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ON A TWO TYPE DIFFERENCE SYSTEM

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Abstract

In this paper, we investigate the periodic character of the solutions of the following difference system:

$$\begin{cases} x_{n+1} = \dfrac{x_{n-1}}{y_n}, \\ y_{n+1} = \dfrac{y_{n-1}}{x_n}, \end{cases} \quad n = 0, 1, 2, \ldots.$$

1. Introduction

In [1], Amleh et al. studied the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots.$$

They used the basic theorems to study the subsequence of this equation.

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In [2], El-Owaidy et al. studied the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, 2, \ldots.$$

They investigated the periodic character and global asymptotically stable character of the equation. The conclusions in [1] are included the conclusions in [2].

In [3], Papaschinopoulos and Papadopoulos studied the existence, the boundedness and the asymptotic behavior of the positive solutions of the fuzzy equation $x_{n+1} = A + \frac{x_n}{x_{n-m}}$.

Moreover, in [4], Zhang et al. studied the following equation:

$$x_{n+1} = \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots.$$

Motivated by [1-7], we investigate the convergence of solutions of following system of difference equations:

$$\begin{cases} x_{n+1} = \dfrac{x_{n-1}}{y_n}, \\ y_{n+1} = \dfrac{y_{n-1}}{x_n}, \end{cases} \quad n = 0, 1, 2, \ldots. \tag{1.1}$$

2. Some Lemmas

Lemma 2.1. *Equations* (1.1) *possess a unique positive equilibrium* $\bar{x} = \bar{y} = 1$.

The proof being easy is left.

Lemma 2.2. *Let* $x_{-1} = p \neq 0$, $x_0 = q \neq 0$, $y_{-1} = h \neq 0$, $y_0 = k \neq 0$, *and* $\{x_n, y_n\}_{n=-1}^{\infty}$ *be a solution of* (1.1). *Then the following statements are true*:

(a) $x_{2n} = \dfrac{q^{a_n}}{h^{b_n}}$, $y_{2n} = \dfrac{k^{a_n}}{p^{b_n}}$, $n = 1, 2, \ldots,$

where

$$a_n = \frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1},$$

$$b_n = \frac{5+3\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \frac{5-3\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}.$$

(b) $x_{2n+1} = \dfrac{p^{c_n}}{k^{d_n}}, \ y_{2n+1} = \dfrac{h^{c_n}}{q^{d_n}}, \ n = 0, 1, 2, \ldots,$

where

$$c_n = \frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1},$$

$$d_n = \frac{15+7\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \frac{15-7\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}.$$

Proof. Part (a).

By equations (1.1), we have

$$x_2 = \frac{q^2}{h},\ y_2 = \frac{k^2}{p};$$

$$x_4 = \frac{q^5}{h^3},\ y_4 = \frac{k^5}{p^3};$$

$$x_6 = \frac{q^{13}}{h^8},\ y_6 = \frac{k^{13}}{p^8};$$

$$\cdots.$$

We assume that

$$x_{2n} = \frac{q^{a_n}}{h^{b_n}}, \quad y_{2n} = \frac{k^{a_n}}{p^{b_n}}, \quad n = 0, 1, 2, \ldots.$$

By induction, we have

$$\begin{cases} b_{n+1} = a_n + b_n, \\ a_{n+1} = a_n + b_{n+1} = 2a_n + b_n, \end{cases} \quad n = 1, 2, ..., \tag{2.1}$$

where $a_1 = 2,\ b_1 = 1$.

(2.1) can be written as

$$\begin{pmatrix} b_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} b_n \\ a_n \end{pmatrix},$$

i.e.,

$$Z_{n+1} = AZ_n, \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad n = 1, 2, \ldots.$$

Obviously, $A = PBP^{-1}$, where

$$P = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix},\ B = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix},\ P^{-1} = \begin{pmatrix} \frac{5-\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{5+\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \end{pmatrix}.$$

So, $Z_{n+1} = PBP^{-1}Z_n$. This equation can be changed into $P^{-1}Z_{n+1} = BP^{-1}Z_n$. Let $Y_n = P^{-1}Z_n$. Then

$$Y_{n+1} = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix} Y_n$$

and $Y_1 = P^{-1}Z_1 = P^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

By induction

$$Y_{n+1} = \begin{pmatrix} \left(\frac{3+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{3-\sqrt{5}}{1}\right)^n \end{pmatrix} Y_1.$$

Therefore,

$$Z_{n+1} = \begin{pmatrix} \frac{5+3\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^n \\ \frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n \end{pmatrix}, \quad n = 0, 1, 2, \ldots.$$

This proves part (a). Part (b) can be proved on similar lines.

