) - h(Y, \nabla_X \xi) = 0.$$

From (3.1), we obtain

$$h(Y, \nabla_X \xi) = 0.$$

Therefore, from (2.11), we get

$$h(Y, \varphi X) = 0. \tag{3.6}$$

Replacing *X* by φX in (3.6) and in view of (2.6) with (3.1), we have

$$h(Y, X) = 0,$$

which gives that *M* is totally geodesic. □

Theorem 3.4. *An invariant submanifold M of a para-Sasakian manifold* $\hat{M}$ *is* 2*-semi-parallel if and only if M is totally geodesic.*

Proof. Assume that *M* is 2-semi-parallel. Thus, from (2.5), we write

$$R^{\perp}(X, Y)(\tilde{\nabla}h)(Z, W, U) - \tilde{\nabla}h(R(X, Y)Z, W, U)$$
$$- \tilde{\nabla}h(Z, R(X, Y)W, U) - \tilde{\nabla}h(Z, W, R(X, Y)U) = 0, \tag{3.7}$$

for any $X, Y, U, W, Z \in \Gamma(TM)$.

By putting $X = W = \xi$ in (3.7) and then by using (2.3), (2.12) and (3.1), we have

$$R^{\perp}(\xi, Y)h(\varphi Z, U) + \eta(Z)h(\varphi Y, U) - \nabla_Z^{\perp}h(Y, U)$$

$$+ h(\nabla_Z R(\xi, Y)\xi, U) + h(Y, \nabla_Z U) - \eta(U)h(\varphi Z, Y) = 0. \tag{3.8}$$

Taking $U = \xi$ in (3.8) and from (2.6) with (3.1), we have

$$h(Y, \nabla_Z \xi) - h(\varphi Z, Y) = 0,$$

which yields

$$h(\varphi Z, Y) = 0. \tag{3.9}$$

Replacing Z by φZ in (3.9) and in view of (2.6) and (3.1), we obtain

$$h(Z, Y) = 0.$$

Hence, M is totally geodesic.

Since the proof of contrary is clear, we omit it. □

4. Semi-invariant Submanifolds

Definition 4.1. A submanifold M of an almost paracontact metric manifold $\hat{M}$ with $\xi \in \Gamma(TM)$ is called a *semi-invariant submanifold*, if there exist two differentiable distributions D and $D^{\perp}$ on M such that [22]

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$,

(ii) $\varphi(D) = D$, that is, D is invariant by φ,

(iii) $\varphi(D^{\perp}) \subseteq T^{\perp}M$, that is, $D^{\perp}$ is anti-invariant by φ.

A semi-invariant submanifold is an invariant submanifold (resp. anti-invariant submanifold) if $D^{\perp} = \{0\}$ (resp. $D = \{0\}$). Also, if $D \neq \{0\} \neq D^{\perp}$, then a semi-invariant submanifold is called *proper*.

For $X \in \Gamma(TM)$, we put

$$X = \omega_1 X + \omega_2 X + \eta(X)\xi, \tag{4.1}$$

where ω_1 and ω_2 are projection operators of TM on D and $D^{\perp}$, respectively.

Moreover, for $X \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, we get

$$\begin{aligned}(\hat{\nabla}_X\varphi)Y &= [(\nabla_X f)Y - A_{wX}Y - Bh(X, Y)] \\ &\quad + [(\nabla_X w)Y + h(X, fY) - Ch(X, Y)],\end{aligned} \tag{4.2}$$

$$\begin{aligned}(\hat{\nabla}_X\varphi)N &= [(\nabla_X B)N - A_{CN}X - fA_N X] \\ &\quad + [(\nabla_X C)N + h(X, BN) - wA_N X],\end{aligned} \tag{4.3}$$

where

$$\varphi X = fX + wX, \quad (fX \in \Gamma(TM), wX \in \Gamma(T^{\perp}M)), \tag{4.4}$$

$$\varphi N = BN + CN, \quad (BN \in \Gamma(TM), CN \in \Gamma(T^{\perp}M)), \tag{4.5}$$

