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DISTRIBUTIONAL BOUNDARY VALUES IN BEURLING ULTRADISTRIBUTIONS

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Abstract

Let $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ be the space of Beurling ultradistributions in the sense of [5] and $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ be the space of the Fourier transform of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$. We define certain analytic functions in tubes in $\mathbb{C}^N$ which are defined by growth conditions and obtain distributional boundary value properties of the analytic functions in $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$. Also, we recover analytic functions by Fourier-Laplace transforms. Further, we show that Beurling ultradistributions of L_p-growth generate the analytic functions which satisfy a boundedness condition and have distributional boundary values in $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$.

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1. Introduction

Let $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$, $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$, $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{D}'_{L_p}(\mathbb{R}^N)$ be spaces of distributions, distributions of compact support, tempered distributions and Schwarz distributions, respectively. Tillmann has obtained a characterization of the analytic functions which represent $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ as boundary values in [21] and similar analysis for the spaces of $\mathcal{D}'_{L_p}(\mathbb{R}^N)$ in [22]. In [23], he has also shown that if functions $f(z)$ which are analytic in an octant have distributional boundary values in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$, then $f(z)$ must satisfy a boundedness condition. In [24], Vladimirov has obtained distributional boundary value results in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ using a boundedness condition similar to Tillmann for functions analytic in a tubular cone of which an octant and a tubular domain are examples. In [3] and [4], Beltrami and Wohlers have obtained distributional boundary value results in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ using a boundedness condition that is less restrictive than that of Tillmann and Vladimirov. In [6] and [7], Bremermann has obtained the representation of elements in $\mathcal{O}'_\alpha$, which are intermediate spaces between $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ as boundary values of analytic functions in half planes and tubes defined by quadrant.

In [8] and [9], Carmichael has obtained distributional boundary value results in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{D}'_{L_p}(\mathbb{R}^N)$ as a subspace of $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ for functions analytic in a tubular radial domain using a boundedness condition which is weaker than that of Tillmann and Vladimirov. His results had as special cases the above mentioned results. Analysis concerning the representation of several kinds of distributions in the sense of Schwartz as boundary values of analytic functions in one and multiple variables was presented in [11].

Let $\mathcal{D}'((M_p), \Omega)$ (resp. $\mathcal{D}'(\{M_p\}, \Omega)$) and $\mathcal{E}'((M_p), \Omega)(\mathcal{E}'(\{M_p\}, \Omega))$ be spaces of Beurling ultradistributions (resp. Roumieu ultradistributions) and infra-hyperfunctions for each open subset $\Omega(\neq \emptyset) \subset \mathbb{R}^N$, respectively. Here (M_p) is a non-quasianalytic sequence and $\{M_p\}$ is a quasianalytic sequence. We can find the boundary value characterizations for the spaces $\mathcal{D}'((M_p), \Omega)(\mathcal{D}'(\{M_p\}, \Omega))$ and $\mathcal{E}'((M_p), \Omega)(\mathcal{E}'(\{M_p\}, \Omega))$ in [19, 17, 18] and [20]. Carmichael et al. have defined ultradistributions of Beurling type $\mathcal{D}'((M_p), L^p)$ and of Roumieu type $\mathcal{D}'(\{M_p\}, L^p)$, both of which generalize $\mathcal{D}'_{L_p}$, in [12] and have found the boundary value characterizations for $\mathcal{D}'((M_p), L^p)$ and $\mathcal{D}'(\{M_p\}, L^p)$ with $p > 1$ in [13], [14] and [10].

In the mean time, Braun et al. [5] have introduced ω-ultradistributions of Beurling type $\mathcal{D}'_{(\omega)}(\Omega)$ and Roumieu type $\mathcal{D}'_{(\omega)}(\Omega)$ and Betancor et al. [1] have introduced Beurling ultradistributions of L_q-growth, $\mathcal{D}'_{L_p,(\omega)}$, $\frac{1}{p} + \frac{1}{q} = 1$, which is an extension of $\mathcal{D}'_{L_p}$. Here ω is a weight function. Fernández et al. studied the representation of distributions (and Beurling ultradistributions) of L_p-growth, $1 \leq p \leq \infty$, on $\mathbb{R}^N$ as boundary values of holomorphic functions on $(\mathbb{C}\backslash\mathbb{R})^N$ in [15].

In this paper, we define the space of the Fourier transform of $\mathcal{D}'_{(\omega)}(\Omega)$, $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$, and a certain analytic functions in tubes in $\mathbb{C}^N$ which satisfy the growth conditions. We prove distributional boundary value properties of the analytic functions in $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ and recovery of the analytic functions by Fourier-Laplace transforms. Also, we show that the elements of $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \leq p < \infty$, generate the analytic functions which satisfy a certain boundedness condition and have boundary values in the topology of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$.

Throughout this paper, we let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in N_0^{\mathbb{N}}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$, $\alpha! = \alpha_1!\alpha_2!\ldots\alpha_N!$, $a\alpha = (a\alpha_1, a\alpha_2, ..., a\alpha_n)$ for $a \in \mathbb{R}$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_N^{\alpha_N}$ for $z \in \mathbb{C}^N$.

