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DEDEKIND DOMAINS AND DEDEKIND MODULES

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Abstract

This work studies an interconnection between Dedekind domains and Dedekind modules. The interconnection will be investigated as an adoption of Noetherian and hereditary properties. Particularly, we show that a domain is a Dedekind domain if and only if its finitely generated torsion free uniform modules are Dedekind. The obtained result includes the case of noncommutative rings.

1. Introduction

The first study of Dedekind notion in the module theory was documented in Naoum and Al-Alwan [6]. It introduced Dedekind modules by generalizing the concept of invertible ideals to invertible submodules. Let R be a commutative ring with identity and M be an R-module.

Let the sets

$$S = \{r \in R \mid rs = 0, s \in R \Rightarrow s = 0\}$$

$$T = \{t \in S \mid rm = 0, m \in M \Rightarrow m = 0\}$$

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denote respectively the set of all regular elements of R and a multiplicative subset of it. We have the set $RT^{-1} = \{rt^{-1} \mid r \in R,\ t \in T\}$ is a quotient ring of R. According to Naoum and Al-Alwan [6], a nonzero submodule N of an R-module M is said to be *invertible* if $N^{-1}N = M$ where $N^{-1} = \{x \in RT^{-1} \mid xN \subseteq M\}$, and the R-module M is called a *Dedekind module* if every nonzero submodule of M is invertible.

Following the above definition several authors were interested in finding properties of invertible submodules and Dedekind modules. For instance, Alkan et al. [1] introduced the concept of integrally closed modules and used it to characterize Dedekind modules and Dedekind domains. In another work, Saraç et al. [7] investigated relations between a finitely generated torsion free Dedekind module and a prime module over an order of the Dedekind module.

Other researchers generalized the properties of Dedekind rings to analyze properties of Dedekind modules. One of these results was obtained by Garminia et al. [2]. They proved that every Dedekind projective R-module is an HNP (hereditary, Noetherian, and prime) module.

Proceeding the above results, in this article we discuss a characterization of Dedekind modules by adapting Noetherian and hereditary concepts. In Noetherian area, it is known that a ring R is a Noetherian ring if and only if every finitely generated R-module is a Noetherian module. Similar to the result, a characterization of hereditary ring is related to its projective modules; a ring R is a hereditary ring if and only if every projective R-module is a hereditary module. Due to those two properties, it is natural to investigate connections between Dedekind modules and Dedekind domains. One may conjecture that an integral domain D is a Dedekind domain if and only if every finitely generated torsion free uniform D-module is a Dedekind module. In this article, we present the answer of this conjecture.

Throughout this note, D will denote an integral domain, $S = D - \{0\}$ the set of all nonzero elements of D, and $K = DS^{-1} = \{ab^{-1} \mid a,\ b \in D,\ b \in S\}$

be the field of fractions of D, and M be a D-module. The set $\langle x_1, x_2, ..., x_n\rangle$ denotes the submodule N of M generated by $\{x_1, x_2, ..., x_n\} \subseteq M$, that is $N = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \,|\, \alpha_i \in D\}$. We follow [6] in using the definitions of invertible submodules and Dedekind modules for the commutative ring case. On the noncommutative term, we define a rather different definition of invertible submodules in the third section.

2. Ideals and the Field of Fractions of a Commutative Dedekind Domain

In this section we clarify some characterizations of ideals and the field of fractions of a commutative Dedekind domain. We will restrict our attention to fractional ideals and finitely generated ideals of a commutative Dedekind domain D regarded as D-modules. The obtained properties will be needed to analyze the main result in the third section.

Lemma 2.1. *Let D be a commutative Dedekind domain. Every nonzero ideal I of D regarded as a D-module is Dedekind.*

Proof. Consider an ideal I of D as a D-module. Let L be a nonzero submodule of I. Then L is an ideal of D. Note that $L_D^{-1} = \{x \in K \,|\, xL \subseteq D\}$ is the inverse of L as an ideal of D and $L_I^{-1} = \{x \in K \,|\, xL \subseteq I\}$ is the inverse of L as a submodule of I. Since D is a Dedekind domain, we have $L_D^{-1}L = D$. It is easy to check that $L_D^{-1}I = L_I^{-1}$. We thus get $L_I^{-1}L = (L_D^{-1}I)L = (L_D^{-1}L)I = DI = I$. That is L as a submodule of I is invertible. □

Now let N be a nonzero submodule of the field of fractions K regarded as a D-module. By definition, we see at once that $N^{-1} = \{x \in K \,|\, xN \subseteq K\} = K$. For any nonzero elements ab^{-1} in K and cd^{-1} in N, we have $ad(bc)^{-1}cd^{-1} = ab^{-1}$. This clearly force $N^{-1}N = KN = K$. This fact provides the following result.