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \quad (\nabla_X w)Y = \nabla_X^{\perp} wY - w\nabla_X Y, \tag{4.6}$$

$$(\nabla_X B)N = \nabla_X BN - B\nabla_X^{\perp} N, \;\; (\nabla_X C)N = \nabla_X^{\perp} CN - C\nabla_X^{\perp} N. \tag{4.7}$$

Example 4.1. Let $\hat{M} = \mathbb{R}^9$ be the 9-dimensional real number space with

$$(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4, z)$$

standard coordinate system. If we set

$$\varphi\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad \varphi\frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^i}, \quad \varphi\frac{\partial}{\partial z} = 0, \quad (1 \le i \le 4),$$

then φ is a tensor field of type $(1, 1)$ on $\mathbb{R}^9$. We define the differential 1-form η and the vector field ξ by

$$\eta = dz \text{ and } \xi = \frac{\partial}{\partial z}.$$

Also, we define metric $\hat{g}$ by

$$\hat{g} = \eta \otimes \eta + \sum_{i=1}^{4} (dx^i \otimes dx^i - dy^i \otimes dy^i).$$

Then the set $(\hat{M}, \xi, \eta, \varphi, \hat{g})$ is an almost paracontact metric manifold.

Now, we define a 4-dimensional submanifold M of $\hat{M}$, with $\xi \in \Gamma(TM)$, by

$$x^1 = -y^1, \;\; x^2 = -y^2, \;\; x^3 = x^4, \;\; y^3 = 0, \;\; y^4 = 0.$$

In this case, TM is spanned by $\{U_i\}_{1 \le i \le 4}$, where

$$U_1 = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial y^1}, \;\; U_2 = \frac{\partial}{\partial x^2} - \frac{\partial}{\partial y^2},$$

$$U_3 = \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4}, \;\; U_4 = \xi = \frac{\partial}{\partial z}.$$

Furthermore, $T^{\perp}M$ is spanned by $\{N_i\}_{1 \le i \le 5}$, where

$$N_1 = -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^1}, \;\; N_2 = -\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2},$$

$$N_3 = -\frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^4}, \;\; N_4 = -\frac{\partial}{\partial y^3} - \frac{\partial}{\partial y^4},$$

$$N_5 = \frac{\partial}{\partial y^3} + \frac{\partial}{\partial y^4}.$$

If we set $D = Span\{U_1, U_2\}$, $D^{\perp} = Span\{U_3\}$, $\overline{D} = Span\{N_1, N_2, N_3, N_4\}$ and $\overline{D}^{\perp} = Span\{N_5\}$, then we have that $\varphi D = D$, $\varphi D^{\perp} = \overline{D}^{\perp} \subset T^{\perp}M$, $\varphi \overline{D} = \overline{D}$. Thus, M is a semi-invariant submanifold of $\hat{M}$.

Theorem 4.1. *Let M be a semi-invariant submanifold of a para-Sasakian manifold* $\hat{M}$*. Then the distribution D is integrable if and only if*

$$h(X, \varphi Y) = h(\varphi X, Y),$$

for all $X, Y \in \Gamma(D)$.

Proof. In view of (2.1), (2.2), (2.10) and symmetry of *h*, then we get

$$\begin{aligned} g([X, Y], \varphi Z) &= g(\nabla_X Y - \nabla_Y X, \varphi Z) \\ &= g(\hat{\nabla}_X Y - \hat{\nabla}_Y X, \varphi Z) \\ &= g(\varphi \hat{\nabla}_Y X - \varphi \hat{\nabla}_X Y, Z) \\ &= g((\hat{\nabla}_X \varphi) Y - \hat{\nabla}_X \varphi Y, Z) \\ &\quad - g((\hat{\nabla}_Y \varphi) X - \hat{\nabla}_Y \varphi X, Z) \\ &= -g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - g(\hat{\nabla}_X \varphi Y, Z) \\ &\quad + g(X, Y)\eta(Z) - g(Y, Z)\eta(X) + g(\hat{\nabla}_Y \varphi X, Z) \\ &= g(\hat{\nabla}_Y \varphi X, Z) - g(\hat{\nabla}_X \varphi Y, Z), \end{aligned} \tag{4.8}$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$. Again using (2.1) in (4.8), we have

$$\begin{aligned} g([X, Y], \varphi Z) &= g(\nabla_Y \varphi X, Z) + g(h(Y, \varphi X), Z) \\ &\quad - g(\nabla_X \varphi Y, Z) - g(h(X, \varphi Y), Z) \\ &= g(h(Y, \varphi X), Z) - g(h(X, \varphi Y), Z) \\ &= g(h(Y, \varphi X) - h(X, \varphi Y), Z). \end{aligned}$$

