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NUMERICAL SOLUTION OF THE INVERSE PROBLEM OF FILTRATION THEORY BY MODULATING FUNCTIONS

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Abstract

One of the important tasks in the oil and gas sector is the problem of determining the parameters of the formation of the observed values of pressure, saturation, and others with monitoring wells. This paper deals with the numerical solution of the inverse problem of determining the permeability of the porous medium equation for unsteady filtration of a homogeneous liquid in an elastic inhomogeneous porous medium. The idea of using the method of modulating functions for the solution of inverse problems goes back to Loeb and Cahen [1, 2].

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In mathematical physics, various *direct problems* are usually considered: given the differential equation and the initial and (or) the boundary conditions to be satisfied by the solution of a differential equation. Setting each of the direct problems assumes a certain number of job functions. Some of these functions are defined by differential equations, for example, the coefficients of the equation, the other parts are defined by the boundary conditions. As a result of the direct problem solution, the given set of functions is associated with a new feature - the direct problem solution. Let some of those functions which are usually set in the direct problem be unknown, though, their finding is of great interest, but instead, some *additional information* about the solution of the direct problem is given. Such problems are called *inverse problems* of mathematical physics. Additional information about the solution of the direct problem (or the solution of a series of direct problems) can be set in a different form. This may be the solution given on some set or the integral characteristics of the solution.

One of the important problems in the oil and gas sector is the problem of determining the parameters of the stratum of the observed values of pressure, saturation, and others with monitoring wells. The main characteristics of an oil reservoir are filtration and capacitive parameters. The idea of using the method of modulating functions for the solution of inverse problems goes back to Loeb and Cahen [1, 2].

The main methods of determining the parameters of systems (such as regression analysis, variation methods, deterministic methods of moments and others), described by differential equations, are based on the solutions of the equations. The weakness of the existing methods for determining the filtration parameters should include the following states:

- Low accuracy of the defined parameters.

- The difficulty of processing the experimental data.

- The impossibility of interpreting difficult cases of filtration movement.

- A large number of design models and formulas.

- Formulas are difficult and not universal, etc.

The reason for all the shortcomings is the use of the solutions of the direct problems.

The basic idea of *modulating functions* (*M*-method) is that the filtration equations are replaced with an integral analog that makes up the algebraic or integral equations for the unknown parameters. Note that the resulting integral expressions are not derivatives of experimental functions. And as we know, the operation of differentiation of experimental functions is incorrect.

Consider the equation of unsteady filtration elastic homogeneous fluid in an inhomogeneous elastic porous medium [3]:

$$\frac{\partial}{\partial x}\left(k(x,\ y)\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(k(x,\ y)\frac{\partial p}{\partial y}\right)=\beta'\mu m\frac{\partial p}{\partial t},$$

$$(x,\ y)\in Q\subset R^2,\quad t\in(0,\ T), \tag{1}$$

where

$p(x,\ y,\ t)$ - the reservoir pressure,

$k(x,\ y)$ - the permeability of the porous medium,

m - the porosity of the medium,

μ - the absolute viscosity of the fluid,

β_m - elastic coefficient of volume expansion of the liquid,

β_c - coefficient of compressibility of the porous medium,

$$\beta'=\beta_m+\frac{\beta_c}{m}, \tag{2}$$

with initial and boundary conditions

$$p(x,\ y,\ 0)=p_0(x,\ y),\quad (x,\ y)\in Q, \tag{3}$$

$$Lp|_{\Gamma}=p_1(x,\ y,\ t),\quad (x,\ y)\in\Gamma,\quad t\in[0,\ T], \tag{4}$$

where L is the operator corresponding to the boundary conditions (for example, L is the identity operator in the case of the first boundary problem;

in the case of the second boundary value problem $L = \frac{\partial}{\partial n}$ - the derivative along the outer normal to the boundary of the region, etc.).

