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COMPLEX HARMONIC SPLINES AND APPLICATIONS

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Abstract

In this paper, we design a new family of orthogonal wavelet transforms that are based on polynomial and discrete splines. We mainly discuss and focus on the cubic spline, which is practical in applications.

1. Introduction

We define a complex harmonic spline function which provides high accuracy to the approximation as compared with other well-known methods.

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The complex spline approximations introduced are applied to the solution of differential equations.

Let Γ be the unit circle and $\Delta : z_0, z_1, ..., z_m$ be points on Γ in a counter-clockwise order. $I_{a,b}$ is called a *circular arc with endpoints* $z_a \neq z_b$ if, when a point z runs along the circle Γ from z_a to z_b counterclockwise, then z describes the arc $I_{a,b}$. If $t_1, t_2 \in I_{a,b}$, then we write $t_1 \ll t_2$ if the point z moving counterclockwise from z_a to z_b, first meets t_1, then t_2. $I_{a,b}$ cannot be a contour. We also use the notation $\gamma_j = \gamma(z_j, z_{j+1})$ for $I_{j,j+1}$ for $j = 0, 1, ..., m$, $z_{m+1} \equiv z_0$. Without loss of generality, we may assume that $1 \in \gamma_m \equiv \gamma(z_m, z_0)$.

If we transform the unit circle Γ to the whole real axis according to the formula $x = \dfrac{i(1+z)}{1-z}$, then the points $\{z_j\}_0^m$ are mapped onto $\{x_j\}_0^m : -\infty < x_0 < x_1 < x_2 < \cdots < x_m < +\infty$.

Let $\Phi_n(\Delta)$ denote the family of the complex splines of degree n with knots Δ. Then $S \in \Phi_n(\Delta)$ satisfies the conditions:

(i) $S \in \pi_n$ on γ_j, $j = 0, 1, ..., m$,

(ii) $S \in C^{n-1}(\Gamma)$,

where π_n represents the family of polynomials with complex variable $z = e^{i\theta}$ of degree n.

The real Fourier series decomposes a given function $f(x)$ which is continuous or piecewise-continuous into terms which are multiples of sin nx and cos nx. When the function $f(x)$ is decomposed into terms which are multiples of e^{inx}, this is called a *complex Fourier series*. In practice, it is so easy to work with complex Fourier series, then at the end convert into a real Fourier series. Complex numbers are basically not related with reality, they can be used to solve science and engineering problems. This can be done in two different ways: First, the parameters from a real world problem can be substituted into a complex form. The second approach is making the complex numbers mathematically equivalent to the physical problem. This approach leads to the complex Fourier transform, a more sophisticated version of the real Fourier transform. Euler's formula $e^{ix} = \cos x + i \sin x$ is the basic bridge that connects trigonometric functions by way of complex numbers. When replacing x by $-x$, we get $e^{-ix} = \cos x - i \sin x$. From these two identities, complex exponentials can always be converted to the trigonometric functions:

$$\cos x = \frac{1}{2} e^{-ix} + \frac{1}{2} e^{ix} \quad \text{and} \quad \sin x = \frac{i}{2} e^{-ix} - \frac{i}{2} e^{ix}.$$

For any nonnegative integer n, we define a circular spline function of degree n as follows:

$$N_{j,n}(z) = (z_{j+n+1} - z_j)[z_j, \ldots, z_{j+n+1}]_s (s - z)_+^n, \quad z \in I_{j,\, j+n+1}, \quad (1)$$

where $j = 0, 1, \ldots, k-1$ and the notation $[z_j, \ldots, z_{j+n+1}]f$ represents the divided difference of the $(n+1)$th order of f with respect to the points $z_j, \ldots, z_{j+n+1}$, and the definition of $(s - z)_+^l$ is:

$$(s - z)_+^l := \begin{cases} (s - z)^l, & s \gg z, \\ 0, & s \ll z \text{ or } s = z, \end{cases} \quad (2)$$

where l is any nonnegative integer. Then $N_{j,n}$ is called *B-spline* which is a polynomial complex spline of degree n and $N_{j,n} \in \Phi_n(\Delta)$, and $\{N_{j,n}\}_{j=1}^k$ forms a basis of $\Phi_n(\Delta)$. The complex B-spline is defined for

$$z^l = \sum_{j=1}^{k} z_j^{(l)} N_{j,n}(z), \quad l = 0, 1, 2, \ldots, n \quad (z \in \Gamma),$$

where $N_{j,n}(z)$ is the new blending function which now involves a new parameter n, which will be defined with the recursion relation

$$N_{j,n}(z) = \frac{z - z_j}{z_{j+n} - z_j} N_{j,n-1}(z) + \frac{z_{j+n+1} - z}{z_{j+n+1} - z_{j+1}} N_{j+1,n-1}(z), \quad z \in \Gamma.$$