Therefore, $[X, Y] \in \Gamma(D)$ if and only if $h(Y, \varphi X) - h(X, \varphi Y) = 0$, which completes the proof. □

Theorem 4.2. *Let M be a semi-invariant submanifold of a para-Sasakian manifold* $\hat{M}$. *Then the distribution* $D^\perp$ *is integrable if and only if*

$$A_{\varphi U} V = A_{\varphi V} U,$$

for all $U, V \in \Gamma(D^\perp)$.

Proof. By using (2.2) and (2.10) with (4.4) and (4.5), for all $U, V \in \Gamma(D^{\perp})$, we get

$$(\hat{\nabla}_U \varphi)V = \hat{\nabla}_U \varphi V - \varphi \hat{\nabla}_U V,$$

$$-g(U, V)\xi + \eta(V)U = -A_{\varphi U}V + \nabla_U^{\perp}\varphi V - \varphi \nabla_U V - \varphi h(U, V),$$

$$-g(U, V)\xi = -A_{\varphi U}V + \nabla_U^{\perp}\varphi V - f\nabla_U V$$

$$- w\nabla_U V - Bh(U, V) - Ch(U, V). \tag{4.9}$$

If we take tangential part of (4.9), then we have

$$-g(U, V)\xi = -A_{\varphi U}V - f\nabla_U V - Bh(U, V). \tag{4.10}$$

In (4.10), replacing U with V, we get

$$-g(V, U)\xi = -A_{\varphi V}U - f\nabla_V U - Bh(V, U). \tag{4.11}$$

In view of (4.10) and (4.11), we obtain

$$A_{\varphi V}U - A_{\varphi U}V + f\nabla_U V - f\nabla_V U = 0,$$

from which, we easily see that

$$A_{\varphi U}V - A_{\varphi V}U = f[U, V].$$

Thus, we obtain that $[U, V] \in \Gamma(D^{\perp})$ if and only if $A_{\varphi U}V - A_{\varphi V}U = 0$, which completes the proof. □

Theorem 4.3. *Let M be a semi-invariant submanifold of a para-Sasakian manifold* $\hat{M}$. *Then the distribution D is integrable and its leaves are totally geodesic in M if and only if*

$$g(h(X, Y), \varphi U) = 0,$$

for all $X, Y \in \Gamma(D), U \in \Gamma(D^{\perp})$.

Proof. By use of (2.1), (2.2), (2.10) with (4.4), we obtain

$$
\begin{aligned}
g(h(X, Y), \varphi U) &= g(\hat{\nabla}_X Y, \varphi U) \\
&= -g(\varphi \hat{\nabla}_X Y, U) \\
&= g((\hat{\nabla}_X \varphi)Y, U) - g(\hat{\nabla}_X \varphi Y, U) \\
&= -g(X, Y)\eta(U) + g(X, U)\eta(Y) - g(\hat{\nabla}_X \varphi Y, U) \\
&= -g(\hat{\nabla}_X \varphi Y, U) = -g(\nabla_X fY, U),
\end{aligned}
$$

for all $X, Y \in \Gamma(D), U \in \Gamma(D^{\perp})$.

Assume that D is integrable and its leaves are totally geodesic in M. Then we have

$$\nabla_X Y \in \Gamma(D),$$

which yields $g(h(X, Y), \varphi U) = 0$.

Conversely, if $g(h(X, Y), \varphi U) = 0$, then we get

$$g(\nabla_X fY, U) = 0.$$

Therefore, $\nabla_X fY \in \Gamma(D)$. □

Definition 4.2. Let M be a semi-invariant submanifold of a para-Sasakian manifold $\hat{M}$. If the second fundamental form of M satisfies

$$h(X, Y) = 0, \quad \forall X, Y \in \Gamma(D),$$

then M is called *D-geodesic submanifold*.

