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APPLICATION OF VARIATIONAL ITERATION METHOD TO THE SOLUTION OF CONVECTION-DIFFUSION EQUATION

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Abstract

In this paper, an algorithm is constructed based on variational iteration method (VIM) to solve convection-diffusion equation. The algorithm converges faster and has proved elegant. Numerical examples are presented to show the efficiency of the method.

1. Introduction

Convection-diffusion equation describes the physical phenomenon where particles, energy and other physical quantities are transferred inside a system

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due to diffusion or convection. This equation is of the form

$$\frac{\partial u}{\partial t} + \varepsilon \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq l, \quad t \geq 0, \tag{1}$$

subject to the initial condition $u(x, 0) = g(x), \quad 0 \leq x \leq 1$ and boundary conditions $u(0, t) = 0, \quad t \geq 0, \quad u(l, t) = 0, \quad t \geq 0,$ where the parameter γ is the viscosity coefficient and ε is the phase speed and both are assumed to be positive. g is a given function of sufficient smoothness.

In [1, 2], Adomian decomposition method was used to solve convection-diffusion (CD) equation, in [3, 4], He's homotopy perturbation method was used and in [5], homotopy analysis method was used to solve convection-diffusion equations. In this paper, the equation was solved by variational iteration method [6-11]. To illustrate the efficiency, applicability and reliability of the method, some examples are presented.

2. Variational Iteration Method

The basic idea of the He's variational iteration method (VIM) [6-11] can be explained by considering the following nonlinear partial differential equations:

$$Lu + Nu = g(x), \tag{2}$$

where L is the linear operator, N is the nonlinear operator and $g(x)$ is the inhomogeneous term. According to the method, we can construct a correction functional as follows.

The corresponding variational iteration method for solving (2) is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Lu_n(s) + N\bar{u}_n(s) - g(s)]ds, \tag{3}$$

where λ is a Lagrange multiplier which can be identified optimally by variational iteration method. The subscript n denotes the nth approximation, $\bar{u}_n$ is considered as a restricted variation, i.e., $\delta\bar{u}_n = 0$. The successive approximation u_{n+1}, $n \geq 0$ of the solution u can be easily obtained by determining the Lagrange multiplier and the initial guess u_0, thus the solution is given by $u = \lim_{n \to \infty} u_n$. $\lambda = -1$ for the problems under consideration.

3. Numerical Examples

In this section, examples of convection-diffusion equation are given and the results will be compared with the exact solutions. Three examples are solved with the VIM algorithm and the results have been generated by Maple 18.

Example 3.1. Consider the CD equation in [5]:

$$u_t - 0.02u_{xx} + 0.1u_x = 0 \tag{4}$$

with the initial condition $u(x, 0) = e^{1.1771243444\ \ 6770x}$. The exact solution of this equation is $u(x, t) = e^{1.1771243444\ \ 6770x - 0.09t}$.

Applying (3), we obtain the following:

$$u_1(x, t) = e^{1.177124344x} - 0.08999999998e^{1.177124344x}t, \tag{5}$$

$$u_2(x, t) = e^{1.177124344x} - 0.0899999999e^{1.177124344x}t$$

$$+ 0.004049999997e^{1.177124344x}t^2, \tag{6}$$

$$u_3(x, t) = e^{1.77124344x} - 0.08999999998e^{1.177124344x}t$$

$$+ 0.004049999997e^{1.177124344x}t^2$$

$$- 0.0001214999999e^{1.177124344x}t^3, \tag{7}$$

$$u_4(x, t) = e^{1.177124344x} - 0.08999999998e^{1.177124344x}t$$

$$+ 0.004049999997e^{1.17724344x}t^2$$

$$- 0.0001214999999\ell^{1.177124344x}t^3$$

$$+ 0.000002733749998e^{1.177124344x}t^4, \tag{8}$$

$$u_5(x, t) = e^{1.177124344x} - 0.08999999998e^{1.177124344x}t$$
$$+ 0.004049999997e^{1.177124344x}t^2$$
$$- 0.0001214999999e^{1.177124344x}t^3$$
$$+ 0.000002733749998e^{1.177124344x}t^4$$
$$- 4.92074999410^{-8}e^{1.177124344x}t^5. \quad (9)$$

Table 1 shows the errors index of the approximate solution at different points (x, t). Also, the graphs of u(exact) and u(approx.) are shown in Figure 1 and Figure 2 when $t = 0.1$ and $t = 1$, respectively. Figure 3 and Figure 4 show the 3-D graphs of u(exact) and u(approx.), respectively.

