Far East Journal of Mathematical Sciences (FJMS)
© 2016 Pushpa Publishing House, Allahabad, India
Published Online: May 2016
http://dx.doi.org/10.17654/MS099091399
Volume 99, Number 9, 2016, Pages 1399-1408 **ISSN: 0972-0871**

SOME RESULTS ON GRADED *N*-PRIME SUBMODULES

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Abstract

In this paper, we consider graded N-prime submodules as introduced by Sanh, and investigate their properties besides characterizations. For example, we prove that (i) if X is a fully invariant graded submodule of M, then the residual ideal of X by M is a graded ideal of S, and (ii) if M is a graded quasi-projective module, X is a graded N-prime submodule of M and $Y \subset X$ is a fully invariant graded submodule of M, then X/Y is a graded N-prime submodule of M/Y.

Also, we characterize graded N-prime submodules.

1. Introduction

Dauns introduces the notion of a prime submodule and investigates some of its properties [4]. Graded rings and graded modules have been studied by Nastasescu and Van Oystaeyen [5]. Moreover, based on the definition

Received: November 24, 2015; Revised: January 22, 2016; Accepted: February 9, 2016
2010 Mathematics Subject Classification: 16W50.
Keywords and phrases: N-prime submodule, graded N-prime submodule, graded N-prime module.
Communicated by K. K. Azad

of prime submodules in the sense of Dauns, Atani and Fazalipour have defined the graded prime submodules of graded modules and investigated some properties [1, 2]. The notion of graded primary submodules has been introduced and studied by Oral et al. [6]. Recently, Sanh [7] introduced the prime submodule of fully invariant submodule of R-module M. Let X be a fully invariant proper submodule of M. Then X is called a *prime submodule* of M if for any ideal I of S and any fully invariant submodule U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. In this paper, we use the definition of prime submodules in the sense of Sanh and call these *N-prime submodules*. Moreover, we define an N-prime submodule in graded R-modules and we call it a *graded N-prime submodule*. We prove that if X is a fully invariant graded submodule of M, then the residual ideal of X by M is a graded ideal of S. It is also shown that if M is a graded quasi-projective module, X is a graded N-prime submodule of M and $Y \subset X$ is a fully invariant graded submodule of M, then X/Y is a graded N-prime submodule of M/Y. Also, we give the characterization of graded N-prime submodule as stated in Theorem 2.2.

Let G be an abelian group with identity e and R be any ring with unit 1_R. The ring R is called a *graded ring* if $R = \oplus_{g\in G} R_g$, where R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for every g, h in G. The summands R_g's are called *homogeneous components*. Also, we write $h(R) = \bigcup_{g\in G} R_g$. If $a \in R$, then a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is a component of a in R_g. In this case, R_e is a subring of R and $1_R \in R_e$.

Let R be a graded ring and M be an R-module. We call M a *graded R-module* if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$. The $R_g M_h$ denotes the additive subgroup of M consisting of all finite sum of elements $r_g s_h$, where $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$

are called to be *homogeneous*. If M is a graded R-module, then M_g is an R_e-module for all $g \in G$. A submodule X of a graded R-module M is called a *graded submodule* of M if $X = \oplus_{g \in G} X_g$, where $X_g = X \cap M_g$ for $g \in G$. In this case, X_g is called the *g-component* of X. Moreover, M/X becomes a graded module with g-component $(M/X)_{g \in G} = ((M_g + X)/X)_{g \in G}$.

2. Main Results

Let R be a graded ring, M and N be graded R-modules and $f : M \to N$ be an R-module homomorphism. Then f is said to be a *graded R-module homomorphism* of degree k if $f(M_g) \subseteq N_{gk}$ for each $g \in G$, where $k \in G$. Graded homomorphism without an indication of degree is understood to have degree zero. Let $(END_R(M))_k$ be the set of graded module homomorphism from M to M of degree k and let $END_R(M) = \oplus_{k \in G}(END_R(M))_k$. Then $END_R(M)$ is a graded ring and $END_R(M)$ is a subring of $End_R(M)$ (see [3, Subsection 9.1, p. 303]). If G is a finite group, then $END_R(M) = End_R(M)$ (see [5]).

Let M be a graded right R-module and $S = END_R(M)$. A graded submodule X of M is called a *fully invariant graded submodule* of M if for any $s \in S$, $s(X) \subset X$. By the definition, the family of all fully invariant graded submodules of a graded module M is non-empty and closed under intersections and sums.