2. Beurling Ultradistributions and its Fourier Transform

In this section, we review Beurling ultradistributions and Beurling ultradistributions of L_p-growth in $\mathbb{R}^N$ and establish some of their properties which will be needed later.

Definition 1 [5]**.** A *weight* function is an increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ with the following properties:

(α) there exists $L \geq 0$ with $\omega(2t) \leq L(\omega(t) + 1)$ for all $t \geq 0$,

(β) $\int_1^\infty (\omega(t)/t^2)dt < \infty$,

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞,

(δ) $\psi : t \to \omega(e^t)$ is convex.

For a weight function ω, we define $\widetilde{\omega} : \mathbb{C}^N \to [0, \infty)$ by $\widetilde{\omega}(z) = \omega(|z|)$ and again call this function ω. Here $|z| = \sum_{i=1}^N |z_i|$.

By (δ), $\psi(0) = 0$ and $\lim_{x\to\infty} x/\psi(x) = 0$. Then we can define the Young conjugate ψ^* of ψ by

$$\psi^* : [0, \infty) \to \mathbb{R}, \quad \psi^*(y) = \sup_{x \geq 0}(xy - \psi(x)).$$

Obviously, we have

Lemma 1 [5]. $y/\psi^*(y)$ *is decreasing and* $\lim_{y\to\infty} y/\psi^*(y) = 0.$

Let ω be a weight function. For a compact set $K \subset \mathbb{R}^N$, we define

$$\mathcal{D}_{(\omega)}(K) = \{f \in \mathcal{D}(K) : \| f \|_{K,k} < \infty \text{ for every } k \in N\},$$

where

$$\| f \|_{K,k} = \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^N} | f^{(\alpha)}(x) | e^{-k\psi^*(|\alpha|/k)}.$$

$\mathcal{D}_{(\omega)}(K)$, endowed with its natural topology, is a Fréchet space. For a fundamental sequence $(K_i)_{i,\mathbb{N}}$ of compact subsets of $\mathbb{R}^N$, we let

$$\mathcal{D}_{(\omega)}(\mathbb{R}^N) = ind_i \mathcal{D}_{(\omega)}(K_i).$$

The dual of $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ will be denoted by $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ and endowed with its strong topology. The elements of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ are called *Beurling ultradistributions*. We denote by $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ the set of all $C^\infty(\mathbb{R}^N)$ functions f such that $\| f \|_{K,\lambda} < \infty$ for every compact K in $\mathbb{R}^N$ and every $\lambda > 0$. For more details about $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$, we refer to [2] and [5].

Betancor et al. [1] introduced Beurling ultradistributions of L_p-growth in $\mathbb{R}^N$ as follows: for every $1 \le p \le \infty$, $k \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{R}^N)$, $\gamma_{k,p}(\phi)$ is defined by

$$\gamma_{k,p}(\phi) = \sup_{\alpha\in\mathbb{N}_0^N} \| \phi^{(\alpha)} \|_p e^{-k\psi^*(|\alpha|/k)},$$

where $\|\cdot\|_p$ denotes the usual norm in $L_p(\mathbb{R}^N)$. ($\| f \|_\infty$ means essential supremum of $|f(t)|$.) If $1 \leq p < \infty$, then the space $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ is the set of all C^∞-functions ϕ on $\mathbb{R}^N$ such that $\gamma_{k,p}(\phi) < \infty$ for each $k \in \mathbb{N}$. A function $\phi \in C^\infty(\mathbb{R}^N)$ is in $\mathcal{B}_{L_\infty,(\omega)}$ when $\gamma_{k,\infty}(\phi) < \infty$ for each $k \in \mathbb{N}$. We denote by $\mathcal{D}_{L_\infty,(\omega)}$ the subspace of $\mathcal{B}_{L_\infty,(\omega)}$ that consists of all those functions $\phi \in \mathcal{B}_{L_\infty,(\omega)}$ for which $\lim_{|x|\to\infty} \phi^{(\alpha)}(x) = 0$. The topology of $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, is generated by the family $\{\gamma_{k,p}(\phi)\}_{k\in\mathbb{N}}$ of seminorms.

The dual of $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ will be denoted by $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$. The elements of $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ are called the *Beurling ultradistributions of* L_q-*growth*, $\frac{1}{p} + \frac{1}{q} = 1$, in the sense of Theorem 2.5 in [1]. For more details about $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, we refer to [1].