Table 1. The errors index of the approximate solution at the points (x, t), $x = 1, 2, 3, ..., 10, t = 0.1$ for Example 3.1

x	0	1	2	3	4	5	6	7	8	9	10
Error	0	0	10^{-8}	0	0	0	10^{-6}	0	0	0	10^{-4}

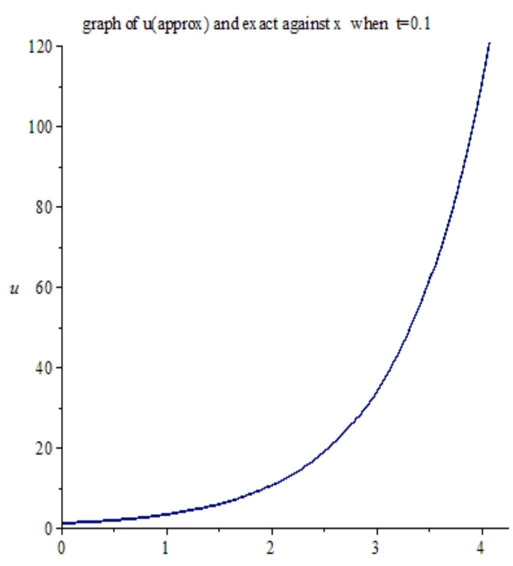

Figure 1. u(exact) and u(approx.) when $t = 0.1$.

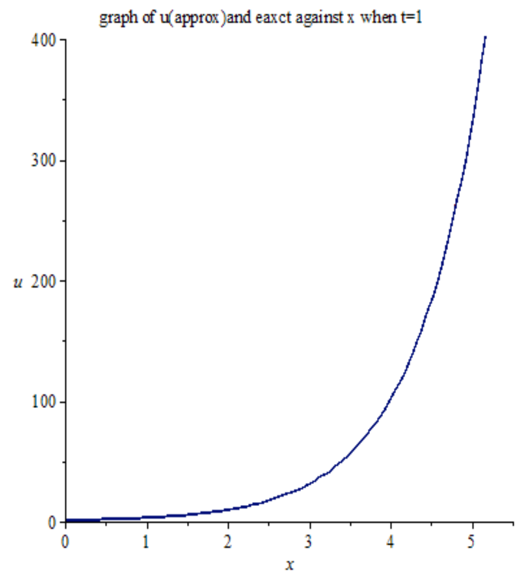

Figure 2. u(exact) and u(approx.) when $t = 1$.

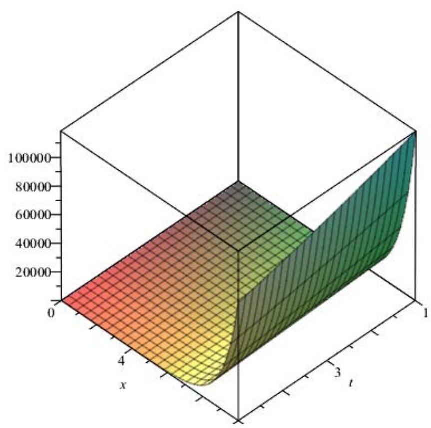

Figure 3. 3-D graph of u(exact).

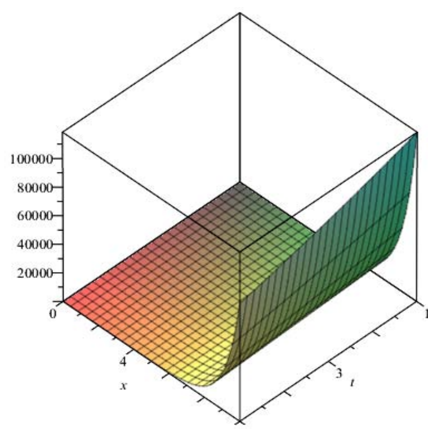

Figure 4. 3-D graph of u(approx.).

