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QUASITRIANGULAR STRUCTURES ON POINTED HOPF ALGEBRAS OF RANK ONE

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Abstract

In this paper, the quasitriangular structures on a finite dimensional pointed Hopf algebra of rank one are investigated. A sufficient and necessary condition for a finite dimensional pointed Hopf algebra of rank one to be quasitriangular is given. As an example, all quasitriangular structures on a finite dimensional pointed Hopf algebra of rank one such that the group of group-like elements is cyclic are completely described. In particular, quasitriangular structures on Sweedler's four dimensional Hopf algebra are recovered.

1. Introduction

Quasitriangular Hopf algebras constitute a very important class of Hopf

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algebras, which were introduced by Drinfeld [5] in order to supply solutions to the quantum Yang-Baxter equation that arises in mathematical physics. The finite dimensional representations of quasitriangular Hopf algebras form a braided rigid tensor category, which are naturally related to low dimensional topology. Furthermore, Drinfeld showed that any finite dimensional Hopf algebra can be embedded in a finite dimensional quasitriangular Hopf algebra, known as its Drinfeld double or quantum double.

A general classification of quasitriangular Hopf algebras is not known yet. However, the problem was solved for triangular Hopf algebras in the semisimple case [8], in the minimal triangular pointed case [11], and more generally for triangular Hopf algebras with the Chevalley property [9].

A family of finite dimensional pointed Hopf algebras, called *finite dimensional pointed Hopf algebras* of rank one, has been studied by many authors. The classification of this family of Hopf algebras over an algebraically closed field $\mathbb{k}$ of characteristic 0 is described by means of a group datum [17], see also [3]. Indeed, a quadruple $\mathcal{D} = (G, \chi, g, \mu)$ is called a *group datum*, if G is a finite group, χ is a $\mathbb{k}$-linear character of G, g is a central element of G, and $\mu \in \mathbb{k}$ satisfying some conditions (see Definition 2.1 below). Given a group datum $\mathcal{D}$, there is a finite dimensional pointed Hopf algebra of rank one, denoted $H_{\mathcal{D}}$, associated to $\mathcal{D}$. Conversely, every finite dimensional pointed Hopf algebra of rank one has the form $H_{\mathcal{D}}$. The representations of $H_{\mathcal{D}}$ have been carried out in [3, 17, 20]. The Hopf algebra $H_{\mathcal{D}}$ is a Nakayama algebra of finite representation type. It is neither unimodular nor symmetric, and it has the Chevalley property if the group datum $\mathcal{D}$ is of nilpotent type (see [20] for details).

In this paper, we focus on quasitriangular structures on $H_{\mathcal{D}}$. As we shall see the quasitriangular structures on $H_{\mathcal{D}}$ are determined largely by the order of $\chi(g)$. If the order of $\chi(g)$ is equal to 1, then $H_{\mathcal{D}}$ is nothing but a group algebra $\mathbb{k}G$. If the order of $\chi(g)$ is greater than 2, then there is no quasitriangular structures on $H_{\mathcal{D}}$. If the order of $\chi(g)$ is equal to 2, then a sufficient and necessary condition for $H_{\mathcal{D}}$ to be quasitriangular is given. In the case where the group G in the group datum $\mathcal{D}$ is cyclic, we completely determine the quasitriangular structures on $H_{\mathcal{D}}$. Especially, we recover all quasitriangular structures on Sweedler's four dimensional Hopf algebra.

2. Preliminaries

In this section, we first recall the construction and classification of any finite dimensional pointed Hopf algebra of rank one in terms of group datum. We then give basic definitions and results about quasitriangular Hopf algebras used in this paper.

A Hopf algebra H is called *pointed* if all its simple left or right comodules are one dimensional. This is equivalent to saying that the coradical of H is a group algebra [14]. Let H_0 be the coradical of Hopf algebra H. We define

$$H_i = \Delta^{-1}(H \otimes H_{i-1} + H_0 \otimes H) \text{ for } i \geq 1.$$

Then $\{H_i \mid i \geq 0\}$ is called the *coradical filtration* of Hopf algebra H. If H is pointed, then its coradical filtration is a Hopf algebra filtration (cf. [14, Lemma 5.2.8]). Coradical filtration is important in the classification of pointed Hopf algebras, more details can be found in [1, 2], etc.

Let $\{H_i \mid i \geq 0\}$ be the coradical filtration of Hopf algebra H. We assume that the coradical H_0 is a Hopf subalgebra of H. Then each H_i is a free H_0-module. Consider $\Bbbk$ as the trivial right H_0-module, if H is generated as an algebra by H_1 and $\dim_{\Bbbk}(\Bbbk \otimes_{H_0} H_1) = n + 1$, then H is called a *Hopf algebra* of *rank n* [17].