Assume that G is an entire function in $\mathbb{C}^N$ such that $\log|G(z)| = O(\omega(|z|))$, as $|z| \to \infty$. The functional T_G on $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ is defined by

$$\langle T_G, \phi \rangle = \sum_{\alpha\in\mathbb{N}_0^N} (-i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} \phi^{(\alpha)}(0), \quad \phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^N).$$

The operator $G(D)$ defined on $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ by

$$G(D) : \mathcal{D}'_{(\omega)}(\mathbb{R}^N) \to \mathcal{D}'_{(\omega)}(\mathbb{R}^N), \quad \mu \to G(D)\mu = \mu * T_G$$

is called an *ultradifferential operator of* (ω)-*class*. When $G(D)$ is restricted to $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$, is a continuous operator from $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ into $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ and, if for every $\phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$,

$$(G(D)\phi)(x) = \sum_{\alpha \in \mathbb{N}_0^N} (i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} \phi^{(\alpha)}(x), \quad x \in \mathbb{R}^N.$$

As in Proposition 2.4 in [16], it can be shown that each ultradifferential operator $G(D)$ of (ω)-class defines a continuous linear mapping from $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ into $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ for every $1 \le p \le \infty$. Thus, $G(D)$ is also a continuous linear operator from $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ into $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$.

Definition 2. An ultradifferential operator $G(D)$ of (ω)-class is said to be *strongly elliptic* if there exist $M > 0$ and $l > 0$ such that $|G(z)| \ge Me^{lw(|z|)}$ when $|\operatorname{Im} z| < M|\operatorname{Re} z|$. In this case, the entire function $G(z)$ is said to be the (l, M)*-strongly elliptic*.

The Fourier transform is not well-defined for $T \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ with $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ since $\hat{\phi}$ may not be a function in $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$. To establish our results in the next section, we need to define a theory of the Fourier transform on the whole of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$.

We define the support function for a compact set $K \in \mathbb{R}^N$ as $H_K(x) = \sup_{y \in K} \langle x, y \rangle$.

Definition 3. Let ω be a weight function. Then $\mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ is the space of all infinitely differentiable functions Φ which can be extended to be entire analytic functions in $\mathbb{C}^N$ such that there exist $A > 0$ for which

$$|\Phi(z)| \le C_k e^{(A|\operatorname{Im} z| - k\omega(z))}, \quad z \in \mathbb{C}^n,$$

for every $k \in \mathbb{N}$. Here *C* depends on *k* and possibly Φ and *A* depend on Φ.

A sequence $\{\Phi_n\}$ converges in $\mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ if

(i) each $\Phi_n \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$;

(ii) there exist constants C_k and A, which are independent n such that for all n

$$|\Phi_n(z)| \leq C_k e^{(A|\operatorname{Im} z| - k\omega(z))}, \quad z \in \mathbb{C}^n,$$

for each k;

(iii) $\{\Phi_n\}$ converges uniformly on every bounded set in $\mathbb{C}^n$.

$\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ is the set of all continuous linear functionals on $\mathcal{Z}_{(\omega)}(\mathbb{R}^N)$. From Proposition 3.4 in [5], we have that the Fourier transform is a topological isomorphism of $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ onto $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$.

We are ready to define the Fourier transform on $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$.

Definition 4. Let $T \in \mathcal{D}'_{(\omega)}$. Then the Fourier transform of T is the element $S = \hat{T} \in \mathcal{Z}'_{(\omega)}$ such that

$$\langle S, \Phi \rangle = \langle T, \check{\phi} \rangle, \quad \phi \in \mathcal{D}_{(\omega)}, \quad \Phi = \hat{\phi} \in \mathcal{Z}_{(\omega)}, \tag{1}$$

where $\check{\phi}(x) = \phi(-x)$.

We have that Fourier transform defined in (1) is an element of $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ and this Fourier transform is a topological isomorphism of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ onto $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$. The inverse Fourier transform is a topological isomorphism of $\mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ onto $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ from which we can define an inverse Fourier transform from $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ to $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ by

$$\langle T, \phi \rangle = \langle S, \Phi \rangle, \ \Phi \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N), \ \phi(t) = \mathcal{F}^{-1}[\Phi(x); t] \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$$

with $\check{\Phi}(x) = \Phi(-x)$, where $S \in \mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ and $T = \mathcal{F}^{-1}[S]$ is the inverse Fourier transform of S. The inverse Fourier transform is a topological isomorphism of $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ onto $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$.

3. Distributional Boundary Values in $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$

Let $C \subset \mathbb{R}^N$ be a cone with vertex at zero, i.e., if $y \in C$ implies $\lambda y \in C$ for all $\lambda > 0$. The intersection of the cone C with the unit sphere $\{y \in \mathbb{R}^N : |y| = 1\}$ is called the *projection* of C and denoted $pr(C)$. If C' and C are cones such that $pr\overline{C}' \subset pr(C)$, then C' will be called a *compact subcone* of C. An open convex cone C such that $\overline{C}$ does not contain any straight line will be called a *regular cone*. For a cone C, $\mathcal{O}(C)$ will denote the convex hull (envelope) of C and $T^C = \mathbb{R}^N + iC \subset \mathbb{C}^N$ is a tube in $\mathbb{C}^N$. If C is open, T^C is called a *tubular domain*. If C is both open and connected, T^C is called a *tubular radial domain*. The set $C^* = \{t \in \mathbb{R}^N : \langle t, y\rangle \geq 0,$ for all $y \in C\}$ is the dual cone of the cone C.