Lemma 2.3. *Let* $x_{-1} = p \neq 0$, $x_0 = q \neq 0$, $y_{-1} = h \neq 0$, $y_0 = k \neq 0$. *Then the following statements are true*:

(a) *If* $h = -1$, $q = 1$, *then the solution* $\{x_{2n}, y_{2n-1}\}_{n=0}^{\infty}$ *of equations* (1.1) *is periodic with period*-3 *as follows*:

$$-1, 1, -1, 1, -1, \ldots.$$

(b) *If* $h = 1$, $q = -1$, *then the solution* $\{x_{2n}, y_{2n-1}\}_{n=0}^{\infty}$ *of equations* (1.1) *is periodic with period*-3 *as follows*:

$$1, -1, -1, 1, -1, -1, \ldots.$$

(c) *If* $h = -1$, $q = -1$, *then the solution* $\{x_{2n}, y_{2n-1}\}_{n=0}^{\infty}$ *of equations* (1.1) *is periodic with period*-3 *as follows*:

$$-1, -1, 1, -1, -1, 1, \ldots.$$

(d) *If* $p = -1$, $k = 1$, *then the solution* $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *of equations* (1.1) *is periodic with period*-3 *as follows*:

$$-1, 1, -1, -1, 1, -1, \ldots.$$

(e) *If* $p = 1$, $k = -1$, *then the solution* $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *of equations* (1.1) *is periodic with period*-3 *as follows*:

$$1, -1, -1, 1, -1, -1, \ldots.$$

(f) *If* $p = -1$, $k = 1$, *then the solution* $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *of equations* (1.1) *is periodic with period*-3 *as follows*:

$$-1,\ -1,\ 1,\ -1,\ -1,\ 1,\ \ldots.$$

The proof is easy.

3. Main Results

Theorem 3.1. *Assume that* $x_{-1} = p > 0$, $x_0 = q > 0$, $y_{-1} = h > 0$, $y_0 = k > 0$ *and* $\{x_n, y_n\}_{n=-1}^{\infty}$ *is a positive solution of equations* (1.1). *Then the following statements are true*:

(a) *If* $h = q^{\frac{1+\sqrt{5}}{2}}$, *then* $\lim_{n\to\infty} x_{2n} = 1$, $\lim_{n\to\infty} y_{2n+1} = 1$.

(b) *If* $h > q^{\frac{1+\sqrt{5}}{2}}$, *then* $\lim_{n\to\infty} x_{2n} = 0$, $\lim_{n\to\infty} y_{2n+1} = +\infty$.

(c) *If* $h < q^{\frac{1+\sqrt{5}}{2}}$, *then* $\lim_{n\to\infty} x_{2n} = +\infty$, $\lim_{n\to\infty} y_{2n+1} = 0$.

(d) *If* $p = k^{\frac{1+\sqrt{5}}{2}}$, *then* $\lim_{n\to\infty} y_{2n} = 1$, $\lim_{n\to\infty} x_{2n+1} = 1$.

(e) *If* $p > k^{\frac{1+\sqrt{5}}{2}}$, *then* $\lim_{n\to\infty} y_{2n} = 0$, $\lim_{n\to\infty} x_{2n+1} = +\infty$.

(f) *If* $p < k^{\frac{1+\sqrt{5}}{2}}$, *then* $\lim_{n\to\infty} y_{2n} = +\infty$, $\lim_{n\to\infty} x_{2n+1} = 0$.

