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# CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN INVERSE OF NORMAL MEANS

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**Abstract**

This paper presents three new confidence intervals for the difference between inverse of normal means. One of the new confidence intervals based on the approximation confidence interval is constructed. In addition, the method of variance estimates recovery (MOVER) and the generalized confidence interval (GCI) are proposed. Monte Carlo simulation is used to assess the performance of these intervals based on their coverage probabilities and expected lengths. An application is included to illustrate our methods.

## 1. Introduction

Statistical estimation of the inverse of normal mean arises in many situations, including the biological sciences, econometrics, and in experimental nuclear physics. Lamanna et al. [1] studied charged particle

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momentum $p = \mu^{-1}$, where $\mu$ is the track curvature of a particle. Zaman [2, 3] discussed this problem in the one-dimensional special case of the single period control problem, and estimation of the structural parameters of a simultaneous equation, as recognized. In econometrics, Zellner [4] estimated the inverse of common mean of structural coefficient of linear structural econometric models. More recently, Withers and Nadarajah [5] proposed the unbiased estimators that are obtained for positive powers of the mean, and estimators of almost exponentially small bias are obtained for negative powers of the mean. Srivastava and Bhatnagar [6] proposed a class of estimators with finite moment for the inverse of the mean. Voinov [7] proposed unbiased estimators of power for the inverse of the mean. Niwitpong and Wongkhao [8] proposed the approximation *t*-distribution to obtain a confidence interval for the inverse of normal mean.

In this paper, an extension of Niwitpong and Wongkhao [8], we propose new confidence intervals for the difference between inverse of normal means. The first confidence interval is constructed based on Casella and Berger [9] who proposed the expectation and variance of the inverse of normal mean by using the Delta method and we now use these estimators to form approximate confidence interval for the difference between inverses of normal means. The second confidence interval is constructed based on the method of variance estimates recovery (MOVER), recently published by Zou and Donner [10]. As review in Zou and Donner [10], Zou et al. [11] and Donner and Zou [12], the MOVER method can be used to obtain confidence interval for parameters from two independent populations in the various situations, see, e.g., the difference and the ratio of parameters. A recent paper by Suwan and Niwitpong [13] recommended the MOVER method is quite convenient and effective approach for constructing confidence intervals for the difference of parameters and they proposed confidence intervals for the difference between variances of the nonnormal distribution that utilizes the kurtosis based on the MOVER method. Following Phonyiem and Niwitpong [14], we also use the GCI to construct the third confidence interval for the inverse of normal means introduced by Weerahandi [15]; see also the book by Weerahandi [16]. Much of researches for constructing confidence intervals based on GCI have been investigated recently, see, e.g., Lee and Lin [17], Lin and Lee [18], Lin et al. [19], Phonyiem and Niwitpong [14] and references therein.

We compared these three confidence intervals based on their coverage probability and expected length via Monte Carlo simulation.

## 2. Confidence Intervals for the Difference between Inverse of Normal Means

Consider two normal populations; $X$ and $Y$ with means $\mu_x$, $\mu_y$ and variances $\sigma_x^2$, $\sigma_y^2$, respectively. Let $\overline{X}$ and $S_x^2$ and $\overline{Y}$ and $S_y^2$ denote the sample means and sample variances of part samples of size $n$ for population 1 and size $m$ for population 2, respectively. We are interested in constructing the confidence interval for the difference between the inverse of normal means, $\delta = \mu_x^{-1} - \mu_y^{-1}$.

### 2.1. Based on approximation confidence interval

The approximate variances of $\delta_1 = \mu_x^{-1}$ and $\delta_2 = \mu_y^{-1}$ by using Delta method are given by Casella and Berger [9]:

$$Var\left(\frac{1}{\overline{X}}\right) \approx \left(\frac{1}{\overline{X}}\right)^4 S_x^2$$

and

$$Var\left(\frac{1}{\overline{Y}}\right) \approx \left(\frac{1}{\overline{Y}}\right)^4 S_y^2.$$