Definition 5. The function

$$U_C(t) = \sup_{y \in pr(C)} (-\langle t, y\rangle)$$

is the indicatrix of the cone.

By the sets S_A and G_M, we shall mean $S_A = \{t : U_C(t) \leq A\}$ and $G_M = \{z \in \mathbb{C}^N : |\operatorname{Im} z| < M|\operatorname{Re} z|\}$, respectively. For a cone C, put $C_* = \mathbb{R}^N \backslash C^*$. The number

$$H_C = \sup_{t \in C_*} \frac{U_{\mathcal{O}(C)}(t)}{U_C(t)}$$

characterizes the nonconvexity of the cone C. In [24, p. 220, Lemma 2 and

Lemma 3], Vladimirov has proved that a cone is convex if and only if $H_C = 1$, and if a cone is open and consists of a finite number of components then $H_C < \infty$.

Lemma 2 [24, p. 223, Lemma 2]**.** *Let C be an open cone and C' be a cone that is compact in $\mathcal{O}(C)$. Then there exist a $\delta = \delta_y > 0$, depending on y, and an open cone C'', depending on C', containing the cone C^* such that*

$$\langle y, t\rangle \geq \delta |y||t|, \quad y \in C', \quad t \in C''. \tag{2}$$

Further, if C' is an arbitrary compact subcone of $\mathcal{O}(C)$, then there exists a $\delta(C') > 0$, depending only on C' and not on $y \in C'$, such that (2) holds for all $y \in C'$ and all $t \in C^*$.

Lemma 3 [24, Sec. 25.1 and Sec. 26.4]**.** *If $t \in C_*$, then*

$$-\langle t, y\rangle \leq U_{\mathcal{O}(C)}(t)|y|, \quad U_{\mathcal{O}(C)}(t) \leq H_C U_C(t), \quad y \in \mathcal{O}(C). \tag{3}$$

If we let C'_* be an arbitrary subcone that is compact in C^* and $t \in C'_*$, then there exists $\gamma = \gamma(C'_*)$ such that

$$\gamma|t| \leq U_C(t) \leq |t|. \tag{4}$$

Let C be an open connected cone in $\mathbb{R}^N$. For any real number $m > 0$ and any compact subcone C' of C, put

$$T(C'; m) = \mathbb{R}^N + i(C' \backslash (C' \cap \overline{N(\overline{0}, m)})),$$

where $N(\overline{0}, m) = \{y \in \mathbb{R}^N : |y| \leq m\}$ and $\overline{0} = (0, 0, ..., 0)$.

Definition 6. Let ω be a weight function and C be an open connected cone in $\mathbb{R}^N$. We shall say that $f(z) \in H_\omega(A; C)$ if $f(z)$ is analytic in tube $T^C = \mathbb{R}^N + iC$ and for every compact subcone C' of C and any real number $m > 0$, there exists a constant $K(C', \sigma, m)$ depending on C', σ and on $m > 0$ such that

$$|f(z)| \leq K(C', \sigma, m)e^{M\omega(z)}e^{(A+\sigma)|y|}, \quad z \in T(C'; m), \tag{5}$$

for all $\sigma > 0$. Here M and A are nonnegative real numbers which do not depend on C' and $m > 0$.

Clearly, $H(A; C) \subset H_\omega(A; C)$ by (γ) in Definition 1, where $H(A; C)$ is defined in Definition 4.7.2 in [11]. We will show that the Fourier-Laplace transform of a certain type of distributions is in $H_\omega(0; C)$.

Theorem 1. *Let C be an open connected cone in* $\mathbb{R}^N$ *and let* C' *be an arbitrary compact subcone of C. Let* $g(t)$ *be a continuous function with support in* C^* *which satisfies*

$$|g(t)| \leq K(C', m, \sigma)\exp(2\pi(\langle s, t\rangle + \sigma|s|)), \quad t \in \mathbb{R}^N \tag{6}$$

for all $\sigma > 0$. *Here* $K(C', m, \sigma)$ *is a constant which depends on* C', $m > 0$, *and* σ *and* (6) *is independent of* $s \in (C' \backslash (C' \cap \overline{N(\overline{0}, m)}))$.

Let $G(z)$ *be an* (l, M)*-strongly elliptic entire function and* $G(D)$ *be an ultradifferential operator of* (ω)*-class corresponding* $G(z)$. *If* $V = G(D)g$, *then*

$$f(z) = \langle V(\cdot), \exp\langle z, \cdot\rangle\rangle$$

is an element of $H_\omega(0; C)$.