Example 3.2. Consider the CD equation [4, 5]:

$$u_t + 0.22u_{xx} - 0.5u_x = 0 \tag{10}$$

with initial condition $u(x, 0) = e^{0.22x} \sin(\pi x)$. The exact solution is

$$u(x, t) = e^{0.22x-(0.024+0.5\pi^2)t} \sin(\pi x).$$

Applying (3), we obtain the following:

$$
\begin{aligned}
u_1(x, t) = {} & e^{0.22x} \sin(3.141592652x) \\
& + 2.270664968e^{0.2200000000x} \sin(3.141592654x)t \\
& + 1.266690158e^{0.2200000000x} \cos(3.141592654x)t, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
u_2(x, t) = {} & e^{0.22x} \sin(3.141592652x) \\
& + 2.270664969e^{0.2200000000x} \sin(3.141592654x)t \\
& + 1.266690158e^{0.2200000000x} \cos(3.141592654x)t \\
& + 1.775707721t^2 e^{0.2200000000x} \sin(3.141592654x) \\
& + 2.876228968t^2 e^{0.2200000000x} \cos(3.141592654), \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
u_3(x, t) = {} & e^{0.22x} \sin(3.141592654x) \\
& + 2.270664969e^{0.2200000000x} \sin(3.141592654)t \\
& + 1.266690158e^{0.2200000000x} \cos(3.141592654x)t \\
& + 1.775707722t^2 e^{0.2200000000x} \sin(3.141592654x) \\
& + 2.876228968t^2 e^{0.2200000000x} \cos(3.141592654x) \\
& + 0.1295821303t^3 e^{0.2200000000x} \sin(3.141592654x) \\
& + 2.926741284t^3 e^{0.2200000000x} \cos(3.141592654x), \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
u_4(x, t) = {} & e^{0.22x} \sin(3.141592654x) \\
& + 2.270664969e^{0.2200000000x} \sin(3.141592654)t \\
& + 1.266690158e^{0.2200000000x} \cos(3.141592654x)t \\
& + 1.775707722t^2 e^{0.2200000000x} \sin(3.141592654x)
\end{aligned}
$$

$$+\, 2.876228968t^2 e^{0.2200000000x} \cos(3.141592654x)$$

$$+\, 0.1295821303t^3 e^{0.2200000000x} \sin(3.141592654x)$$

$$+\, 2.926741284t^3 e^{0.2200000000x} \cos(3.141592654x)$$

$$-\, 0.8532591938t^4 e^{0.2200000000x} \sin(3.141592654x)$$

$$+\, 1.702447328t^4 e^{0.2200000000x} \cos(3.141592654x), \qquad (14)$$

$$u_5(x,\, t) = e^{0.22x} \sin(3.141592654x)$$

$$+\, 2.270664969e^{0.2200000000x} \sin(3.141592654)t$$

$$+\, 1.266690158e^{0.2200000000x} \cos(3.141592654x)t$$

$$+\, 1.775707722t^2 e^{0.2200000000x} \sin(3.141592654x)$$

$$+\, 2.876228968t^2 e^{0.2200000000x} \cos(3.141592654x)$$

$$+\, 0.1295821303t^3 e^{0.2200000000x} \sin(3.141592654x)$$

$$+\, 2.926741284t^3 e^{0.2200000000x} \cos(3.141592654x)$$

$$-\, 0.8532591938t^4 e^{0.2200000000x} \sin(3.141596254x)$$

$$+\, 1.702447328t^4 e^{0.2200000000x} \cos(3.141592654x)$$

$$-\, 0.8187878072t^5 e^{0.2200000000x} \sin(3.141592654x)$$

$$+\, 0.5569744970t^5 e^{0.2200000000x} \cos(3.141592654x). \qquad (15)$$

Table 2. The errors index of the approximate solution at the points $(x,\, t)$, $x = 1,\, 2,\, 3,\, \ldots,\, 10,\; t = 0.1$ for Example 3.2

x	0	1	2	3	4	5	6	7	8	9	10
Error	0	10^{-12}	10^{-12}	0	0	0	10^{-6}	0	0	0	10^{-9}

The graphs of u(exact) and u(approx.) are shown in Figure 5 and Figure 6 when $t = 0.1$ and $t = 1$, respectively. Figure 7 and Figure 8 show the 3-D graphs of u(exact) and u(approx.), respectively.