Proof. To complete the proof, note that by Lemma 2.2,

$$x_{2n} = \frac{q^{a_n}}{h^{b_n}}$$

$$= \frac{q^{\frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}{h^{\frac{5+3\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{5-3\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}$$

$$= \left(\frac{q^{\frac{1+\sqrt{5}}{2}}}{h}\right)^{\frac{5+3\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}} q^{\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}} h^{\frac{3\sqrt{5}-5}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}},$$

$$y_{2n} = \frac{k^{a_n}}{p^{b_n}}$$

$$= \frac{k^{\frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}{p^{\frac{5+3\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{5-3\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}$$

$$= \left(\frac{k^{\frac{1+\sqrt{5}}{2}}}{p}\right)^{\frac{5+3\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}} q^{\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}} p^{\frac{3\sqrt{5}-5}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}},$$

$$x_{2n+1} = \frac{p^{c_n}}{k^{d_n}}$$

$$= \frac{p^{\frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}{k^{\frac{15+7\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{15-7\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}$$

$$= \left(\frac{p}{k^{\frac{1+\sqrt{5}}{2}}}\right)^{\frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}} q^{\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}} k^{\frac{7\sqrt{5}-15}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}},$$

$$y_{2n+1} = \frac{h^{c_n}}{q^{d_n}}$$

$$= \frac{h^{\frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}{q^{\frac{15+7\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}+\frac{15-7\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}}$$

$$= \left(\frac{h}{q^{\frac{1+\sqrt{5}}{2}}}\right)^{\frac{5+2\sqrt{5}}{5}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}} h^{\frac{5-2\sqrt{5}}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}} q^{\frac{7\sqrt{5}-15}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}.$$

Noting that $\frac{3+\sqrt{5}}{2} > 1,\ 0 < \frac{3-\sqrt{5}}{2} < 1.$

Theorem 3.2. *Assume that* $x_{-1} = p \neq 0,\ \ x_0 = q \neq 0,\ \ y_{-1} = h \neq 0,$ $y_0 = k \neq 0,$ *and* $\{x_n,\ y_n\}_{n=-1}^{\infty}$ *is a positive solution of equations* (1.1). *Then the following statements are true*:

(a) *If* $|\,h\,| = |\,q\,|^{\frac{1+\sqrt{5}}{2}},$ *and at least one of* h *and* q *is less than* 0, *then* $\{x_{2n},\ y_{2n-1}\}_{n=0}^{\infty}$ *of equations* (1.1) *converges to a period*-3 *solution of equations* (1.1) *as one of Lemma* 2.3(a)-(c), *that is*:

(i) *if* $h < 0,\ \ q > 0,\ \ \{x_{2n},\ y_{2n-1}\}_{n=0}^{\infty}$ *converges to a period*-3 *solution as* (a);

(ii) *if* $h > 0,\ \ q < 0,\ \ \{x_{2n},\ y_{2n-1}\}_{n=0}^{\infty}$ *converges to a period*-3 *solution as* (b);

(iii) *if* $h < 0,\ \ q < 0,\ \ \{x_{2n},\ y_{2n-1}\}_{n=0}^{\infty}$ *converges to a period*-3 *solution as* (c).

(b) *If* $|h| > |q|^{\frac{1+\sqrt{5}}{2}}$, *and at least one of* h *and* q *is less than* 0, *then* $\{x_{2n}, y_{2n-1}\}_{n=0}^{\infty}$ *of equations* (1.1) *has the following properties*:

$$\lim_{n\to\infty} x_{2n} = 0, \quad \lim_{n\to\infty} |y_{2n-1}| = +\infty;$$

(i) *if* $h < 0$, $q > 0$, *then*

$$\lim_{n\to\infty} y_{6n-1} = -\infty, \quad \lim_{n\to\infty} y_{6n+1} = -\infty, \quad \lim_{n\to\infty} y_{6n+3} = +\infty;$$

(ii) *if* $h > 0$, $q < 0$, *then*

$$\lim_{n\to\infty} y_{6n-1} = +\infty, \quad \lim_{n\to\infty} y_{6n+1} = -\infty, \quad \lim_{n\to\infty} y_{6n+3} = +\infty;$$

(iii) *if* $h < 0$, $q < 0$, *then*

$$\lim_{n\to\infty} y_{6n-1} = -\infty, \quad \lim_{n\to\infty} y_{6n+1} = +\infty, \quad \lim_{n\to\infty} y_{6n+3} = +\infty.$$