The inverse problem is to find the coefficient $k(x, y)$ in equation (1) with known functions m, μ, β'. We expand the function $k(x, y)$ by the Maclaurin formula

$$k(x, y) = k_0 + k_1 \cdot x + k_2 \cdot y + k_3 \cdot xy + k_4 \cdot x^2 + k_5 \cdot y^2 + \cdots. \tag{5}$$

After substituting (5) into (1), we have

$$\begin{aligned} &\frac{\partial}{\partial x}[k_0 p'_x + k_1 \cdot x \cdot p'_x + k_2 \cdot y \cdot p'_x + k_3 \cdot xy \cdot p'_x + k_4 \cdot x^2 \cdot p'_x + \cdots] \\ &+ \frac{\partial}{\partial y}[k_0 \cdot p'_y + k_1 \cdot x \cdot p'_y + k_2 \cdot y \cdot p'_y + k_3 \cdot xy \cdot p'_y \\ &\qquad + k_4 \cdot x^2 \cdot p'_y + k_5 \cdot y^2 \cdot p'_y + \cdots] \\ &= \beta' m\mu \frac{\partial p}{\partial t}. \end{aligned} \tag{6}$$

Multiply both sides of (6) by the smooth functions $\varphi_i(x)$, $\varphi_i(y)$, $\varphi_i(t)$, $i = 0, 1, \ldots, n$ (where $\varphi_i(x)$, $\varphi_i(y)$ are functions of class C^2, $\varphi_i(t)$ are functions of class C^1) and integrate, respectively, over $x_0 \le x \le x_1$, $y_0 \le y \le y_1$, $t_0 \le t \le t_1$. For our case, $t_0 = 0$. Then

$$\begin{aligned} &\int_{t_0}^{t_1}\int_{y_0}^{y_1} \varphi_i(t)\varphi_i(y)dydt\Bigg[k_0 \cdot \int_{x_0}^{x_1} p''_{xx} \cdot \varphi_i(x)dx + k_1 \cdot \int_{x_0}^{x_1} (x \cdot p'_x)'_x \\ &\qquad \cdot \varphi_i(x)dx + k_2 \cdot \int_{x_0}^{x_1} yp^2_{xx} \cdot \varphi_i(x)dx + \cdots\Bigg] \end{aligned}$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\varphi_i(t)\cdot\varphi_i(x)dxdt$$

$$\cdot\Bigg[k_0\int_{y_0}^{y_1}p_{yy}^{//}\cdot\varphi_i(y)dy+k_1$$

$$\cdot\int_{y_0}^{y_1}x\cdot p_{yy}^{//}\cdot\varphi_i(y)dy+k_2\cdot\int_{y_0}^{y_1}(y\cdot p_y')'_y\cdot\varphi_i(y)dy+\cdots\Bigg]$$

$$=\beta' m\mu\int_{y_0}^{y_1}\int_{x_0}^{x_1}\varphi_i(y)\cdot\varphi_i(x)dxdy\int_{t_0}^{t_1}p_t'\cdot\varphi_i(t)dt,\quad i=\overline{0,n}.$$

Choosing modulating functions $\varphi_i(x)$, $\varphi_i(y)$, $\varphi_i(t)$ satisfying the conditions $\varphi_i(x_0)=\varphi_i(x_1)=0$, $\varphi_i(y_0)=\varphi_i(y_1)=0$, $\varphi_i(t_0)=\varphi_i(t_1)=0$, $\varphi_i'(x_0)=\varphi_i'(x_1)=0$, $\varphi_i'(y_0)=\varphi_i'(y_1)=0$, $\overline{i=0,n}$ and applying the formula for integration by parts twice in the integrals in which P_{xx}'', P_{yy}'' are present and once to integrals, in which there are P_x', P_y', P_t', we obtain

$$\int_{t_0}^{t_1}\int_{y_0}^{y_1}\varphi_i(t)\varphi_i(y)dydt\cdot\Bigg[k_0\cdot\int_{x_0}^{x_1}p\cdot\varphi_i''(x)dx+k_1\int_{x_0}^{x_1}p(x\varphi_i'\cdot(x))'dx$$

