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BORELIAN DENSITIES IN NUMBER THEORY: BORELIAN DENSITIES OF SUBSETS OF $\mathbb{N}$

Noureddine Daili

Department of Mathematics
Cité Des 300 Lots, Yahiaoui
51 Rue Harrag Senoussi
19000, Sétif, Algeria
e-mail: nourdaili_dz@yahoo.fr

Abstract

In this paper, we introduce Borelian density of non-empty subset E of the set $\mathbb{N}$ of natural numbers and make comparative study of this density with others. Further, we provide certain criteria and also the applications of this newly proposed density.

1. Introduction

The probabilistic Poisson law is the law of random variables which count the number of occurrences of a rare event. Its domain of application was limited for a long time to that of rare events. But, for some decades, its field of application considerably widened. It got used in telecommunications, the statistical quality control, study of the phenomena connected to the radioactive splitting, the biology, the medicine, the meteorology, the industry, and elsewhere.

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The network of Poisson probability is met under diverse forms. That is, in several contexts. It is defined for all the subsets of all the sets $\mathbb{N} = \{0, 1, 2, ...\}$.

Throughout the paper, E denotes a nonempty subset of the set $\mathbb{N}$, and I_E denotes the indicator function on E.

In this work, we consider the family of measures of probability on $\mathbb{N}$, defined by

$$b_\lambda(E;\ k) = \frac{1}{k!}\lambda^k e^{-\lambda} I_E(k), \quad k \in \mathbb{N}.$$

Definition 1.1. The family of Borelian or Poisson probability measures on the integers is defined as follows:

$$b_\lambda(E;\ k) = \begin{cases} \frac{1}{k!}\lambda^k e^{-\lambda} I_E(k), & \text{for } k = 0, 1, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

These can be made diffuse by letting $\lambda \nearrow +\infty$.

We calculate the measure of probability of E. Then we diffuse this measure by passing onto the limit on λ, as λ goes to $+\infty$, namely, $\lambda \nearrow +\infty$. We obtain what we shall call the *Borelian density*.

We then study the link of these densities, for E, with the binomial, asymptotic and H^p, $p \in \mathbb{N}^*$ densities.

The Borelian density related to the Poisson probabilistic law is strictly defined in the next section.

2. Main Results

2.1. Borelian density of subset E of $\mathbb{N}$

We calculate the measure of probability of E. Then we diffuse this measure by passing onto the limit on λ, as λ goes to $+\infty$, namely, $\lambda \nearrow +\infty$. Thus, we obtain the *Borelian density*.

We then study the link of these densities with the binomial, asymptotic and H^p, $p \in \mathbb{N}^*$ densities introduced in [3, 6, 7] and [9]. More precisely, we show that the binomial density implies the Borelian density for *E*. The converse is false, in general. So, if we denote a class of subsets of $\mathbb{N}$ which admits binomial density by $\mathfrak{B}$ and a class of subsets of $\mathbb{N}$ which admits the Borelian density by $\mathfrak{P}$, then we obtain a strict inclusion $\mathfrak{B} \subset \mathfrak{P}$.

Definition 2.1. Let

$$B_\lambda(E) = e^{-\lambda} \sum_k \frac{\lambda^k}{k!} I_E(k).$$

We say that *E* has the number ℓ as the Borelian density related to the Poisson measure probabilistic law if $\lim\limits_{(\lambda \nearrow +\infty)} B_\lambda(E)$ exists and equals ℓ as $(\lambda \nearrow +\infty)$. If this is the case, then we shall denote this density by $\beta(E)$ and write

$$\lim_{(\lambda \nearrow +\infty)} B_\lambda(E) = \ell = \beta(E).$$

We write $\mathfrak{P}$ for a class of subsets of $\mathbb{N}$ which having the Borelian density.

Theorem 2.1 (Comparison between densities)**.** *If E admits a binomial density* $\mathfrak{b}(E) = \ell$ [3], *then it has the Borelian density* $\beta(E) = \ell$.