Theorem 4.4. *Let M be a semi-invariant submanifold of a para-Sasakian manifold $\hat{M}$. Then D is integrable and its leaves are totally geodesic in $\hat{M}$ if and only if M is D-geodesic submanifold.*

Proof. We suppose that D is integrable and its leaves are totally geodesic in $\hat{M}$. Then we have

$$\hat{\nabla}_X Y \in \Gamma(D),$$

for all $X, Y \in \Gamma(D)$. Hence, we get

$$g(h(X, Y), N) = g(\hat{\nabla}_X Y, N) = 0,$$

for $N \in \Gamma(T^{\perp} M)$, which shows M is a D-geodesic submanifold.

Contrary, let M be a D-geodesic submanifold. Then we get

$$g(\hat{\nabla}_X Y, N) = g(h(X, Y), N) = 0. \tag{4.12}$$

The assertion is proved by virtue of (4.12). □

Definition 4.3. Let M be a semi-invariant submanifold of para-Sasakian manifold $\hat{M}$. If the second fundamental form of M satisfies

$$h(X, Y) = 0, \quad X \in \Gamma(D), \quad Y \in \Gamma(D^{\perp}),$$

then M is called *mixed-geodesic submanifold*.

Theorem 4.5. *Let M be a semi-invariant submanifold of para-Sasakian manifold $\hat{M}$. Then M is mixed-geodesic submanifold if and only if*

$$A_N \varphi X \in \Gamma(D),$$

for any $X \in \Gamma(D)$ and $N \in \Gamma(T^{\perp} M)$.

Proof. Using (2.1) and (2.2), we obtain

$$\begin{aligned} g(h(\varphi X, V), N) &= g(\hat{\nabla}_V \varphi X, N) = -g(\hat{\nabla}_V N, \varphi X) \\ &= g(A_N V, \varphi X) = g(V, A_N \varphi X), \end{aligned}$$

for any $X \in \Gamma(D), V \in \Gamma(D^{\perp})$ and $N \in \Gamma(T^{\perp} M)$. Thus, we get $h(\varphi X, V) = 0$ if and only if $A_N \varphi X \in \Gamma(D)$. □

Definition 4.4. Let M be a semi-invariant submanifold of para-Sasakian manifold $\hat{M}$. If there exists a normal vector field H such that the second fundamental form of M is given by

$$h(X, Y) = g(\varphi X, \varphi Y)H + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \tag{4.13}$$

for any $X, Y \in \Gamma(TM)$, then M is called *totally paracontact umbilical submanifold*. If we have $H = 0$ in (4.13), that is,

$$h(X, Y) = \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \tag{4.14}$$

then we say M is *totally paracontact geodesic*.

Theorem 4.6. *Any totally paracontact umbilical semi-invariant submanifold M of a para-Sasakian manifold is totally paracontact geodesic*.

Proof. By using (2.6), we get

$$(\hat{\nabla}_X \varphi)\varphi X = -\varphi(\hat{\nabla}_X \varphi)X - (\hat{\nabla}_X \eta)(X)\xi - \eta(X)\hat{\nabla}_X \xi,$$

for any $X \in \Gamma(T\hat{M})$. Moreover, from (2.10), we get

$$(\hat{\nabla}_X \varphi)X = -g(X, X)\xi + \eta(X)X$$

and

$$g(\hat{\nabla}_X \xi, X) = (\nabla_X \eta)(X) = 0.$$

Thus, we obtain

$$(\hat{\nabla}_X \varphi)\varphi X = -\eta(X)\varphi X - \eta(X)\hat{\nabla}_X \xi. \tag{4.15}$$

Now, we assume that $X \in \Gamma(D)$. So, from (2.10), we get

$$g((\hat{\nabla}_X \varphi)\varphi X, H) = 0. \tag{4.16}$$

Using (2.6) in (4.16), we obtain

$$
\begin{aligned}
0 &= g(\hat{\nabla}_X^2 \varphi X - \varphi \hat{\nabla}_X \varphi X, H) \\
&= g(\hat{\nabla}_X X, H) + g(\hat{\nabla}_X \varphi X, \varphi H) \\
&= -g(X, \hat{\nabla}_X H) + g(\hat{\nabla}_X \varphi X, \varphi H) \\
&= -g(A_H X, X) + g(A_{\varphi H} X, \varphi X),
\end{aligned}
$$

from which, we have

$$0 = g(h(X, \varphi X), \varphi H) - g(h(X, X), H). \tag{4.17}$$

Thus, using (4.14) in (4.17), we obtain

$$g(X, X) g(H, H) = 0,$$

and consequently, $H = 0$, which proves the result. □

Acknowledgment

The author thanks the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

References

[1] S. Kaneyuki and M. Konzai, Paracomplex structure and affine symmetric spaces, Tokyo J. Math. 8 (1985), 301-318.