Now we have the fact from the Central Limit Theorem:

$$\frac{\left(\left(\frac{1}{\overline{X}} - \frac{1}{\overline{Y}}\right) - \left(\frac{1}{\mu_x} - \frac{1}{\mu_y}\right)\right)}{\sqrt{\frac{1}{n}\left(\frac{1}{\overline{X}}\right)^4 S_x^2 + \frac{1}{m}\left(\frac{1}{\overline{Y}}\right)^4 S_y^2}} \sim N(0, 1).$$

Therefore, the new confidence interval for $\delta$ is given by

$$CI_{lm} = \left[\frac{1}{\overline{\overline{X}}} - \frac{1}{\overline{\overline{Y}}} \pm c\left(\frac{1}{n}\left(\frac{1}{\overline{\overline{X}}}\right)^4 S_x^2 + \frac{1}{m}\left(\frac{1}{\overline{\overline{Y}}}\right)^4 S_y^2\right)^{1/2}\right],$$

where $c$ is an upper $1 - \alpha/2$ quantile of the standard normal distribution.

**2.2. Based on the method of variance estimates recovery (MOVER)**

We propose the new confidence interval based on the MOVER, introduced by Zou and Donner [10]. For the difference between inverse of normal means, $\delta = \delta_1 - \delta_2$, the $100(1-\alpha)\%$ two-sided confidence interval for $\delta$ is given by

$$(L', U') = \Big((\hat{\delta}_1 - \hat{\delta}_2) - \sqrt{(\hat{\delta}_1 - l_1)^2 + (u_2 - \hat{\delta}_2)^2},$$

$$(\hat{\delta}_1 - \hat{\delta}_2) + \sqrt{(u_1 - \hat{\delta}_1)^2 + (\hat{\delta}_2 - l_2)^2}\Big). \tag{1}$$

Let $(l_i', u_i')$, $i = 1, 2$ be the confidence limits for $\delta_1$, $\delta_2$, respectively. Then the confidence intervals for $\delta_1$ and $\delta_2$ are given by

$$(l_1', u_1') = \left[\frac{\sqrt{n}}{dS_x + \sqrt{n}\overline{\overline{X}}}, \frac{\sqrt{n}}{-dS_x + \sqrt{n}\overline{\overline{X}}}\right], \tag{2}$$

where $d = t_{1-\alpha/2, n-1}$ and

$$(l_2', u_2') = \left[\frac{\sqrt{m}}{d'S_y + \sqrt{m}\overline{\overline{Y}}}, \frac{\sqrt{m}}{-d'S_x + \sqrt{m}\overline{\overline{Y}}}\right], \tag{3}$$

where $d' = t_{1-\alpha/2, m-1}$.

From equation (1), we set $\hat{\delta}_1 = \frac{1}{\overline{\overline{X}}}$, $\hat{\delta}_2 = \frac{1}{\overline{\overline{Y}}}$, $l_1 = l_1'$, $l_2 = l_2'$, $u_1 = u_1'$ and $u_2 = u_2'$. Therefore, the new confidence interval for $\delta = \delta_1 - \delta_2$ is given by

$$CI_{MV} = \left[\left(\frac{1}{\overline{\overline{X}}} - \frac{1}{\overline{\overline{Y}}}\right) - \sqrt{(\hat{\delta}_1 - l_1')^2 + (u_2' - \hat{\delta}_2)^2},\right.$$

$$\left.\left(\frac{1}{\overline{\overline{X}}} - \frac{1}{\overline{\overline{Y}}}\right) + \sqrt{(u_1' - \hat{\delta}_1)^2 + (\hat{\delta}_2 - l_2')^2}\right].$$

**2.3. Based on generalized confidence interval (GCI)**

Let $S_1^2 = n^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$, $S_2^2 = m^{-1}\sum_{i=1}^{n}(Y_j - \overline{Y})^2$, $U_1 = (nS_1^2/\sigma_x^2) \sim \chi_{n-1}^2$, $V_1 = (mS_2^2/\sigma_y^2) \sim \chi_{m-1}^2$ and apply them into below expression. Consider how to derive the mean of $X$:

$$\begin{aligned}\mu_x &\approx \bar{x} - (\overline{X} - \mu_x)\\ &\approx \bar{x} - \frac{(\overline{X} - \mu_x)}{\sigma_x/\sqrt{n}}\sigma_x/\sqrt{n}\\ &\approx \bar{x} - Z\sigma_x/\sqrt{n}, \quad Z \sim N(0, 1)\\ &\approx \bar{x} - Zs_1/\sqrt{U_1}.\end{aligned}$$

Similarly, the mean of $Y$ can be defined as $\mu_y \approx \bar{y} - Zs_2/\sqrt{V_1}$.