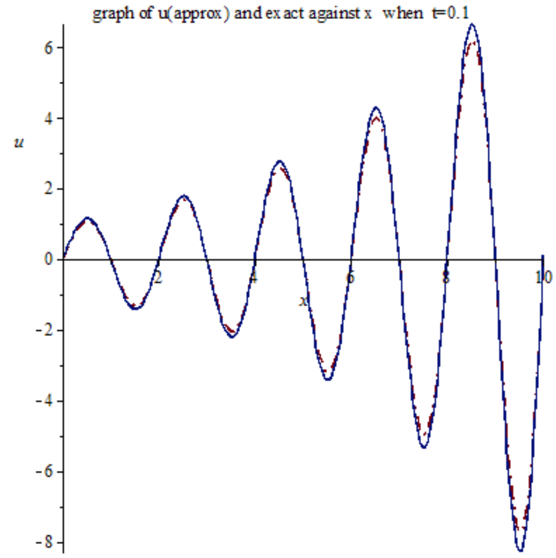

Figure 5. u(exact) and u(approx.) when $t = 0.1$.

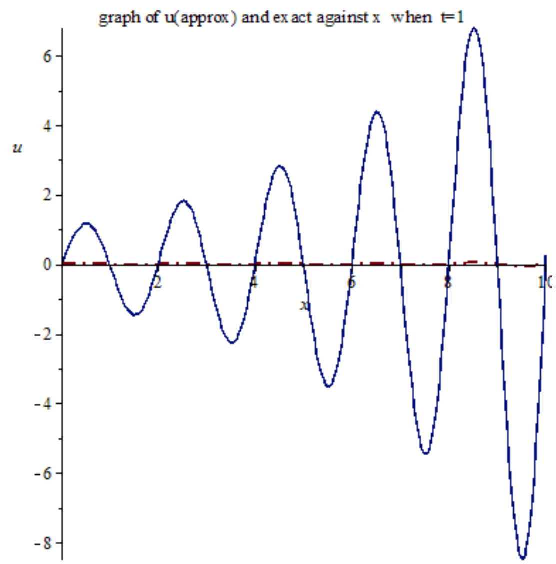

Figure 6. u(exact) and u(approx.) when $t = 1$.

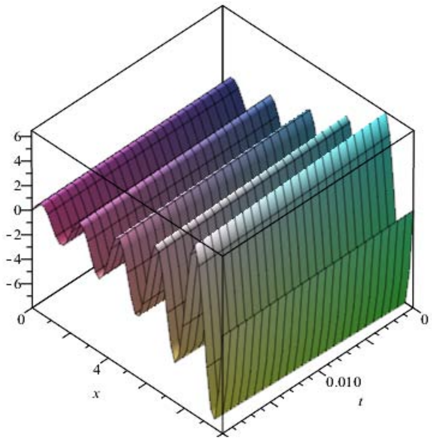

Figure 7. 3-D graph of u(exact).

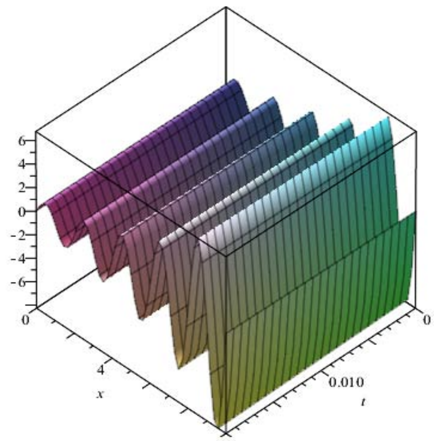

Figure 8. 3-D graph of u(approx.).