$$+k_2\cdot\int_{x_0}^{x_1}yp\cdot\varphi_i''(x)dx+\cdots\Bigg]$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\varphi_i(t)\cdot\varphi_i(x)dxdt$$

$$\cdot\Bigg[k_0\cdot\int_{y_0}^{y_1}p\varphi_i''(y)dy$$

$$+k_1\int_{y_0}^{y_1}xp\varphi_i''(y)dy+k_2\int_{y_0}^{y_1}p(y\cdot\varphi_i'(y))'dy+\cdots\Bigg]$$

$$=-\beta' m\mu\int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1}p\varphi_i(x)\varphi_i(y)\varphi_i'(t)dxdydt,\quad\overline{i=0,n}.\tag{7}$$

The system of equations (7) can be rewritten as:

$$k_0\int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p[\varphi_i''(x)\varphi_i(y)+\varphi_i''(y)\cdot\varphi_i(x)]\varphi_i(t)dxdydt$$

$$+\,k_1\int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p[(x\varphi_i'(x))'\cdot\varphi_i(y)+x\varphi_i''(y)\varphi_i(x)]\varphi_i(t)dxdydt$$

$$+\,k_2\int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p[(y\cdot\varphi_i'(y))'\varphi_i(x)+y\cdot\varphi_i''(x)\cdot\varphi_i(y)]\varphi_i(t)dxdydt+\cdots$$

$$=-\beta' m\mu\int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p\varphi_i(x)\varphi_i(y)\varphi_i'(t)dxdydt,\quad i=\overline{0,\,n}. \tag{8}$$

To determine the coefficient $k(x,\,y)$, we need to find $k_0,\,k_1,\,...,\,k_n,\,...$ from the system of linear algebraic equations (8).

As modulating functions, we can use the functions $\varphi(x)=\alpha(x)(x-x_0)\cdot(x_1-x)$ in the equations of the form where there are derivatives of first order and $\varphi(x)=\alpha(x)(x-x_0)^2(x_1-x)^2$ in the equations with derivatives of second order, where $\alpha(x)$ is some weighting function. The factor $\alpha(x)$ provides the necessary form of modulating functions to $[x_0,\,x_1]$.

The Algorithm of the Numerical Solution of the Inverse Problem of Filtration Theory by Modulating Functions

In the expansion (5) considering the first $(n+1)$ summands, we write a system of linear algebraic equations (8) in a matrix form

$$A\cdot K=B, \tag{9}$$

where the elements of the matrix $A=(a_{ij})$ are known integrals of the unknowns $k_0,\,k_1,\,...,\,k_n,$ for example,

$$a_{i0}=\int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p[\varphi_i''(x)\varphi_i(y)+\varphi_i''(y)\cdot\varphi_i(x)]\varphi_i(t)dxdydt,$$

$$a_{i1} = \int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p[(x\varphi_i'(x))' \cdot \varphi_i(y) + x\varphi_i''(y)\varphi_i(x)]\varphi_i(t)dxdydt,$$

$$a_{i2} = \int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p[(y \cdot \varphi_i'(y))' \varphi_i(x) + y \cdot \varphi_i''(x) \cdot \varphi_i(y)]\varphi_i(t)dxdydt, \ldots,$$

$$i = \overline{0, n}, \quad (10)$$

$$B = (b_0, b_1, \ldots, b_n)^T,$$

$$b_i = -\beta' m\mu \int_{t_0}^{t_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} p\varphi_i(x)\varphi_i(y)\varphi_i'(t)dxdydt, \quad i = \overline{0, n}, \quad (11)$$

$K = (k_0, k_1, \ldots, k_n)^T$ is the vector of the unknowns k_i, $i = \overline{0, n}$.

To uniquely determine the $(n+1)$ unknown k_i, $i = \overline{0, n}$, we have $(n+1)$ equations, and the determinant of the linear algebraic equations (9) must not be equal to zero.