Let I, J be graded ideals of S and X be a graded submodule of M. We define

$$IJ = \left\{ \sum_{1 \le i \le n} x_i y_i \mid x_i \in h(I),\ y_i \in h(J),\ n \in \mathbb{N} \right\} \text{ and } I(X) = \sum_{f \in h(I)} f(X).$$

For any graded right R-module M and any graded right ideal I of graded ring R, the set MI is a fully invariant graded submodule of M (see [2, Subsection 1]).

Definition 2.1. Let M be a graded right R-module and X be a proper fully invariant graded submodule of M. Then X is called a *graded N-prime submodule* of M if for any graded ideal I of S and any fully invariant graded submodule U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$.

Especially, if we take M is the R-module R, a graded ideal P of R is a graded prime ideal if for any graded ideals I, J of R with $IJ \subset P$ implies $I \subset P$ or $J \subset P$. From now on, a graded R-module M means a graded right R-module.

The following theorem gives some characterization of graded N-prime submodule.

Theorem 2.2. *Let M be a graded R-module and X be a proper fully invariant graded submodule of M. Then the following are equivalent*:

(1) *X is a graded N-prime submodule of M.*

(2) *For any graded right ideal I of S and graded submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$.*

(3) *For any $\varphi \in h(s)$ and fully invariant graded submodule U of M, if $\varphi(U) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$.*

Proof. (1⇒2) Suppose X is a graded N-prime submodule of M. Take any graded right ideal I of S and a graded submodule U of M where $I(U) \subset X$. Since I is a graded right ideal of S, $IS \subset I$ and SI is a graded ideal of S. Since U is a graded submodule of M, $S(U)$ is a fully invariant graded submodule of M. If $I(U) \subset X$, then $(SI)(S(U)) = (SIS)(U) \subset S(I(U)) \subset S(X) \subset X$. From assumptions that X is a graded N-prime submodule of M, we have $SI(M) \subset X$ or $S(U) \subset X$. Hence, either $I(M) \subset X$ or $U \subset X$.

(2⇒3) Obvious.

(3⇒1) Take any graded ideal I of S and any fully invariant graded submodule U of M where $I(U) \subset X$. Since I is a graded ideal, I has a set of homogeneous generators. By (3), we obtain $I(M) \subset X$ or $U \subset X$. □

Let M be a graded R-module, $S = END_R(M)$ and X be a fully invariant submodule of M. We define the set $I_X = \{f \in S \mid f(M) \subset X\}$. The set I_X is a graded ideal if X is a fully invariant graded submodule as we give in the following lemma.

Lemma 2.3. *Let M be a graded R-module and* $S = END_R(M)$. *Suppose that X is a fully invariant graded submodule of M. Then the set* I_X *is a graded ideal of S.*

Proof. Take any $\varphi \in S$ and $f \in I_X$. It is clear that $(I_X, +)$ is an abelian group. Then $\varphi f(M) \subset \varphi(X) \subset X$ and $f\varphi(M) \subset f(M) \subset X$. So φf, $f\varphi$ in I_X, and we prove that I_X is an ideal of S. Furthermore, we will prove that I_X is a graded ideal of S, i.e., $I_X = \oplus_{g \in G}(I_X \cap S_g)$ for every $g \in G$. For every $g \in G$, $I_X \cap S_g \subset I_X$, so we obtain $\oplus_{g \in G}(I_X \cap S_g) \subset I_X$. Take any $f \in I_X$. Then $f = \sum_{g \in G} f_g$ and $f(M) = \left(\sum_{g \in G} f_g\right)(M) \subset X$. We will prove that $f \in \oplus_{g \in G}(I_X \cap S_g)$. It is clear that $f_g \in S_g$, so we have to prove that $f_g \in I_X$ for every $g \in G$. Without loss of generality, we may assume that $f = \sum_{i=1}^{m} f_{g_i}$, where $f_{g_i} \neq 0$ for all $i = 1, 2, ..., m$ and $f_g = 0$ for all $g \notin \{g_1, g_2, ..., g_m\}$. Since M is a graded module, we assume that $m = \sum_{j=1}^{l} m_{h_j}$, where $m_{h_j} \neq 0$ for all $j = 1, 2, ..., l$. Since $f(M) \subset X$ and $m_{h_j} \in M_{h_j} \subset M$ for all j, we obtain $f(m_{h_j}) \in X$ for all j. Then $\sum_{i=1}^{m} f_{g_i}(m_{h_j}) \in X$, where $f_{g_i}(m_{h_j}) \in M_{g_i h_j}$. Since X is a graded submodule, we obtain $f_{g_i}(m_{h_j}) \in M_{g_i h_j} \cap X \subset X$. Thus, $f_{g_i}(m_{h_j}) \in X$ for all j so $f_{g_i}(M) \subset X$ and $f_{g_i} \in I_X$ for all i, as required. □