According to Weerahandi [15], one of the potentials generalized can be defined as

$$\begin{aligned}W(X, Y, x, y, \mu_x, \mu_y, \varsigma) &= \frac{1}{\bar{x} - Zs_1/\sqrt{U}} - \frac{1}{\bar{y} - Zs_2/\sqrt{V}}\\ &= \frac{1}{\bar{x} - T_1s_1/\sqrt{n-1}} - \frac{1}{\bar{y} - T_2s_2/\sqrt{m-1}},\end{aligned}$$

where $T_1 \sim t_{n-1}$, $T_2 \sim t_{m-1}$, $\varsigma = (\sigma_x^2, \sigma_y^2)$ and $\bar{x}$, $\bar{y}$, $s_1^2$, $s_2^2$ are observed value of $\overline{X}, \overline{Y}, S_1^2, S_2^2$.

It is easy to see that $W(X, Y, x, y, \mu_x, \mu_y, \varsigma) = \delta$ and does not depend on the nuisance parameter. Therefore, the $100(1-\alpha)\%$ generalized confidence interval for δ is given by

$$CI_{PV} = [W_{\delta(\alpha/2)}, W_{\delta(1-\alpha/2)}],$$

where $W_{\delta(\alpha/2)}$ is the percentile $\alpha/2\text{th}$ of $W_{\delta}$.

## 3. Simulation Studies

In this section, we examine the performance of the propose confidence intervals for the difference between inverse of normal means. In terms of coverage probability and expected length, we compare the new confidence interval based on approximation confidence interval, GCI and the MOVER method. Simulation studies using different values of sample sizes $(n = m = 10)$, $(n = 10, m = 20)$, $(n = m = 20)$, $(n = 20, m = 40)$, $(n = 40, m = 20)$, $(n = m = 40)$ and standard deviations

$$(\sigma_x, \sigma_y) = (0.05, 0.05), (0.05, 0.15), (0.10, 0.10), (0.10, 0.20),$$
$$(0.20, 0.20), (0.20, 0.30), (0.30, 0.30), (0.30, 0.40)$$

are considered. Without loss of generality, the population mean is set to 1, we consider samples taken from populations that have $N(1, \sigma_x^2)$ and $N(1, \sigma_y^2)$.

The results via Monte Carlo simulation with 10,000 runs for each combination of $n$, $m$, $\sigma_x$ and $\sigma_y$, using written function in $R$, are summarized in Table 1. The generalized computations based on 500 pivotal quantities are also used to compute the performance of confidence intervals, $CI_{PV}$. By detailing the estimated coverage probabilities and the expected lengths for the 95% confidence interval based on three methods including sample sizes and corresponding standard deviations, Table 1 presents the simulation results.

**Table 1.** Coverage probability and expected length (in parentheses) of 95% confidence intervals for the difference between inverse of normal means