Proof. From (6), $g(t)$ defines an element of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$, hence $V = G(D)g$ is well defined and

$$\begin{aligned} f(z) &= \langle V(t), \exp\langle z, t\rangle\rangle \\ &= \langle g * T_G(t), \exp\langle z, t\rangle\rangle \\ &= \langle g(\eta), < G(z), \exp\langle z, \eta\rangle\rangle \\ &= G(z)\int_{C^*} g(t)\exp\langle z, t\rangle dt, \quad z \in T^C. \end{aligned}$$

By the proof of Theorem 4.7.1 in [11, pp. 181-183], $g(t)$ is analytic and

$$\left|\int_{C^*} g(t)\exp\langle z, t\rangle dt\right| \le K(C', m, \sigma)\exp\left(2\pi\left(\frac{\sigma}{2}\right)|y|\right),$$

for $z \in T(C'; m)$, where $K'(C', m, \sigma)$ is a constant depending on C' and on $m > 0$; here σ is arbitrary. Since $G(z)$ is an entire function in $\mathbb{C}^N$ such that $\log|G(z)| = \mathcal{O}(\omega(|z|))$, $f(z) \in H_\omega(0; C)$. $\square$

If $f \in H_\omega(A; C)$, then $f(\cdot + iy)$ defines an element of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ (even an element in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$) for each $y \in C$. Throughout this paper, we assume that

$$\lim_{y \to 0,\, y \in C'} \langle f(\cdot + iy), \phi(\cdot)\rangle, \quad \phi \in \mathcal{D}_\omega \text{ or } \mathcal{Z}'_\omega$$

exists (which is different from the existence of the limits in weak topology.) Then the limit defines an element of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ or $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ by Banach-Steinhaus theorem. Now we are ready to study distributional boundary value results in $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ for functions analytic in a tubular radial domain.

Theorem 2. *Let $f(z) \in H_\omega(A; C)$. Then there exist an element $U \in \mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ such that $f(z) \to U$ in the weak topology of $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C' \subset C$ and an element $V \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ having support in $S_A = \{t : U_C(t) \le A\}$ such that $U = \mathcal{F}[\check{V}]$ in $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$.*

Proof. Let $z = x + iy$. Take an integer l such that $N - l < -1 - \varepsilon$ for any $\varepsilon > 0$. Let $G(z)$ be an (l, M)-strongly elliptic entire function. Then

$$\left|\frac{f(z)}{G(z)}\right| \le K(C')e^{(-1-\varepsilon)\omega(z)}e^{(A+\sigma)|y|}, \quad z \in T(C'; m) \cap G_M. \tag{7}$$

Consider

$$g(t) = \int_{\mathbb{R}^N} \frac{f(z)}{G(z)} e^{-i\langle z, t\rangle} dx, \quad z \in T(C'; m) \cap G_M. \tag{8}$$

From (7), $g(t)$ is well-defined and is continuous function of t for each fixed $y \in (C' \backslash (C' \cap \overline{N(\overline{0}, m)}))$. Let C'' be an arbitrary compact subdomain of C. From (7), for $z \in G_M$,

$$\int_{C''} \frac{f(z)}{G(z)} e^{-i\langle z, t\rangle} dy \to 0 \tag{9}$$

as $|x| \to \infty$. Applying (7) and Cauchy's theorem to integrand in (9), we have that $g(t)$ is independent of $y \in C''$, hence of $y \in C$. (8) can be rewritten as

$$e^{-\langle y, t\rangle} g(t) = \mathcal{F}^{-1}\left[\frac{f(z)}{G(z)}; t\right], \quad z \in T(C'; m) \cap G_M.$$

We have from (7) that $f(z)/G(z) \in L_1 \cap L_2$ as a function of $x \in \mathbb{R}^N$ for $y \in (C' \backslash (C' \cap \overline{N(\overline{0}, m)}))$ arbitrary. By the Plancherel theorem,

$$\frac{f(z)}{G(z)} = \mathcal{F}[e^{-\langle y, t\rangle} g(t); (x)], \quad z \in T(C'; m) \cap G_M \tag{10}$$

in L_2. Only replacing $\int_{\mathbb{R}^N} (1 + |x|)^{-n-\varepsilon} dx$ by $\int_{\mathbb{R}^N} \exp((-n - \varepsilon)\omega(x)) dx$, we can show $supp(g) \subseteq S_A$ by the exactly the same line in the proof of Theorem 4.7.2 in [11].

Let $\Phi \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ be such that $\Phi = \hat{\phi}$. Since $e^{-\langle y, t\rangle} g(t) \in \mathcal{D}'(\mathbb{R}^N) \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$,

$$\left\langle \frac{f(z)}{G(z)}, \Phi(x) \right\rangle = \langle e^{-\langle y, t\rangle} g(t), \check{\phi}(t)\rangle, \quad z \in T(C'; m) \cap G_M, \tag{11}$$

where C' is an arbitrary compact subcone of C. Since $e^{-\langle y,t\rangle}g(t) \to g(t)$ in $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C' \subset C$ and $g(t) \in \mathcal{D}'(\mathbb{R}^N) \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$, we have

$$\langle e^{-\langle y,t\rangle}g(t),\ \check{\phi}(t)\rangle \to \langle g(t),\ \check{\phi}(t)\rangle = \langle \hat{g}(x), \Phi(x)\rangle \tag{12}$$