Example 3.3. Consider the CD equation [4, 5]:

$$u_t - 0.2u_{xx} + 0.1u_x = 0, \quad 0 \le x \le 1, \quad t \ge 0 \tag{16}$$

with initial condition $u(x, 0) = e^{0.25x}\sin(\pi x)$. The exact solution of the problem is

$$u(x, t) = e^{0.25x-(0.0125+0.2\pi^2)t}\sin(\pi x).$$

Applying (3), we obtain the following:

$$\begin{aligned} u_1(x, t) = {} & e^{0.25x}\sin(3.141592654x) \\ & - 0.2211420880e^{0.2500000000x}\sin(3.141592654x)t \\ & - 0.2827433388e^{0.2500000000x}\cos(3.141592654x)t, \end{aligned} \tag{17}$$

$$\begin{aligned} u_2(x, t) = {} & e^{0.25x}\sin(3.141592654x) \\ & - 0.2211420880e^{0.2500000000x}\sin(3.141592654x)t \\ & - 0.2827433388e^{0.2500000000x}\cos(3.141592654x)t \end{aligned}$$

$$- 0.01551998626t^2e^{0.2500000000x}\sin(3.141592654x)$$

$$+ 0.06252645235t^2e^{0.2500000000x}\cos(3.141592654x), \tag{18}$$

$$u_3(x, t) = e^{0.25x}\sin(3.141592654x)$$

$$- 0.2211420880e^{0.2500000000x}\sin(3.141592654x)t$$

$$- 0.2827433388e^{0.2500000000x}\cos(3.141592654x)t$$

$$- 0.01551998626t^2e^{0.2500000000x}\sin(3.141592654x)$$

$$+ 0.06252645235t^2e^{0.2500000000x}\cos(3.141592654x)$$

$$+ 0.007037020026t^3e^{0.2500000000x}\sin(3.141592654x)$$

$$- 0.003146352499t^3e^{0.2500000000x}\cos(3.141592654x), \tag{19}$$

$$u_4(x, t) = e^{0.25x}\sin(3.141592654x)$$

$$- 0.2211420880e^{0.2500000000x}\sin(3.141592654x)t$$

$$- 0.2827433388e^{0.2500000000x}\cos(3.141592654x)t$$

$$- 0.01551998626t^2e^{0.2500000000x}\sin(3.141592654x)$$

$$+ 0.06252645235t^2e^{0.2500000000x}\cos(3.141592654x)$$

$$+ 0.007037020026t^3e^{0.2500000000x}\sin(3.141592654x)$$

$$- 0.003146352499t^3e^{0.2500000000x}\cos(3.141592654x)$$

$$- 0.0006114478782t^4e^{0.2500000000x}\sin(3.141592654x)$$

$$- 0.0003234698942t^4e^{0.2500000000x}\cos(3.141592654x), \tag{20}$$

$$
\begin{aligned}
u_5(x, t) = {} & e^{0.25x} \sin(3.141592654x) \\
& - 0.2211420880e^{0.2500000000x} \sin(3.141592654x)t \\
& - 0.2827433388e^{0.2500000000x} \cos(3.141592654x)t \\
& - 0.01551998626t^2 e^{0.2500000000x} \sin(3.141592654x) \\
& + 0.06252645235t^2 e^{0.2500000000x} \cos(3.141592654x) \\
& + 0.007037020026t^3 e^{0.2500000000x} \sin(3.141592654x) \\
& - 0.003146352499t^3 e^{0.2500000000x} \cos(3.141592654x) \\
& - 0.0006114478782t^4 e^{0.2500000000x} \sin(3.141592654x) \\
& - 0.0003234698942t^4 e^{0.2500000000x} \cos(3.141592654x) \\
& + 0.000008751580532t^5 e^{0.2500000000x} \sin(3.141592654x) \\
& + 0.00004888312448t^5 e^{0.2500000000x} \cos(3.141592654x). \quad (21)
\end{aligned}
$$

Table 3. The errors index of the approximate solution at the points (x, t), $x = 1, 2, 3, \ldots, 10$, $t = 0.1$ for Example 3.3

x	0	1	2	3	4	5	6	7	8	9	10
Error	10^{-15}	10^{-10}	10^{-9}	10^{-9}	10^{-8}	10^{-9}	10^{-9}	10^{-8}	10^{-9}	10^{-8}	10^{-8}

The graphs of u(exact) and u(approx.) is shown in Figure 9 and Figure 10 when $t = 0.1$ and $t = 1$, respectively. Figure 11 and Figure 12 show the 3-D graphs of u(exact) and u(approx.), respectively.