For this, select the modulating functions as follows:

$$\varphi_i(x) = x^i(x - x_0)^2(x_1 - x)^2, \quad x \in [x_0, x_1],$$

$$\varphi_i(y) = y^i(y - y_0)^2(y_1 - y)^2, \quad y \in [y_0, y_1],$$

$$\varphi_i(t) = t^i(t - t_0)(t_1 - t), \quad t \in [t_0, t_1], \quad i = \overline{0, n}. \quad (12)$$

From equations (9)-(12), it is clear that the question of the numerical realization of the problem comes down to the question of the numerical integration of triple integrals. Then we obtain the following *algorithm* for the numerical solution of the inverse problem of filtration theory:

1. Reception and processing of input data. To solve the problem, we first need to have experimental data on $p(x, y, t)$ values at the nodes of a cubic lattice. In the absence of data in all the lattice points, the value of $p(x, y, t)$ can be interpolated.

2. Selection of the modulating functions $\varphi_i(x)$, $\varphi_i(y)$, $\varphi_i(t)$, satisfying conditions (12). It is convenient to present the modulating functions as follows:

$$\varphi_j(x) = x^j \sum_{v=0}^{4} \omega_v^x x^v, \quad \varphi_j(y) = y^j \sum_{v=0}^{4} \omega_v^y y^v,$$

$$\varphi_j(t) = t^j \sum_{v=0}^{2} \omega_v^t t^v, \quad j = 0, 1, \ldots, n,$$

where ω_v^x, ω_v^y, ω_v^t, respectively, are:

$$\omega_0^x = x_1^2 x_2^2, \quad \omega_1^x = -2(x_1 + x_2) x_1 x_2, \quad \omega_2^x = (x_1 + x_2)^2 + 2x_1 x_2,$$

$$\omega_3^x = -2(x_1 + x_2), \quad \omega_4^x = 1, \quad \omega_0^y = y_1^2 y_2^2, \quad \omega_1^y = -2(y_1 + y_2) y_1 y_2,$$

$$\omega_2^y = (y_1 + y_2)^2 + 2y_1 y_2, \quad \omega_3^y = -2(y_1 + y_2), \quad \omega_4^y = 1,$$

$$\omega_0^t = -t_1 t_2, \quad \omega_1^t = (t_1 + t_2), \quad \omega_2^t = -1.$$

3. Filling in the values of the elements of the matrices A and B (9), namely the values of a_{ij} and b_i (10)-(11), which are triple integrals. Calculation of triple integrals in (10)-(11) reduces to the calculation of repeated integrals using known formulas of numerical integration (Newton-Cotes formulas, etc.).

4. Finding $\text{cond}(A)$ - the condition number of the matrix A:

$$\text{cond}(A) = \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right|,$$

where $\lambda_{\max}$ is the maximum eigenvalue and $\lambda_{\min}$ is the minimum eigenvalue of A. To overcome the difficulties associated with incorrect ill-conditioned systems, we must ensure the "closeness" of the condition number of A to 1. This is achieved by varying modulating functions or averaging original data.

5. The solution of linear algebraic equations (9).

Suitably selecting modulating function, we can ensure the condition of non-degeneracy of the matrix A: $\det A \neq 0$.

To solve the system (9), for example, the well-known Gaussian elimination with pivoting can be used. As a result, for the solution of (9), we find the vector of unknowns $K = (k_0, k_1, ..., k_n)^T$.

6. Substituting the obtained values of k_i, $i = \overline{0, n}$ in the expansion (5) of the required functions $k(x, y)$.

Conclusion

1. M-method allows us to move from incorrect differentiation experimental function to correct operation of integration.

2. In the algorithms for solving inverse problems using the M-method, the use of solutions of the direct boundary value problems is not the main source of error.

3. A practical source of errors can be numerical methods for solving integrals and approximation of the experimental functions.

4. One of the main advantages of this method is its simplicity and efficiency of practical application.

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