It is worth pointing out that I_X is a graded prime if X is a graded N-prime, as we give in the following theorem.

Theorem 2.4. *Let M be a graded R-module,* $S = END_R(M)$ *and X be a fully invariant proper graded submodule of M. If X is a graded N-prime submodule, then* I_X *is a graded prime ideal of S.*

Proof. Let K, L be graded ideals of I_X such that $KL \subset I_X$. Then $KL(M) \subset I_X(M) \subset X$. If we assume that $K \not\subset I_X$, then $K(M) \not\subset X$. Since submodule X is a graded N-prime submodule, $L(M) \subset X$, so we obtain $L \subset I_X$. Thus, I_X is a graded prime ideal of S. □

We define the set $I(M) = \sum_{f \in I} f(M)$. If $I(M) \subset X$, then $I \subset I_X$ and the converse is also true as we prove in the following proposition.

Proposition 2.5. *Let M be a graded R-module, X be a fully invariant graded submodule of M and I be a graded ideal of S. Then* $I(M) \subset X$ *if and only if* $I \subset I_X$.

Proof. Take any $f \in I$, $f(M) \subseteq I(M)$. Since $I(M) \subset X$, we have $f(M) \subset X$. So we have $f \in I_X$. Conversely, consider the set $I(M)$. Since $I \subset I_X$, we have $\sum_{f \in I} f(M) \subset \sum_{f \in I_X} f(M) \subset X$. □

We conclude from Proposition 2.5 and Definition 2.1 and obtain the following theorem.

Theorem 2.6. *Let M be a graded R-module and X be a fully invariant proper graded submodule of M. Then X is a graded N-prime submodule if and only if for any graded ideal I of S and any fully invariant graded submodule U of M such that* $I(U) \subset X$ *implies* $I \subset I_X$ *or* $U \subset X$.

Proof. Let X be a graded N-prime submodule. By Definition 2.1, for any graded ideal I of S and any fully invariant graded submodule U of M such that $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. According to Proposition 2.5, $I(M) \subset X$ is equivalent to $I \subset I_X$. □

Definition and some properties of a graded *N*-prime module are given as follows.

Definition 2.7. A graded *R*-module *M* is called an *N-prime module* if 0 is a graded *N*-prime submodule of *M*.

We can characterize *N*-prime module using the annihilator as the following proposition.

Proposition 2.8. *Let M be a graded R-module and* $S = END_R(M) = \oplus_{k \in G} END_R(M)_k$. *A module M is an N-prime module if and only if* $Ann_S(M) = Ann_S(X)$ *for all nonzero graded submodules X of M.*

Proof. (⇒) Let *M* be a graded *N*-prime module. Then 0 is a graded *N*-prime submodule of *M*. Since *X* is a nonzero graded submodule of *M*, $Ann_S(M) \subseteq Ann_S(X)$. Take any $f \in Ann_S(X)$, hence $f(X) = 0$. Since *X* is a nonzero graded submodule and 0 is a graded *N*-prime submodule of *M*, we have $f(M) = 0$. Equivalently, $f \in Ann_S(M)$. So we obtain $Ann_S(M) \supseteq Ann_S(X)$ and moreover $Ann_S(X) = Ann_S(X)$.

(⇐) Take any graded ideal *I* of *S* and a nonzero fully invariant graded submodule *X* of *M* where $I(X) = 0$. Since $Ann_S(M) = Ann_S(X)$, $I(M) = 0$. So we obtain 0 is a graded *N*-prime submodule of *M*. It is proved that *M* is a graded *N*-prime module. □

Proposition 2.9. *Let M be a graded N-prime R-module. Then* $S = END_R(M) = \oplus_{k \in G} END_R(M)_k$ *is a prime ring.*

Proof. Let *M* be a graded *N*-prime module. Then 0 is a graded *N*-prime submodule of *M*. Based on Theorem 2.4, I_0 is a graded prime ideal of *S*, so *S* is a prime ring. □

The following proposition states the relations between a graded module homomorphism of M and a graded module homomorphism of M/Y.