The basic characteristic feature of the B-spline function is equally spaced knots, and this is referred to as uniform knot spacing. Using the binomial coefficient relations

$$\binom{n}{l} z_j^{(l)} = \sum \{z_{j+1}^{\alpha_1} \cdots z_{j+n}^{\alpha_n} : \alpha_1 + \alpha_2 + \cdots + \alpha_n = l,$$

$$\alpha_i = 0 \text{ or } 1,\ i = 1, 2, \ldots, n\}.$$

See a more detailed explanation in [1] and references therein. Using the following well-known notations and representation:

$$z_j = \omega_l^j, \quad \omega_l = \exp(ih_l), \quad h_l = \frac{2\pi}{K(l)}, \quad K(l) = 2^l K, \quad K \geq 2 + n, \quad l \in \mathbb{N},$$

where $\mathbb{N}$ is the set of nonnegative integers, we can express a circular spline function in the form

$$N_{j,n}^{(l)}(z) = (-1)^{(n+1)}(\omega_l^{j+n+1} - \omega_l^j)[\omega_l^j, \ldots, \omega_l^{j+n+1}]_s (z - s)_+^n. \tag{3}$$

The Fourier expansion of (3) can be represented as

$$N_{j,n}^{(l)}(e^{i\theta}) = K_n^{(l)} \sum_{\nu \in \mathbb{Z}} C_\nu^{(l)} C_{\nu-1}^{(l)} \cdots C_{\nu-n}^{(l)} e^{i\nu(\theta - jh_l)}, \tag{4}$$

where

$$K_n^{(l)} = \frac{n!(2\pi i)^n}{\prod_{\nu=1}^n (\omega_l^\nu - 1)}, \qquad C_j^{(l)} = \frac{1 - \omega_l^j}{2\pi ji} \ (j \neq 0),$$

$$C_0^{(l)} = \frac{1}{K(l)} \quad \text{and} \quad K_0^{(l)} = 1.$$

Every function $S(z) \in \Phi_n(\Delta)$ can be written in the form

$$S(z) = \begin{cases} P(z), & P(z) \in \pi_n, \quad z \in \gamma_0, \\ P(z) + \sum_{k=1}^{j} C_k (z - z_k)^n, & z \in \gamma_j, \quad j = 1, 2, \ldots, m, \end{cases} \tag{5}$$

where π_n is the family of polynomials of degree n with complex variable z, and $\{C_k\}$ are constants satisfying $\sum_{k=1}^{m} C_k (z_0 - z_k)^{n-l} = 0,\ l = 0, 1, \ldots, n-1.$ The system of equations [1]

$$\begin{cases} S(z_j) = f_j, & j = 0, 1, \ldots, m, \\ \sum_{k=1}^{m} C_k (z_0 - z_k)^{n-l} = 0, & l = 0, 1, \ldots, n-1 \end{cases} \tag{6}$$

is equivalent to the following system:

$$\begin{cases} t(x_j) = f_j(x_j)(x_j + i)^n, & j = 0, 1, \ldots, m, \\ \sum_{k=1}^{m} \widetilde{C}_k (x_0 - x_k)^{n-l} = 0, & l = 0, 1, \ldots, n-l, \end{cases} \tag{7}$$

where

$$t(x) = (x + i)^n S\left(\frac{x - i}{x + i}\right), \quad \widetilde{C}_k = \frac{(2i)^n C_k}{(i + x_k)^n}, \quad k = 0, 1, \ldots, m,$$

i is the imaginary unit. The real and imaginary parts of these systems have $m + n + 1$ equations each and the two systems possess the same matrix, therefore, we need to consider only one of them. Assume that

$$T_n(x_0, x_1, \ldots, x_m)$$

is a family of real spline functions and $f(x) \in T_n(x_0, x_1, \ldots, x_m)$ and $f(x)$ satisfies the Lipschitz condition $| f(u) - f(v) | \leq L| u - v |$ for all $u, v \in \mathbb{R}$, and $L > 0$ is a real number. Let $t(x) \in C^{n-1}(-\infty, +\infty)$ and $\pi_n(x)$ be a family of polynomials of degree n defined as follows [1]:

$$\begin{cases} t(x) = Q(x), \quad Q(x) \in \pi_n(x), & x \in (-\infty, +\infty), \\ t(x) = Q(x) + \sum_{k=0}^{m} C_k (x - x_k)_+^n, & x \in [x_0, x_m]. \end{cases} \tag{8}$$