as $y \to 0$, $y \in C' \subset C$. From (11) and (12),

$$\frac{f(z)}{G(z)} \to \hat{g}(x) \tag{13}$$

in $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C' \subset C$. Also, since $\log|G(z)| = O(\omega(|z|))$, as $|z| \to \infty$, if $\Phi \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$, $G(z)\Phi(x) \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ as a function of $z = \operatorname{Re} z \in \mathbb{R}^N$ for $y = \operatorname{Im} z \in \mathbb{C}^N$. Hence we have from (13) that

$$\begin{aligned}\langle f(z), \Phi(x)\rangle &= \left\langle \frac{f(z)}{G(z)}, G(z)\Phi(x)\right\rangle \\ &\to \langle G(z)\hat{g}(x), \Phi(x)\rangle \end{aligned} \tag{14}$$

as $y \to 0$, $y \in C' \subset C$. Put

$$V = G(D)g,$$

where $G(D)$ is an ultradifferential operator of (ω)-class corresponding to $G(z)$. By the properties of $G(z)$, $G(D)$ defines an ultradifferential operator of (ω)-class. Since $g \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ and $G(D)$ defines on $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ into $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$, $V \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ and $supp(V) = supp(g) \subseteq S_A$. Also, if we let $\Phi \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ such that $\Phi = \hat{\phi}$, then for $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$,

$$\begin{aligned}&\langle \mathcal{F}[V](x), \Phi(x)\rangle \\ &= \langle G(D)g(t),\ \check{\phi}(t)\rangle \\ &= \langle g * T_G(t),\ \hat{\hat{\phi}}(t)\rangle\end{aligned}$$

$$= \langle g(\eta), \langle T_G(\xi), \langle \hat{\phi}(s), e^{-i\langle s, \eta+\xi\rangle}\rangle\rangle\rangle$$

$$= \left\langle g(\eta), \sum_{\alpha\in\mathbb{N}_0^N} (-i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} \frac{\partial^\alpha}{\partial\xi^\alpha} \langle \hat{\phi}(s), e^{-i\langle s, \eta+\xi\rangle}\rangle|_{\xi=0} \right\rangle$$

$$= \langle g(\eta), G(-s)\langle \hat{\phi}(s), e^{-i\langle s, \eta\rangle}\rangle\rangle$$

$$= \langle \hat{g}(s), G(-s)\Phi(s)\rangle$$

$$= \langle G(-s)\hat{g}(s), \Phi(s)\rangle, \tag{15}$$

where we have used the fact that $\Phi \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ implies $G(z)\Phi \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ as a function of $x = \operatorname{Re} z \in \mathbb{R}^N$. Then we have from (14) and (15) that for $\Phi \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ and $y \in C$,

$$\begin{aligned}\lim_{y\to 0}\langle f(x+iy), \Phi(x)\rangle &= \lim_{y\to 0}\left\langle \frac{f(z)}{G(z)}, G(z)\Phi(x)\right\rangle \\ &= \langle \hat{g}(x), G(z)\Phi(x)\rangle \\ &= \langle \mathcal{F}[\check{V}](x), \Phi(x)\rangle. \end{aligned} \tag{16}$$

If we put $U = \mathcal{F}[\check{V}] \in \mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$, then (16) proves that $f(x+iy) \to U$ in the weak topology of $\mathcal{Z}'_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C$. The proof is complete. □

Now we will study distributional boundary values results in $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ for functions analytic in a tubular radial domain. Let $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \le p < \infty$, and $supp(U) \subset S_A$ for some open connected cone C. Take $\alpha_\varepsilon(t) \in C^\infty$ such that

$$\alpha_\varepsilon(t) = \begin{cases} 1 \text{ on an } \varepsilon\text{-neighborhood of } S_A, \\ 0 \text{ on a complement of a } 2\varepsilon\text{-neighborhood of } S_A \end{cases} \tag{17}$$

and

$$|D_t^\gamma \alpha_\varepsilon(t)| \leq M_\gamma, \quad t \in \mathbb{R}^N, \tag{18}$$

where M_γ is a constant depending on γ. We will show that $\alpha_\varepsilon(t)e^{i\langle z, t\rangle}$, $z \in T^{C'}$, is in $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \leq p < \infty$ as a function of t. Consider

$$(I) = \int_{\mathbb{R}^N} |D^{(\beta)}\alpha_\varepsilon(t)D^{(\gamma)}(e^{i,\langle z,t\rangle})|^p e^{-k\psi^*(|\alpha|/k)}dt, \tag{19}$$

where $\beta + \gamma = \alpha$ and $k \in \mathbb{N}$. Then

$$(I) \leq M_\beta \int_{\mathbb{R}^N} |D^{(\gamma)}(e^{i,\langle z,t\rangle})|^p e^{-k\psi^*(|\alpha|/k)}dt$$

$$= M_\beta e^{|\gamma| p \log |z|} e^{-k\psi^*(|\alpha|/k)} \int_{S_A} e^{-p\langle y,t\rangle}dt, \tag{20}$$

where M_β is a constant in (18). If $|z| \leq 1$, then

$$e^{|\gamma| p \log |z|} e^{-k\psi^*(|\alpha|/k)} \leq 1. \tag{21}$$

Let $|z| > 1$ and $|\gamma| p > k$. Then

$$e^{|\gamma| p \log |z|} e^{-k\psi^*(|\alpha|/k)} \leq e^{k \log |z|} e^{-k\psi^*(|\alpha|/k)} \leq e^{k\omega(z)}. \tag{22}$$