Lemma 2.2. *For any integral domain D, the field of fractions K is a Dedekind D-module.*

Now we recall another relation between a Dedekind module and its field of fractions (or the total quotient ring) which was stated by Alkan et al. [1] as follows.

Lemma 2.3. *Let M be a Dedekind module and let m be a nonzero element of M. Then M is isomorphic to the R-submodule* $(Rm)^{-1}$ *of* RT^{-1}.

The result ensures that every Dedekind torsion free D-module M is isomorphic to a D-submodule of field of fractions K. This lemma makes it legitimate to study the properties of Dedekind modules by observing the class of submodules of K.

One may ask which class of D-submodules of K are Dedekind D-modules. To answer this question, we begin with the class of fractional D-ideals.

Definition 2.4. Let R be a ring and RS^{-1} be a quotient ring of R. A submodule I of RS^{-1} is a fractional right R-ideal if the following conditions hold:

- $aI \subseteq R$,
- $bR \subseteq I$

for some units a, b in RS^{-1}.

McConnell and Robson [4] introduced and studied properties of inverse of fractional ideals. This inspires to the investigation of invertibility of any fractional ideals.

Lemma 2.5. *Let D be a commutative Dedekind domain. Every fractional D-ideal I viewed as D-module is Dedekind.*

Proof. Suppose that J is a submodule of I. We have $aI \subseteq D$ and $bD \subseteq I$ for some units $a, b \in K$. Note that aI is an ideal of D and so aI is a Dedekind D-module.

Hence aJ, being a submodule of aI, is invertible. Denote $(aJ)_{aI}^{-1} =$

$\{x \in K \mid xaJ \subseteq aI\}$. We have $(aJ)_{aI}^{-1}(aJ) = aI$. Meanwhile the inverse of J as a D-submodule of I is

$$\begin{aligned} J^{-1} &= \{x \in K \mid xJ \subseteq I\} \\ &= \{x \in K \mid xaJ \subseteq aI\} \\ &= (aJ)_{aI}^{-1}. \end{aligned}$$

Hence $J^{-1}aJ = (aJ)_{aI}^{-1}aJ = aI$. Which results in $J^{-1}J = I$. □

The next interest is to investigate the class of finitely generated D-submodules of K and their relation with the class of fractional D-ideals where D is Dedekind.

Let I be a fractional D-ideal. We thus have $aI \subseteq D$ and $bD \subseteq I$ for some units $a, b \in K$. It is a fact that any ideal in a Dedekind domain D is generated by two elements. Hence aI is an ideal of the form $aI = \langle n_1, n_2 \rangle$, for some $n_1, n_2 \in aI$. For every $am \in aI$, we obtain $am = \alpha_1 n_1 + \alpha_2 n_2$, for some $\alpha_1, \alpha_2 \in D$. Therefore $m = \alpha_1 \frac{n_1}{a} + \alpha_2 \frac{n_2}{a}$ and $I = \left\langle \frac{n_1}{a}, \frac{n_2}{a} \right\rangle$. This yields I is a finitely generated D-module. Thus any fractional D-ideal is finitely generated D-module.

Conversely, let L be a finitely generated submodule of K.

We have $L = \left\langle \frac{x_1}{y_1}, \frac{x_2}{y_2}, ..., \frac{x_n}{y_n} \right\rangle$, for some $x_i, y_i \in D$, $y_i \neq 0$.

For any $\alpha_i \in D$, we obtain

$$y_1 y_2 \cdots y_n \left(\alpha_1 \frac{x_1}{y_1} + \alpha_2 \frac{x_2}{y_2} + ... + \alpha_n \frac{x_n}{y_n} \right)$$

$$= \alpha_1 x_1 \prod_{i \neq 1} y_i + \alpha_2 x_2 \prod_{i \neq 2} y_2 + ... + \alpha_n x_n \prod_{i \neq n} y_n \in D.$$

For every $u \in D$, we get

$$\left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + \ldots + \frac{x_n}{y_n}\right)u$$

$$= \left(\frac{x_1 \prod_{i \neq 1} y_1 + x_2 \prod_{i \neq 2} y_2 + \ldots + x_n \prod_{i \neq n} y_n}{y_1 y_2 \cdots y_n}\right)u \in L.$$

By choosing

$$a = y_1 y_2 \cdots y_n$$

and

$$b = \frac{x_1 \prod_{i \neq 1} y_1 + x_2 \prod_{i \neq 2} y_2 + \ldots + x_n \prod_{i \neq n} y_n}{y_1 y_2 \cdots y_n}$$

we may get $aL \subseteq D$ and $bD \subseteq L$.

Finally, *L* is a fractional *D*-ideal.