Assume that $f(z)$ is continuous on Γ, and $S(z)$ is the complex cubic spline with knots $\{z_j\}$ such that $s(z_j) = f(z_j)$, $j = 1, 2, \ldots, N$. Then we have [1, 2]:

$$| s(z) - f(z) | < K(R)\omega(f, |\Delta|), \quad z \in \Gamma,$$

where $R = \dfrac{\max_j | z_{j+1} - z_j |}{\min_j | z_{j+1} - z_j |}$, and $\omega(f, |\Delta|)$ is the modulus of continuity of f on Γ, and

$$K(R) = \min(5.13R + 7.13,\ 0.07R^2 + 1.5) \ \text{ and } \ |\Delta| = \max_j | z_{j+1} - z_j |.$$

2. Main Results and Application of the Complex Spline Method for the Solution of a Differential Equation

Differential equations are one of the elegant and useful tools for a great number of dynamical systems to describe and investigate. A detailed survey, definition and historical developments and application of spline functions can be seen through the given references [3-8] and references therein. Let us consider the following initial value problem for the differential equation

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 = \varphi(0), \quad x \in [0, T], \tag{9}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, and $(x, u) \to f(x, u)$ is continuous in x, u, satisfies the Lipschitz condition $| f(x, u_1) - f(x, u_2) | \le L | u_1 - u_2 |$.

Let us consider a uniform partition defined by the knots

$$0 = x_0 < x_1 < \cdots < x_{k-1} < x_k < \cdots < x_N = T,\ x_j = jh,\ h = \frac{x_N - x_0}{N}.$$

We denote the linear spaces of the polynomial spline functions of degree m and continuity class C^{m-1} with knots x_j by S_m and on the first interval $[x_0, x_1]$ the spline component is defined by

$$s_0(x) = \varphi(0) + \frac{\varphi'(0)}{1!}x + \frac{\varphi''(0)}{2!}x^2 + \cdots + \frac{\varphi^{m-1}(0)}{(m-1)!}x^{m-1} + \frac{a_0}{m!}x^m, \quad (10)$$

where $0 \le x \le h$ and with the last coefficient undetermined. Now, to determine a_0 by requiring that s_0 should satisfy equation (9) for $x = x_1 = h$, this substitution gives the equation

$$s_0'(x_1) = f(x_1, s_0(x_1)) \quad (11)$$

to be solved for a_0. Having determined the polynomial (10) on the interval $[x_1, x_2]$, we define

$$s_0(x) = \sum_{j=0}^{m-1} \frac{s_0^{(j)}(x_1)}{j!}(x - x_1)^j + \frac{a_m}{m!}(x - x_1)^m, \quad x \in [x_1, x_2], \quad (12)$$

where $s_0^{(j)}(x_1)$, $0 \le j \le m-1$ are left-hand limits of derivatives as $x \to x_1$ of the segment of s defined in (10) on $[x_0, x_1]$ and a_1 is determined as to satisfy the equation

$$s_0'(x_2) = f(x_2, s_0(x_2)).$$

Continuing in this manner, we obtain a spline function $s_0 : [\zeta_0, \zeta_1] \to \mathbb{R}$ $(\zeta_0 = x_0, \zeta_1 = x_N)$ of degree m and class C^{m-1} which approximates the solution y of the initial value problem (IVP) and satisfies the equation

$$s_0'(x_{k-1}) = f(x_{k+1}, s_0(x_{k+1})), \quad k = 0, 1, 2, ..., N-1.$$

Consider the interval $[\zeta_j, \zeta_{j+1}]$ $(j = \overline{0, M-1})$ which is also uniformly partitioned with knots

$$x_k = x_0 + kh,\ k = 0, 1, ..., N,\ x_0 = \zeta_j,\ x_N = \zeta_{j+1},\ h = \frac{\zeta_{j+1} - \zeta_j}{N}.$$

Denoting by $s \in S_m = Q$ the complex spline function approximating the solution of the (IVP) (9), then on the interval $[x_k, x_{k+1}]$, s is defined by

$$s(x) = \sum_{i=0}^{m-1} \frac{s^{(i)}(x_k)}{i!}(x - x_k)^i + \frac{a_k}{m!}(x - x_k)^m, \quad x_k \le x \le x_{k+1}, \tag{13}$$

where $s^{(i)}(x_k)$, $0 \le i \le m-1$ are left-hand limits of the derivatives as $x \to x_k$ of the segment of s defined on $[x_{k-1}, x_k]$ and the parameter a_k is determined so that

$$s'(x_{k+1}) = f(x_{k+1}, s(x_{k+1})). \tag{14}$$

This procedure generates a polynomial spline function of degree m and class C^{m-1} over the entire interval $[\zeta_j, \zeta_{j+1}]$ with the knots $\{x_k\}$.