| $(n, m)$ | $(\sigma_x, \sigma_y)$ | $CI_{lm}$ | $CI_{MV}$ | $CI_{PV}$ |
|---|---|---|---|---|
| (10, 10) | (0.05, 0.05) | 0.9362 (0.0866) | 0.9636 (0.1001) | 0.9621 (0.0999) |
| | (0.05, 0.15) | 0.9259 (0.1930) | 0.9554 (0.2257) | 0.9537 (0.2259) |
| | (0.10, 0.10) | 0.9388 (0.1735) | 0.9661 (0.2020) | 0.9613 (0.2012) |
| | (0.10, 0.20) | 0.9309 (0.2757) | 0.9562 (0.3261) | 0.9558 (0.3256) |
| | (0.20, 0.20) | 0.9392 (0.3515) | 0.9635 (0.4199) | 0.9620 (0.4180) |
| | (0.20, 0.30) | 0.9433 (0.4531) | 0.9633 (0.8431) | 0.9615 (0.5541) |
| | (0.30, 0.30) | 0.9460 (0.5370) | 0.9636 (0.8431) | 0.9619 (0.8386) |
| | (0.30, 0.40) | 0.9484 (0.6413) | 0.9628 (0.8386) | 0.9612 (0.8327) |
| (20, 20) | (0.05, 0.05) | 0.9429 (0.0616) | 0.9575 (0.0685) | 0.9566 (0.0675) |
| | (0.05, 0.15) | 0.9374 (0.1376) | 0.9539 (0.1477) | 0.9514 (0.1479) |
| | (0.10, 0.10) | 0.9483 (0.1235) | 0.9595 (0.1324) | 0.9572 (0.1320) |
| | (0.10, 0.20) | 0.9428 (0.1956) | 0.9561 (0.2109) | 0.9535 (0.2107) |
| | (0.20, 0.20) | 0.9493 (0.2483) | 0.9594 (0.2689) | 0.9555 (0.2682) |
| | (0.20, 0.30) | 0.9458 (0.3185) | 0.9537 (0.3483) | 0.9533 (0.3476) |
| | (0.30, 0.30) | 0.9444 (0.3177) | 0.9556 (0.3474) | 0.9524 (0.3464) |
| | (0.30, 0.40) | 0.9502 (0.3771) | 0.9584 (0.4158) | 0.9570 (0.4143) |
| (30, 30) | (0.05, 0.05) | 0.9476 (0.0504) | 0.9562 (0.0526) | 0.9553 (0.0525) |
| | (0.05, 0.15) | 0.9440 (0.1125) | 0.9538 (0.1178) | 0.9517 (0.1179) |
| | (0.10, 0.10) | 0.9432 (0.1009) | 0.9500 (0.1055) | 0.9496 (0.1055) |
| | (0.10, 0.20) | 0.9455 (0.1596) | 0.9544 (0.1675) | 0.9528 (0.1675) |
| | (0.20, 0.20) | 0.9453 (0.2023) | 0.9516 (0.2130) | 0.9492 (0.2127) |
| | (0.20, 0.30) | 0.9506 (0.2592) | 0.9549 (0.2746) | 0.9541 (0.2743) |
| | (0.30, 0.30) | 0.9509 (0.3061) | 0.9559 (0.3258) | 0.9503 (0.3250) |
| | (0.30, 0.40) | 0.9495 (0.3612) | 0.9517 (0.3877) | 0.9504 (0.3870) |
| (50, 50) | (0.05, 0.05) | 0.9466 (0.0391) | 0.9530 (0.0401) | 0.9521 (0.0401) |
| | (0.05, 0.15) | 0.9457 (0.0873) | 0.9511 (0.0897) | 0.9499 (0.0897) |
| | (0.10, 0.10) | 0.9465 (0.0783) | 0.9508 (0.0803) | 0.9495 (0.0803) |
| | (0.10, 0.20) | 0.9474 (0.1237) | 0.9522 (0.1272) | 0.9525 (0.1273) |
| | (0.20, 0.20) | 0.9460 (0.1567) | 0.9493 (0.1615) | 0.9477 (0.1613) |
| | (0.20, 0.30) | 0.9473 (0.2004) | 0.9509 (0.2072) | 0.9502 (0.2073) |
| | (0.30, 0.30) | 0.9545 (0.2365) | 0.9567 (0.2452) | 0.9555 (0.2451) |
| | (0.30, 0.40) | 0.9541 (0.2793) | 0.9548 (0.2911) | 0.9518 (0.2909) |
| (100, 100) | (0.05, 0.05) | 0.9490 (0.0276) | 0.9521 (0.0280) | 0.9504 (0.0280) |
| | (0.05, 0.15) | 0.9498 (0.0617) | 0.9539 (0.0626) | 0.9519 (0.0627) |
| | (0.10, 0.10) | 0.9466 (0.0553) | 0.9490 (0.0560) | 0.9488 (0.0561) |
| | (0.10, 0.20) | 0.9516 (0.0877) | 0.9550 (0.0889) | 0.9530 (0.0890) |
| | (0.20, 0.20) | 0.9513 (0.1108) | 0.9527 (0.1125) | 0.9518 (0.1439) |
| | (0.20, 0.30) | 0.9501 (0.1415) | 0.9527 (0.1438) | 0.9518 (0.1439) |
| | (0.30, 0.30) | 0.9530 (0.1666) | 0.9544 (0.1696) | 0.9520 (0.1697) |
| | (0.30, 0.40) | 0.9507 (0.1968) | 0.9522 (0.2007) | 0.9509 (0.2008) |