Above relation between fractional *D*-ideals and finitely generated *D*-submodules of *K* is summarized in the following proposition.

Proposition 2.6. *Let D be a commutative Dedekind domain and K be the quotient field of D. A submodule I of K is a fractional D-ideal if and only if I as a D-submodule of K is finitely generated.*

3. Main Results

This section discusses a relation between Dedekind property of modules and Dedekind domains. A part of this relation was discussed by Khoramdel and Hesari [3]. However our results focus on the property of a class of modules being Dedekind and the property of its underlying ring being Dedekind. In addition we show that torsion-freeness of modules is necessary for being Dedekind, hence the class of modules under consideration is torsion free.

This section is divided into two subsections. In the first subsection we restrict our discussion for the case the underlying domain is commutative. We show that uniform property is a necessary condition for any module being Dedekind. Then we use this class of modules to investigate our main result. In the second subsection we generalize the obtained results to the case of noncommutative rings.

3.1. Commutative Dedekind domains

By analyzing properties of invertible submodules, we find a fact that every invertible submodule is an essential submodule. We know that the concept of essential submodules is related to uniform modules. This fact provides a relation between Dedekind modules and uniform modules.

Lemma 3.1.1. *Every Dedekind module is a uniform module.*

Proof. Let M be a Dedekind D-module and $m_1, m_2 \in M$. Denote $N = \langle m_1 \rangle$. Since M is Dedekind, $m_2 = \sum_{i=1}^{k} \alpha_i n_i$, for some $\alpha_i = \frac{r_i}{t_i} \in N^{-1}$, $n_i \in N$. Thus $m_2 = \sum_{i=1}^{k} \alpha_i s_i m_1 = \left(\sum_{i=1}^{k} \frac{r_i}{t_i} s_i\right) m_1$, for some $r_i, s_i \in D$, $t_i \in T$ which implies $\left(\prod_{i=1}^{k} t_i\right) m_2 = \left(\sum_{i=1}^{k} \left(\prod_{j=1,\, j \neq i}^{k} t_j\right) r_i s_i\right) m_1$ that the set $\{m_1, m_2\}$ is linearly dependent. As a consequent M is uniform. □

The above lemma provides a criterion of Dedekind modules. That is the class of Dedekind modules is included in the class of uniform modules. By this fact, for the rest of this paper, we restrict our discussion to the class of uniform submodules.

Lemma 3.1.2. *Let D be a principal ideal domain* (*PID for short*) *and P be a nonzero prime ideal in D. For any positive integer e, the D-module $\frac{D}{P^e}$ is not Dedekind.*

Proof. Consider $N = \frac{DP}{P^e}$ as a D-submodule of $\frac{D}{P^e}$. It can be shown that for any $x \in DT^{-1}$, $xN \subseteq \frac{D}{P^e}$ implies $x \in D$. Hence $N^{-1} = D$ and $N^{-1}N = DN = N \subsetneq \frac{D}{P^e}$. So, the D-submodule $\frac{DP}{P^e}$ is not invertible. □

Corollary 3.1.3. *Let D be a commutative Dedekind domain and M be a finitely generated torsion D-module. If M is not simple, then M is not Dedekind.*

Proof. Without loss of generality suppose $M = \frac{D}{P^e}$ for some $P \subseteq D$ a prime ideal and e a positive integer [5]. Thus, M is not Dedekind by Lemma 3.12. □

The results above lead us to our main theorem. This theorem expresses an interconnection between Dedekind domain and Dedekind module.

Theorem 3.1.4. *A ring D is a commutative Dedekind domain if and only if every finitely generated torsion free uniform D-module is Dedekind.*

Proof. By regarding D as a D-module, it is clear that D is finitely generated, torsion free, and uniform. Hence D is Dedekind. Conversely, let D be a commutative Dedekind domain and M be a finitely generated torsion free uniform D-module. Note that MS^{-1} is a vector space over K with $\dim(MS^{-1}) = 1$. The transformation $\varphi : MS^{-1} \to K$ with $\varphi(ms^{-1}) = 1$ is a K-isomorphism, particularly being a D-isomorphism. Hence $MS^{-1} \approx K$ and $\varphi(M) \approx M$ as D-modules. We obtain $\varphi(M)$ is a finitely generated submodule of K. By Proposition 2.6 and Lemma 2.5, $\varphi(M)$ is a Dedekind D-module, which implies M is a Dedekind D-module. □

The above paragraph clarifies a whole class of Dedekind module over a commutative Dedekind domain. The explanation provides us that every module of a commutative Dedekind domain needs to be torsion free for being Dedekind.

3.2. Dedekind domains

In this part, we generalize the obtained characterization above to the class of modules such that the underlying ring is noncommutative. First, we need to define the concept of Dedekind module over a domain (do not need to be commutative).