Proof. Suppose *E* has a binomial density ℓ. Then

$$\lim_{(n \to +\infty)} B_n(E;\, p) = \ell = \mathfrak{b}(E).$$

We have

$$e^{-\lambda} \sum_k \frac{\lambda^k}{k!} I_E(k) \to \ell \text{ as } (\lambda \nearrow +\infty)$$
$$\Updownarrow$$
$$e^{-p\lambda} \sum_k \frac{(p\lambda)^k}{k!} I_E(k) \to \ell \text{ as } (\lambda \nearrow +\infty).$$

Moreover,

$$e^{q\lambda}\sum_k \frac{(p\lambda)^k}{k!} I_E(k) = \left(\sum_j \frac{(q\lambda)^j}{j!}\right)\left(\sum_k \frac{(p\lambda)^k}{k!} I_E(k)\right) = \sum_n c(n)\lambda^s,$$

where

$$c(n) = \frac{q^n}{n!} I_E(0) + \frac{pq^{n-1}}{(n-1)!} I_E(1) + \cdots + \frac{p^n}{n!} I_E(n)$$

or

$$c(n) = \frac{1}{n!} B_n(E;\ p).$$

Therefore,

$$e^{-p\lambda}\sum_k \frac{(p\lambda)^k}{k!} I_E(k) = e^{-p\lambda}e^{-q\lambda}e^{q\lambda}\sum_k \frac{(p\lambda)^k}{k!} I_E(k)$$

$$= e^{-\lambda}\sum_n B_n(E;\ p)\frac{\lambda^n}{n!}.$$

But

$$\lim_{(n\to+\infty)} B_n(E;\ p) = \ell,$$

according to the Toeplitz lemma (see [10, Chap. 3]). Hence

$$e^{-\lambda}\sum_n B_n(E;\ p)\frac{\lambda^n}{n!} \to \ell \text{ as } (\lambda \nearrow +\infty).$$

So, the subset E has the Borelian density $\beta(E) = \ell$. □

Remark 2.1. It is not well known, at the moment, if the binomial density is strictly weaker than that of the Borelian density.

Proposition 2.2. *The set E of natural integers of multiples of m has the Borelian density $\beta(E)$ and $\beta(E) = \frac{1}{m}$.*

Proof. Follows from Proposition 2.5 [3, p. 138] and Theorem 2.1. □

Definition 2.2. We have

(a) Césaro summability means that

$$\sum_{k} b(k) = \ell(C;\ s);$$

(b) Hölder's summability means that

$$\sum_{k} b(k) = \ell(H;\ s).$$

We need following theorems connecting the asymptotic density and the H^{p}-density for $p \in \mathbb{N}^{*}$:

Theorem 2.3 [6, 7: Theo. 2.1, p. 63]**.** *For* $\ell \in [0, 1]$, *the following two statements are equivalent*:

(p_1) *E admits* ℓ *as* H^{1}*-density and* $(d_1(E) = \ell)$;

(p_2) *E admits* ℓ *as* H^{p}*-density for all* $p \in \mathbb{N}^{*}$ *and* $(d_{p}(E) = \ell)$.

More generally, we have the following theorem:

Theorem 2.4. *For* $\ell \in [0, 1]$ *and* $s \in \mathbb{N}^{*}$, *the following two statements are equivalent*:

(p_1) *E admits* ℓ *as an asymptotic density of order s and* $(d_{s}^{C}(E) = \ell)$;

(p_2) *E admits* ℓ *as* H^{s}*-density and* $(d_{s}^{H}(E) = \ell)$.

Theorem 2.5 [10, p. 211]**.** *If* $b(n) = o(\sqrt{n})$ *and* $\sum_{k} b(k) = \ell(B) = \beta(E)$, *then*

$$\sum_{k} b(k) = \ell(C;\ 2) = d_{2}^{C}(E).$$

Theorem 2.6. *If E admits the Borelian density* $\beta(E)$, *then it admits an asymptotic density* $d(E)$ *and the two limit values are equal, namely,* $\beta(E) = d(E) = \ell$.