[2] S. Zamkovoy, Canonical connections on paracontact manifolds, Ann. Global Anal. Geom. 36 (2009), 37-60.

[3] D. V. Alekseevski, V. Cortés, A. S. Galaev and T. Leistner, Cones over pseudo-Riemannian manifolds and their holonomy, J. Reine Angew. Math. 635 (2009), 23-69.

[4] D. V. Alekseevski, C. Medori and A. Tomassini, Maximally homogeneous para-CR-manifolds, Ann. Global Anal. Geom. 30 (2006), 1-27.

[5] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, Special geometry of Euclidean supersymmetry 1, vector multiplets, J. High Energy Phys. 03 (2004), 028.

[6] V. Cortés, M. A. Lawn and L. Schäfer, Affine hyperspheres associated to special para-Kähler manifolds, Int. J. Geom. Methods Mod. Phys. 3 (2006), 995-1009.

[7] S. Erdem, On almost (para)contact (hyperbolic) metric manifolds and harmonicity of (φ, φ')-holomorphic maps between them, Houston J. Math. 28 (2002), 21-45.

[8] J. Deprez, Semi-parallel surfaces in the Euclidean space, J. Geom. 25 (1985), 192-200.

[9] F. Dillen, Semi-parallel hypersurfaces of a real space form, Israel J. Math. 75 (1991), 193-202.

[10] H. Endo, Invariant submanifolds in contact metric manifolds, Tensor (N.S.) 43(1) (1986), 83-87.

[11] M. Kon, Invariant submanifolds of normal contact metric manifolds, Kodai Math. Sem. Rep. 25 (1973), 330-336.

[12] K. Arslan, Ü. Lumiste, C. Murathan and C. Özgür, 2-semiparallel surfaces in space forms, I. two particular cases, Proc. Estonian Acad. Sci. Phys. Math. 49(3) (2000), 139-148.

[13] A. Bejancu and N. Papaghiuc, Semi-invariant submanifolds of a Sasakian manifold, An. St. Univ. Al. I. Cuza, Iaşi Sect. I Math. 27(1) (1981), 163-170.

[14] C. Calin, Contact CR-submanifolds of a nearly Sasakian manifold, Demonstratio Math. 29 (1996), 337-348.

[15] J. S. Kim, X. Liu and M. M. Tripathi, On semi-invariant submanifolds of nearly trans-Sasakian manifolds, Int. J. Pure Appl. Math. Sci. 1 (2004), 15-34.

[16] K. Yano and M. Kon, On contact CR-submanifolds, J. Korean Math. Soc. 26 (1989), 231-262.

[17] K. Matsumoto, M. H. Shahid and I. Mihai, Semi-invariant submanifolds of certain almost contact manifolds, Bull. Yamagata Univ. Natur. Sci. 13 (1994), 183-192.

[18] M. Kobayashi, Semi-invariant submanifolds of a certain class of almost contact manifold, Tensor (N.S.) 43 (1986), 28-36.

[19] B. Y. Chen, Geometry of submanifolds, Pure Appl. Math., Vol. 22, Marcel Dekker Inc., New York, 1973.

[20] D. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. 509, Springer-Verlag, 1976.

[21] D. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhauser, Boston, 2002.

[22] A. Bejancu, Geometry of CR-submanifolds, D. Reidel Publ. Co., Holland, 1986.

[23] S. Kaneyuki and F. L. Willams, Almost paracontact and parahodge structure on manifolds, Nagoya Math. J. 99 (1985), 173-187.

[24] I. Sato, On a structure similar to the almost structure, II, Tensor (N.S.) 31 (1977), 199-205.