Let R be a domain and M be an R-module. That means the set of all regular elements is $S = \{r \in R \,|\, rs = 0,\ s \in R \Rightarrow s = 0\} = R - \{0\}$ and let $T = \{t \in S \,|\, rm = 0,\ m \in M \Rightarrow m = 0\}$. Hence we have the set $RT^{-1} = \{rt^{-1} \,|\, r \in R,\ t \in T\}$ is a quotient ring of R. Adopting the definition of the inverse of ideals in [4], for any submodule N of M we define the left and right inverses of N as follows.

$$N_l^{-1} = \{x \in RT^{-1} \,|\, xN \subseteq M\}$$

$$N_r^{-1} = \{x \in RT^{-1} \,|\, Nx \subseteq M\}.$$

A nonzero submodule N of M is said to be *invertible* if $N_l^{-1}N = M = NN_r^{-1}$ and an R-module M is called a *Dedekind module* if every nonzero submodule of M is invertible.

Applying the above definition we examine the previous properties in the section 2 and subsection 3.1 for the class of noncommutative rings. This leads us to the following results.

Lemma 3.2.1. *Let R be a Dedekind domain. Every ideal* (*two sided ideal*) *I of R regarded as an R-module is Dedekind*.

Proof. Consider R as an R-module. Since R is a domain, we have $T = S$ and RS^{-1} is the total quotient ring of R. Consider an ideal I as an R-module. Let L be a submodule of I. Then L is an ideal of R. The set $L_{l_R}^{-1} = \{x \in$

$RS^{-1} \mid xL \subseteq R\}$ is the left inverse of L as an ideal of R and $L_{l_I}^{-1} = \{x \in RS^{-1} \mid xL \subseteq I\}$ is the left inverse of L as a submodule of I. The similar definition for the right. We see at once that $IL_{l_R}^{-1} = L_{l_I}^{-1}$ and $L_{r_R}^{-1}I = L_{r_I}^{-1}$. This clearly leads to $L_{l_I}^{-1}L = IL_{l_R}^{-1}L = IR = I$ and $LL_{r_I}^{-1} = LL_{r_R}^{-1}I = RI = I$. So, L is an invertible submodule of I. □

Lemma 3.2.2. *For any Dedekind domain R, the quotient ring* RS^{-1} *is a Dedekind R-module.*

The proof of the above lemma is straightforward. It follows easily that the left and right inverses of any submodule of the quotient ring RS^{-1} is RS^{-1} itself.

Replacing the class of Dedekind modules by uniform modules we can generalize Lemma 2.3 as follows.

Lemma 3.2.3. *Let R be a domain and M be a torsion free uniform R-module. Then M is isomorphic to a submodule of the quotient ring* RS^{-1}.

Proof. Let $0 \neq n \in M$. For any $0 \neq m \in M$ there are non-zero elements $\alpha, \beta \in R$ such that $\alpha m = \beta n$ or $m = \frac{\beta}{\alpha} n$. By an R-monomorphism $f : M \to RS^{-1}$ defined by $f(m) = \frac{\beta}{\alpha}$ we get $M \approx f(M)$, and the lemma is proved. □

Lemma 3.2.4. *Let R be a Dedekind domain. Every fractional R-ideal* (*two sided fractional R-ideal*) *I viewed as an R-module is Dedekind.*

Proof. Let J be a submodule of I. We have $Ia \subseteq R$ and $bI \subseteq R$ for some units $a, b \in RS^{-1}$. Both Ia and bI are ideals of R and so they are Dedekind R-modules. It is clear that $(Ja)_{l_{Ia}}^{-1} \subseteq J_{l_I}^{-1}$ and $(bJ)_{r_{bI}}^{-1} \subseteq J_{r_I}^{-1}$. As

Ia and bI are Dedekind R-modules we have $Ia = (Ja)^{-1}_{l_{Ia}}(Ja) \subseteq J^{-1}_{l_I}(Ja)$ and $bI = (bI)(bJ)^{-1}_{r_{bI}} \subseteq (bJ)J^{-1}_{r_I}$. Therefore $I \subseteq J^{-1}_{l}J$ and $I \subseteq JJ^{-1}_{r}$. This is the desired conclusion. □

The same manner as the proof of Proposition 2.6 enables us to write the following properties.

Proposition 3.2.5. *Let R be a Dedekind domain and* RS^{-1} *be the quotient ring of R. A submodule I of* RS^{-1} *is a fractional R-ideal if and only if I is finitely generated.*

We can now use the above results to formulate our last main theorem. The proof is immediate by applying the above properties.

Theorem 3.2.6. *A ring R is a Dedekind domain if and only if every finitely generated torsion free uniform R-module is a Dedekind module.*

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