Proposition 2.10. *Let M be a graded module, Y be a fully invariant graded submodule of M. If $f : M \to M$ is a graded module homomorphism of degree zero, then $\varphi : M/Y \to M/Y$ with $\varphi(m+Y) = f(m) + Y$ is a graded module homomorphism of degree zero.*

Proof. (i) We will show that φ is a mapping. Take any $m_1 + Y$, $m_2 + Y \in M/Y$ with $m_1 + Y = m_2 + Y$, so $m_1 - m_2 \in Y$. Since Y is a fully invariant graded submodule of M, $f(m_1 - m_2) = f(m_1) - f(m_2) \in Y$, it means $f(m_1) + Y = f(m_2) + Y$. In other words, $\varphi(m_1 + Y) = \phi(m_2 + Y)$.

(ii) It is clear that φ is a module homomorphism.

(iii) We show that ϕ is a graded module homomorphism of degree zero. Take any $m_g + K \in (M_g + Y)/Y$ for some $g \in G$, a homogeneous element of degree g in M/Y. We will prove that $\varphi(M_g + Y) \in (M_g + Y)/Y$. Based on definition of φ, $\varphi(M_g + Y) = f(M_g) + Y$. Since f is a graded module homomorphism of degree zero, $f(m_g) \in M_g$. In other words, $\varphi(m_g + Y) \in (M_g + Y)/Y$. It is proved that $\varphi((M_g + Y)/Y) \subseteq (M_g + Y)/Y$ or φ is a graded module homomorphism of degree zero. □

We will look more closely at the properties of graded N-prime submodule of quotient module.

Lemma 2.11. *Let M be a graded module, X, Y be graded submodules of M and $Y \subset X$. Then X/Y is a graded submodule of M/Y.*

Proof. It is clear that X/Y is a submodule of M/Y. Furthermore, we will show that X/Y is a graded submodule. It means, we will show that $X/Y = \oplus_{g\in G} X/Y \cap (M/Y)_g$. The condition $\oplus_{g\in G} X/Y \cap (M/Y)_g \subseteq X/Y$

is obvious. Let $\overline{m} = \sum_{g \in G} \overline{m}_g \in X/Y$. It is sufficient to show that $\overline{m}_g = m_g + Y \in X/Y \cap (X/Y)_g$ for each $g \in G$. Since X is a graded submodule of M, $m_g \in X \cap M_g$ for each $g \in G$, so $m_g \in X$ and $m_g \in M_g$. Then $m_g + Y \in X/Y$ and $m_g + Y \in (M_g + Y)/Y = (M/Y)_g$. Hence, $\overline{m}_g = m_g + Y \in X/Y \cap (M/Y)_g$. □

Theorem 2.12. *Let M be a graded quasi-projective module, X be a graded N-prime submodule of M and* $Y \subset X$ *be a fully invariant graded submodule of M. Then* X/Y *is a graded N-prime submodule of* M/Y.

Proof. Let $\overline{S} = END_R(M/Y)$. Let φ be a homogeneous element of degree zero in $\overline{S}$ and U/Y be a fully invariant graded submodule of M/Y with $Y \subset U$ and $\varphi(U/Y) \subset X/Y$. Since M is quasi-projective, we can find $f \in h(S) = h(END_R(M))$, f is a homogeneous element of degree zero in S such that $\varphi v = v f$, where $v : M \to M/Y$ is the graded canonical projection. Then $\varphi(U/Y) = \varphi v(U) = v f(U) = (f(U) + Y)/Y \subset X/Y$. It follows that $f(U) \subset X$. Since Y is a fully invariant graded submodule of M and U/Y is a fully invariant graded submodule of M/Y, U is a fully invariant graded submodule of M. By the primeness of X, we have $f(M) \subset X$ or $U \subset X$. Thus, $(f(M) + Y)/Y = v f(M) = \varphi v(M) = \varphi(M/Y) \subset X/Y$ or $U/Y \subset X/Y$, that is, X/Y is a graded N-prime submodule of M/Y. □

Acknowledgements

This work was supported by Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta, Indonesia.

The authors also thank the anonymous referees for their valuable suggestions which let to the improvement of the manuscript.

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