As seen in Table 1, the new confidence interval $CI_{lm}$ provides coverage probabilities much different from nominal confidence level 0.95 and closed to 0.95 in situations in which the sample sizes are large $(\text{i.e., } (n, m) = 30, 50, 100)$. When the values of standard deviations are high $(\text{i.e., } (\sigma_x, \sigma_y) = (0.20, 0.30), (0.30, 0.30), (0.30, 0.40))$, the coverage probabilities of $CI_{lm}$ are close to the nominal confidence level of 0.95. Between the new confidence intervals $CI_{MV}$ and $CI_{PV}$, the coverage probabilities of these intervals are not significantly different and are about 0.95. Furthermore, for $(n, m) = 10$, both new intervals are higher than 0.95. By comparing the expected lengths of two confidence intervals, $CI_{MV}$ and $CI_{PV}$, $CI_{MV}$ has slightly longer widths than that of confidence interval $CI_{PV}$ in most cases. However, it can be seen that the expected lengths of these intervals have longer widths than that of confidence interval $CI_{lm}$.

## 4. Application

In this section, we use data from "cyclic strengths compared for two sampling techniques" (see Devore [20]) to exemplify our methods for difference between inverses of normal means. Data was obtained in a study to evaluate the liquefaction potential at a proposed nuclear power station. Before cyclic strength testing, soil samples were gathered using two sampling methods, a pitcher tube method and a block method. The resulting in the following observed values of dry density (lb/ft$^3$):

| | | | | | |
|---|---|---|---|---|---|
| Pitcher sampling | 101.1 | 111.1 | 107.6 | 98.1 | 99.5 |
| | 98.7 | 103.3 | 108.9 | 109.1 | 104.1 |
| | 110.0 | 98.4 | 105.1 | 104.5 | 105.7 |
| | 103.3 | 100.3 | 102.6 | 101.7 | 105.4 |
| | 99.6 | 103.3 | 102.1 | 104.3 | |
| Block sampling | 107.1 | 105.0 | 98.0 | 97.9 | 103.3 |
| | 104.6 | 100.1 | 98.2 | 97.9 | 103.2 |
| | 96.9 | | | | |

Pitcher sampling method yields $\bar{x} = 103.6304$, $s_x = 3.8190$, while blocker sampling yields $\bar{y} = 101.1091$, $s_y = 3.6934$. The 95% confidence interval for the difference between inverses of normal means is obtained from approximation normal distribution as $(-0.00049, 1.16 \times 10^{-5})$ with the expected length equal to 0.00050. By equations (2) and (3), the two confidence limits for inverse of normal mean of two sampling methods are estimated as $(0.0095, 0.0098)$ and $(0.0096, 0.0101)$, respectively. Thus, the 95% confidence interval for the difference between inverse of normal means is obtained from the MOVER method as $(-0.00052, 3.64 \times 10^{-5})$ with the expected length equal to 0.00056. Based on GCI method, the 95% confidence interval for difference between inverse of normal means is given by $(-0.00049, 2.53 \times 10^{-5})$ with the expected length equal to 0.00052. Note that data sets from two sampling methods are tested for normality by the Kolmogorov test.

## 5. Discussion and Conclusions

This paper has proposed confidence intervals for the difference between inverse of normal means. We proposed three methods for constructing confidence intervals for the difference between inverse of normal means and apply them into a variety of situations.

The results in Table 1 show the confidence interval based on MOVER and GCI performing better than that of confidence interval based on approximation confidence interval in terms of coverage probabilities especially in situation in which sample sizes are small. As a result of the expected lengths, both approaches tend to be clearly wider as larger standard deviations and tend to be slightly narrower as larger sample sizes. In addition, confidence interval based on MOVER has slightly longer widths than that of interval based on GCI in most cases. However, confidence interval based on the MOVER method is also easy to use more than the confidence interval based on GCI which is based on a computational approach. In conclusion, we recommended that the MOVER method is considered as an alternative to construct the confidence interval for the difference between inverse means.

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