Let $|z| > 1$ and $|\gamma| p > k$. By Lemma 1, since $\psi^*(t)/t$ is increasing,

$$e^{|\gamma| p \log |z|} e^{-k\psi^*(|\alpha|/k)} \leq e^{|\gamma| p \log |z|} e^{-|\gamma| p\psi^*(|\alpha|/(|\gamma| p))} \leq e^{|\gamma| p\omega(z)}. \tag{23}$$

From (21), (22) and (23),

$$e^{|\gamma| p \log |z|} e^{-k\psi^*(|\alpha|/k)} \leq e^{\max(k,|\gamma| p)\omega(z)}. \tag{24}$$

Let C'' and $C'_* = \mathbb{R}^N \backslash C''$ be cones in which inequality (2) in Lemma 2 and inequality (4) in Lemma 3 are verified, respectively,

$$\int_{S_A} e^{-p\langle y,\, t\rangle} dt \le \sup_{t\in S_A}\left\{ e^{-p\langle y,\, t\rangle}(1+|\,t\,|)^2 \int_{S_A} \frac{1}{(1+|\,t\,|)^2}\, dt \right\}$$

$$\le K\left[\sup_{t\in C''}\{e^{-\langle y,t\rangle}(1+|\,t\,|)^2\} + \sup_{t\in S_A\cap C'_*}\{e^{-\langle y,t\rangle}(1+|\,t\,|)^2\}\right], \quad (25)$$

where K is a constant. Using (3) in Lemma 3, we continue the inequality

$$\left[\sup_{\rho\ge 0} e^{-\delta\rho|\,y\,|}(1+\rho)^2 + \sup_{\gamma|\,t\,|\le U_C(t)\le A} e^{H_C A|\,y\,|}(1+|\,t\,|)^2\right]$$

$$\le K(C')\left[\sup_{\rho\ge 0} e^{-\delta\rho|\,y\,|}\rho^2 + \left(1+\frac{A}{\gamma}\right)^2 e^{H_C A|\,y\,|}\right]$$

$$\le K_1(C')(|\,y\,|^{-2} + e^{H_C A|\,y\,|})$$

$$\le K_1(C')|(1+|\,y\,|^{-2})e^{H_C A|\,y\,|}, \quad z\in T^{C'}, \quad (26)$$

where $K(C')$ and $K_1(C')$ constants, depending on C'. Here we have used the fact that $f(\rho) = e^{-\delta\rho|\,y\,|}\rho^2$ has a maximum $(4e^{-2}/\rho^2)|\,y\,|^2$ at $\rho = 2/(\delta|\,y\,|)$,

$$(I) \le M_B K_1(C') e^{\max(k_0,\,|\,\gamma\,|\,p)\,\omega(z)}(1+|\,y\,|^{-2})e^{H_C A|\,y\,|}. \quad (27)$$

Thus, $\alpha_\varepsilon(t)e^{i\langle z,t\rangle}$ is in $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1\le p<\infty$, as a function of t for a fixed $z\in T^{C'}$, hence

$$\langle U(t)e^{i\langle z,t\rangle}\rangle = \langle U(t)\alpha_\varepsilon(t)e^{i\langle z,t\rangle}\rangle, \quad z\in T^{C'}. \quad (28)$$

is well-defined for t.

Theorem 3. *Let $U\in\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2\le p<\infty$, and $supp(U)\subseteq S_A$ for some open connected cone C. Then there exists a function $f(z)$ such that $f(z)$ is analytic in $T^{\mathcal{O}(C)}$ and for any arbitrary compact subcone C' of*

$\mathcal{O}(C)$, there exist an (l, M)-strong elliptic entire function $G(z)$ and $m \in \mathbb{N}$, depending on U such that

$$|D_t^{(\alpha)} f(z)| \le K(\alpha, C')|G(z)|(1+|y|^{-m})e^{AH_C|y|}, \quad z \in T^{C'} \cap G_M,$$

where $K(\alpha, C')$ is a constant, depending on α and C'. Also, $f(z) \to \hat{U}$ in $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C' \subset \mathcal{O}(C)$.

Proof. Let $2 \le p < \infty$. We note that, since $U_C(t)$ is a convex function, S_A is convex. Furthermore, by Lemma 3,

$$\langle t, y\rangle \ge -U_{\mathcal{O}(C)}(t)|y| \ge -H_C U_C(t)|y| \ge -H_C A|y|, \tag{29}$$

for $(y, t) \in \mathcal{O}(C) \times S_A$.