Theorem 1. *If h is small enough and the function f satisfies the assumed condition, then there exists a unique complex spline approximating the solution of problem* (9) *given by the above construction.*

To prove that a_k can be uniquely determined from equation (14) and replacing s determined by (13) and (14), we obtain

$$a_k = \frac{(m-1)!}{h^{m-1}}\left[f\left(x_{k+1}, s(x_k) + \frac{h}{1!}s'(x_k) + \cdots + \frac{h^{m-1}}{(m-1)!}s^{m-1}(x_k) + \frac{h^m}{m!}\right)\right.$$

$$\left. - s'(x_k) - \cdots - \frac{s^{(m-1)}(x_k)}{(m-1)!}h^{m-2}\right]. \tag{15}$$

If $h < \frac{m}{2L}$, then the right-hand side of (15) is a contraction mapping. The proof of the theorem is similar as in [6].

Remarks. The proposed procedure can be applied to nonlinear differential equations and systems of differential equations in a similar way. For instance, for differential equations with a deviating argument

$$y'(x) = f(x, y(x), y(g(x))), \quad x \in [0, T],$$

$$y(x) = \varphi(x), \quad x \in [\alpha, 0], \quad \alpha < 0, \tag{16}$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R},\ (x, u, v) \to f(x, u, v)$ is continuous in x, u, v, satisfies the Lipschitz condition

$$| f(x, u_1, v_1) - f(x, u_2, v_2) | \leq (| u_1 - u_2 | + | v_1 - v_2 |)$$

and $g \in C[0, T],\ g(x) \leq x - a,\ a > 0,\ x \in [0, T]$.

Because of the physical nature and functional purposes, the wide practical applications of impulsive differential equations explain the growing interest in technical problems. Impulsive differential equations have appeared in mathematical models of many processes in biology, medicine, and technology. The study of dynamical systems with impulse effects has assumed importance in view of the fact that many evolutionary processes experience an abrupt change of state at certain moments of time.

Let us denote the sets of real and integer numbers with $\mathbb{R}, \mathbb{Z}$, respectively, and let $X \in \mathbb{R}$ and $T = \{t_k \,|\, k \in \mathbb{Z}\} \subset \mathbb{R}$, where $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$, $t_k \to +\infty$ when $k \to +\infty$ and $t_k \to -\infty$ when $k \to -\infty$ and $t_k^+ = t_k + 0,\ t_k^- = t_k - 0$.

Assume that $\Omega \subset \mathbb{R}$ is any real interval containing t_0 and

$$\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$$

is a vector of the state function, $\boldsymbol{f}(t, x) : \Omega \times X \to X$ and

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}), \quad t \neq t_k,$$

$$\Delta \boldsymbol{x}|_{t=t_k} = \boldsymbol{x}(t_k^+) - \boldsymbol{x}(t_k^-) = I_k(\boldsymbol{x}(t_k)),\ t = t_k, \text{ where } I_k : X \to X. \tag{17}$$

Let $\boldsymbol{x}(t)$ be a solution of the system (1) which satisfies the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Then the solution can be represented in the form

$$\boldsymbol{x}(t) = \begin{cases} \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}(s))ds + \sum\limits_{t_0 < t_k < t} I_k(\boldsymbol{x}(t_k)), & t \in \Omega^+, \\ \\ \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}(s))ds - \sum\limits_{t < t_k < t_0} I_k(\boldsymbol{x}(t_k)), & t \in \Omega^-, \end{cases} \tag{18}$$

where Ω^+ and Ω^- are the maximal intervals on which the solution can be continued to the right and to the left of the point t_0, respectively. Let $t_q > t_0$, i.e., $t_q \in \Omega^+$. Then an approximate solution $[x_1(t_q), x_2(t_q), ..., x_n(t_q)]^T$ of the system (17) can be obtained using the above algorithm for the solution of the impulsive differential equations.

References

[1] Han-Lin Chen, Complex Harmonic Splines, Periodic Quasi-wavelets, Theory and Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[2] Han-Lin Chen and T. Hvaring, Approximation of complex harmonic functions by complex harmonic splines, Math. Comp. 42 (1984), 151-164.

[3] J. H. Ahlberg, E. N. Nilson and J. L. Walsh, The Theory of Splines and their Applications, Academic Press, New York, 1967.

[4] H. Akça and G. Micula, Numerical solutions of systems of differential equations by spline functions, Babeş-Bolyai University Research Seminar, 1990, pp. 93-104.

[5] C. De Boor, A Practical Guide to Splines, Springer-Verlag, New York, 1978.

[6] G. Micula and H. Akça, Numerical solution of differential equations with deviating argument using spline functions, Studia Univ. Babeş-Bolyai Math. 33(2) (1988), 45-57.

[7] G. Micula and H. Akça, Approximate solutions of the second order differential equations with deviating argument by spline functions, Mathematica (Cluj) 30(53)(1) (1988), 37-46.

[8] G. Micula and S. Micula, Handbook of Spline, Kluwer Academic Publishers, Dordrecht, London, Boston, 1999.