Proof. From Borel summability,

$$\sum_k b(k) = \ell(B).$$

This implies that

$$e^{-\lambda} \sum_k B(k) \frac{\lambda^k}{k!} \to \ell \text{ (as } \lambda \nearrow +\infty),$$

where

$$B(k) = \sum_{i=1}^{k} b(i).$$

Thus, E has an asymptotic density ℓ if and only if

$$\sum_k b(k) = \ell(C; 1) = d(E),$$

where

$$b(k) = \Delta a(k) = a(k) - a(k-1).$$

To see this, consider

$$\sum_k b(k) = \ell(C; 1) \Leftrightarrow \frac{\sum_{k=0}^{n} B(k)}{n+1} \to \ell \Leftrightarrow \frac{\sum_{k=0}^{n} b(k)}{n+1} \to \ell.$$

Suppose E admits the Borelian density ℓ. Because $B(k) = a(k)$,

$$e^{-\lambda} \sum_k B(k) \frac{\lambda^k}{k!} \to \ell \text{ (as } \lambda \nearrow +\infty).$$

Also,

$$b(n) = 0 \text{ or } +1.$$

Thus, from Theorems 2.3 and 2.4, we have

$$\sum_k b(k) = \ell(C;\ 2),$$

and hence

$$\sum_k b(k) = \ell(H;\ 2).$$

Finally, according to [6, Theorem 2.1, p. 63] or Theorem 2.3 above, we have

$$\sum_k b(k) = \ell(C; 1).$$

Thus, E admits an asymptotic density. □

2.2. Applications

Criterion. Let $E = \bigcup_{n \geq 1} [p_n,\ q_n[$ be a subset of $\mathbb{N}^*$ neither finite nor cofinite and

$$\rho_n = q_n - p_n, \quad \sigma_n = q_n - q_{n-1} \quad (n \geq 1)\ (q_0 = 1).$$

Let the following properties hold:

(p_1) $p_n \sim q_{n-1}$ as $(n \to +\infty)$;

(p_2) $\dfrac{\rho_n}{\sigma_n} \to \ell$ as $(n \to +\infty)$.

Then E admits the Borelian density $\beta(E)$ and $\beta(E) = \ell.$ If $\ell = 0,$ then the only condition (p_2) implies that $\beta(E) = 0.$

As applications, we have the following:

Proposition 2.7. *Let* $E = \bigcup_{k \geq 1} [P(k), Q(k)[$, *where* $P(k)$ *and* $Q(k)$ *are polynomials with positive integral coefficients in k, defined by*

$$\begin{cases} P(k) = ak^n + bk^{n-1} + o(k^{n-1}), \\ Q(k) = ak^n + ck^{n-1} + o(k^{n-1}), \end{cases}$$

where $n \in \mathbb{N}^*$ *is the common degree of P and Q.*

Suppose that

$$0 < \frac{c - b}{na} < 1.$$

Then E admits the Borelian density $\beta(E)$ *and* $\beta(E) = \dfrac{c - b}{na}$.

Proof. Follows from previous Criterion, Proposition 4.1 [7, p. 527] and the above theorem (Theorem 2.6). □

We have the following particular case of Proposition 2.7.

Proposition 2.8. *Let* $E = \bigcup_{n \geq 1} [2n - 1, 2n[$ *be the set of odd integers. Then E admits the Borelian density* $\beta(E)$ *and* $\beta(E) = \dfrac{1}{2}$.

Proof. For this purpose, we put

$$\begin{cases} p_n = 2n - 1, \quad q_n = 2n, \\ \rho_n = q_n - p_n = 1, \\ \sigma_n = q_n - q_{n-1} = 2, \end{cases}$$

and verify that

(p_1) $p_n \sim q_{n-1}$ as $(n \to +\infty)$;

(p_2) $\dfrac{\rho_n}{\sigma_n} \to \dfrac{1}{2}$ as $(n \to +\infty)$. □

Proposition 2.9. *Let* E_2 *be the set of square-free numbers. Then* E_2 *has the Borelian density* $\beta(E_2)$ *and* $\beta(E_2) = \frac{1}{\zeta(2)}$.

Proof. Follows from Proposition 2.8 [3, p. 140] and Theorem 2.1. □

Remark 2.2. This gives a new proof of the well known fact [11, Theorem 333, p. 269] that the square-free numbers have an asymptotic density $d(E_2) = \frac{6}{\pi^2} = \frac{1}{\zeta(2)}$.

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