Now, since $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, there exists a strongly elliptic ultradifferential operator $G(D)$ of (ω)-class and $h \in L_p$, $\frac{1}{p} + \frac{1}{q} = 1$ such that $U = G(D)h$. Take $m \in \mathbb{N}$, depending on *h* (hence on *U*) such that

$$\int_{\mathbb{R}^N} \frac{h(t)}{(1+|t|)^m} < \infty. \tag{30}$$

Put

$$f(z) = \langle e^{-\langle y, t\rangle}U, e^{i\langle x, t\rangle}\rangle = \langle U, \alpha_\varepsilon(t)e^{i\langle z, t\rangle}\rangle, \quad z = x + iy \in T^{C'},$$

where $\alpha_\varepsilon(t)$ satisfies (17) and (18). From (28), $f(z)$ is well-defined and analytic in $T^{\mathcal{O}(C)}$. Then, for $z \in T^{C'} \cap G_M$, there exists a strongly elliptic entire function $G(z)$ corresponding to $G(D)$ such that

$$\begin{aligned} |f(z)| &= |\langle e^{-\langle y, t\rangle}U, \alpha_\varepsilon(t)e^{i\langle x, t\rangle}\rangle| \\ &= |\langle h(t), G(D)(\alpha_\varepsilon(t)e^{i\langle z, t\rangle})\rangle| \end{aligned}$$

$$\leq |\,G(z)\,| \int_{\mathbb{R}^N} e^{-\langle y,t\rangle} |\,h(t)\,| dt$$

$$\leq |\,G(z)\,| \sup_{t \in S_A} \{e^{-\langle y,t\rangle}(1+|\,t\,|)^m\} \int_{\mathbb{R}^N} \frac{h(t)}{(1+|\,t\,|)^m} dt$$

$$\leq K|\,G(z)\,| \left[\sup_{t \in C''}\{e^{-\langle y,t\rangle}(1+|\,t\,|)^m\} + \sup_{t \in S_A \cap C'_*}\{e^{-\langle y,t\rangle}(1+|\,t\,|)^m\}\right], \quad (31)$$

where m is as in (30), K is a constant and C'' and C'_* are cones in which inequality (2) in Lemma 2 and inequality (3) in Lemma 3 are verified, respectively. Using (3) in Lemma 3, we continue the inequality (31)

$$|\,f(z)\,| \leq K|\,G(z)\,| \left[\sup_{\rho \geq 0} e^{-\delta\rho|\,y\,|}(1+\rho)^m + \sup_{\gamma|\,t\,| \leq U_C(t) \leq A} e^{H_C A}(1+|\,t\,|)^m\right]$$

$$\leq K(C')|\,G(z)\,|(1+|\,y\,|^{-m})e^{H_C A|\,y\,|}, \quad z \in T^{C'} \cap G_M, \quad (32)$$

where $K(C')$ is a constant, depending on C'. Since

$$|\,D_t^{(\alpha)} f(z)\,| \leq N(|\,\alpha\,|) \max_{1 \leq j \leq |\,\alpha\,|} M_j (1+|\,z\,|)^{|\,\alpha\,|} \langle U, \alpha_\varepsilon(t) e^{i\langle z,t\rangle}\rangle,$$

where $N(\alpha)$ is a constant depending on α and M_j is as in (29), we have from (32) and the property of (l, M)-strong elliptic entire function that

$$|\,D_t^{(\alpha)} f(z)\,| \leq K(\alpha, C')|\,G(z)\,|(1+|\,y\,|)^2 e^{AH_C|\,y\,|}, \quad z \in T^{C'} \cap G_M,$$

where $K(\alpha, C')$ is a constant, depending on α and C'.

Now we will show that $f(z) \to \hat{U}$ in $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C' \subset \mathcal{O}(C)$. Let $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$. Then $\hat{\phi} \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ and $e^{-\langle y,t\rangle}\hat{\phi} \in \mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ for $y \in T^C$ and $t \in S_A$. Since $\mathcal{Z}_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $\langle U(t), \alpha_\varepsilon(t) \cdot e^{-\langle y,t\rangle}\hat{\phi}(t)\rangle$ is well-defined. Since $\alpha_\varepsilon(t)e^{-\langle y,t\rangle}\hat{\phi}(t) \to \alpha_\varepsilon(t)\hat{\phi}(t)$ in $\mathcal{Z}_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C' \subset \mathcal{O}(C)$, for $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and $(z, t) \in$

$T^C \times S_A$,

$$\langle f(z), \phi(x)\rangle = \langle\langle U(t), e^{i\langle z, t\rangle}\rangle, \phi(x)\rangle$$
$$= \langle U(t), \alpha_\varepsilon(t)e^{-\langle y, t\rangle}\hat{\phi}(t)\rangle$$
$$\to \langle U(t), \alpha_\varepsilon(t)\hat{\phi}(t)\rangle = \langle \hat{U}(x), \phi(x)\rangle.$$

The proof is complete. □

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