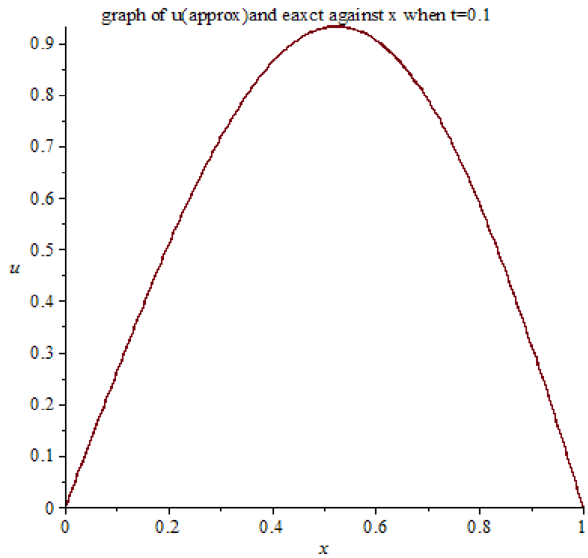

Figure 9. u(exact) and u(approx.) when $t = 0.1$.

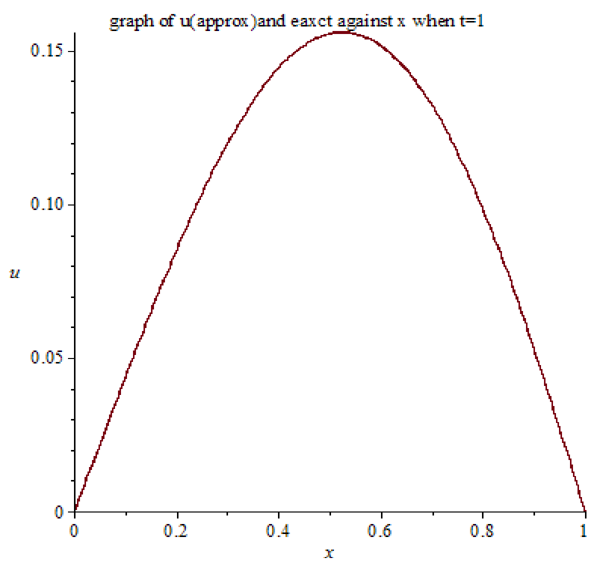

Figure 10. u(exact) and u(approx.) when $t = 1$.

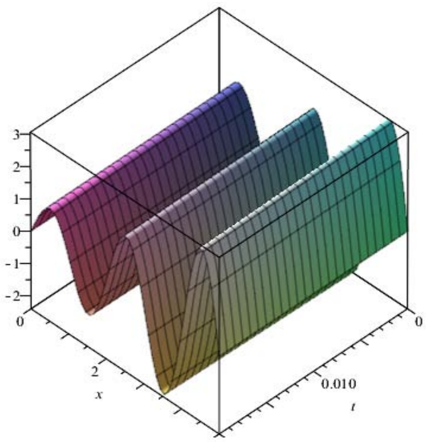

Figure 11. 3-D graph of u(exact).

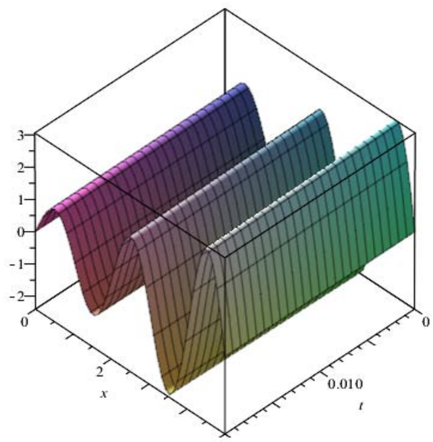

Figure 12. 3-D graph of u(approx.).

4. Conclusion

In this paper, VIM was used for solving the convection-diffusion equations. The result obtained in comparison with the exact solution admits a remarkable efficiency. The computations associated with the examples in the paper were performed using Maple 18.

Tables 1-3 and Figures 1-12 justify that the method is reliable and can be applied to nonlinear convection-diffusion equations of different parameters.

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