(c) *If* $|h| < |q|^{\frac{1+\sqrt{5}}{2}}$, *and at least one of* h *and* q *is less than* 0, *then* $\{x_{2n}, y_{2n-1}\}_{n=0}^{\infty}$ *of equations* (1.1) *has the following properties*:

$$\lim_{n\to\infty} |x_{2n}| = +\infty, \quad \lim_{n\to\infty} y_{2n-1} = 0;$$

(i) *if* $h < 0$, $q > 0$, *then*

$$\lim_{n\to\infty} x_{6n} = +\infty, \quad \lim_{n\to\infty} x_{6n+2} = -\infty, \quad \lim_{n\to\infty} x_{6n+4} = -\infty;$$

(ii) *if* $h > 0$, $q < 0$, *then*

$$\lim_{n\to\infty} x_{6n} = -\infty, \quad \lim_{n\to\infty} x_{6n+2} = +\infty, \quad \lim_{n\to\infty} x_{6n+4} = -\infty;$$

(iii) *if* $h < 0$, $q < 0$, *then*

$$\lim_{n\to\infty} x_{6n} = -\infty, \quad \lim_{n\to\infty} x_{6n+2} = -\infty, \quad \lim_{n\to\infty} x_{6n+4} = +\infty.$$

(d) *If* $|p| = |k|^{\frac{1+\sqrt{5}}{2}}$, *and at least one of* p *and* k *is less than* 0, *then* $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *of equations* (1.1) *converges to a period*-3 *solution of equations* (1.1) *as one of Lemma* 2.3(d)-(f), *that is*:

(i) *if* $p < 0$, $k > 0$, $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *converges to a period*-3 *solution as* (d);

(ii) *if* $p > 0$, $k < 0$, $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *converges to a period*-3 *solution as* (e);

(iii) *if* $p < 0$, $k < 0$, $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *converges to a period*-3 *solution as* (f).

(e) *If* $|p| > |k|^{\frac{1+\sqrt{5}}{2}}$, *and at least one of* p *and* k *is less than* 0, *then* $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *of equations* (1.1) *has the following properties*:

$$\lim_{n\to\infty} |x_{2n-1}| = +\infty, \quad \lim_{n\to\infty} y_{2n} = 0;$$

(i) *if* $p < 0$, $k > 0$, *then*

$$\lim_{n\to\infty} x_{6n-1} = -\infty, \quad \lim_{n\to\infty} x_{6n+1} = -\infty, \quad \lim_{n\to\infty} x_{6n+3} = +\infty;$$

(ii) *if* $p > 0$, $k < 0$, *then*

$$\lim_{n\to\infty} x_{6n-1} = +\infty, \quad \lim_{n\to\infty} x_{6n+1} = -\infty, \quad \lim_{n\to\infty} x_{6n+3} = -\infty;$$

(iii) *if* $p < 0$, $k < 0$, *then*

$$\lim_{n\to\infty} x_{6n-1} = -\infty, \quad \lim_{n\to\infty} x_{6n+1} = +\infty, \quad \lim_{n\to\infty} x_{6n+3} = -\infty.$$

(f) *If* $|p| < |k|^{\frac{1+\sqrt{5}}{2}}$, *and at least one of* p *and* k *is less than* 0, *then* $\{x_{2n-1}, y_{2n}\}_{n=0}^{\infty}$ *of equations* (1.1) *has the following properties*:

$$\lim_{n\to\infty} x_{2n-1} = 0, \quad \lim_{n\to\infty} |y_{2n}| = +\infty;$$

(i) *if* $p < 0,\ k > 0,$ *then*

$$\lim_{n\to\infty} y_{6n} = +\infty, \quad \lim_{n\to\infty} y_{6n+2} = -\infty, \quad \lim_{n\to\infty} y_{6n+4} = -\infty;$$

(ii) *if* $p > 0,\ k < 0,$ *then*

$$\lim_{n\to\infty} y_{6n} = -\infty, \quad \lim_{n\to\infty} y_{6n+2} = +\infty, \quad \lim_{n\to\infty} x_{6n+4} = -\infty;$$

(iii) *if* $p < 0,\ k < 0,$ *then*

$$\lim_{n\to\infty} y_{6n} = -\infty, \quad \lim_{n\to\infty} y_{6n+2} = -\infty, \quad \lim_{n\to\infty} y_{6n+4} = +\infty.$$

The result follows easily by Lemma 2.3 and Theorem 3.1.

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