Krop and Radford defined the notion rank so as to give a measure of complexity for Hopf algebras. One family of pointed Hopf algebras mentioned here is so-called finite dimensional pointed Hopf algebra of rank one. The (generalized) Taft algebras, the Radford Hopf algebras and the half quantum group [12] are typical examples of such Hopf algebras. Every finite dimensional pointed Hopf algebra of rank one can be obtained via a group datum stated as follows (cf. [3, 17]):

Definition 2.1. A quadruple $\mathcal{D} = (G, \chi, g, \mu)$ is called a *group datum* if G is a finite group, χ is a $\Bbbk$-linear character of G, g is an element in the center of G, and $\mu \in \Bbbk$ subject to $\chi^n = 1$ or $\mu(g^n - 1) = 0,$ where n is the order of $\chi(g)$. If $\mu(g^n - 1) = 0,$ then the group datum $\mathcal{D}$ is said to be of *nilpotent type*. If $\mu(g^n - 1) \neq 0,$ and $\chi^n = 1,$ then it is of non-nilpotent type.

For any group datum $\mathcal{D} = (G, \chi, g, \mu),$ denote by $H_{\mathcal{D}}$ the associative algebra generated by y and all h in G such that $\Bbbk G$ is a subalgebra of $H_{\mathcal{D}}$ and

$$y^n = \mu(g^n - 1), \quad yh = \chi(h)hy \tag{2.1}$$

for any $h \in G$. In addition, $H_{\mathcal{D}}$ is endowed with a Hopf algebra structure, where the comultiplication Δ, the counit ε and the antipode S are given, respectively, by

$$\Delta(y) = y \otimes g + 1 \otimes y, \quad \varepsilon(y) = 0, \quad S(y) = -yg^{-1},$$

$$\Delta(h) = h \otimes h, \quad \varepsilon(h) = 1, \quad S(h) = h^{-1}$$

for all $h \in G$.

The Hopf algebra $H_{\mathcal{D}}$ is finite dimensional with a canonical $\Bbbk$-basis $\{y^i h \mid h \in G, 0 \leq i \leq n-1\}$. Thus, $\dim H_{\mathcal{D}} = n|G|,$ where $|G|$ is the order of G. It is easy to see that G is the group of group-like elements of $H_{\mathcal{D}}$ and $H_{\mathcal{D}}$ is a finite dimensional pointed Hopf algebra of rank one.

Remark 2.2. Note that if the order of $\chi(g)$ is $n = 1,$ then $H_{\mathcal{D}}$ is nothing but the group algebra $\Bbbk G$. To avoid this, we always assume that $n \geq 2$

throughout this paper. In this situation $\chi(g) \neq 1$, this implies that $g \neq 1$ and $\chi \neq \varepsilon$.

Example 2.3. Let G be a cyclic group of order m with a generator g, $\omega \in \mathbb{k}$ a primitive mth root of unity and χ a $\mathbb{k}$-linear character of G given by $\chi(g) = \omega$.

(1) The group datum $\mathcal{D} = (G, \chi, g, \mu)$ is of nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is nothing but a Taft algebra [4].

(2) Suppose $d > 1$ and is a divisor of m. Then the group datum $\mathcal{D} = (G, \chi, g^d, \mu)$ is of nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is a generalized Taft algebra [13].

(3) Suppose $d > 1$ and is a divisor of m. Then the group datum $\mathcal{D} = (G, \chi^d, g, \mu)\ (\mu \neq 0)$ is of non-nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is exactly a Radford Hopf algebra [15].

The $\mathbb{k}$-linear character χ induces an automorphism σ of $\mathbb{k}G$ as follows:

$$\sigma(a) = \sum \chi(a_1)a_2 \tag{2.2}$$

for any $a \in \mathbb{k}G$ with the comultiplication $\Delta(a) = \sum a_1 \otimes a_2$. In view of this, we have

$$y^j a = \sigma^j(a) y^j \text{ for any } j \geq 0. \tag{2.3}$$

The family of finite dimensional pointed Hopf algebras of rank one coincides with the family of non-semisimple monomial Hopf algebras discussed in [3]. The classification of such Hopf algebras over an algebraically closed field $\mathbb{k}$ of characteristic 0 has been given, respectively, in [3, 17]. We follow the work of Krop and Radford in [17, Theorem 1] and present the classification results of such Hopf algebras as follows:

Proposition 2.4. *We have the following classification result*:

(1) *For any group datum $\mathcal{D}$, the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is a finite dimensional pointed Hopf algebra of rank one.*

(2) *Every finite dimensional pointed Hopf algebra of rank one over an algebraically closed field* $\Bbbk$ *of characteristic* 0 *is isomorphic to* $H_{\mathcal{D}}$ *for some group datum* $\mathcal{D}$.

(3) *Let* $\mathcal{D} = (G, \chi, g, \mu)$ *and* $\mathcal{D}' = (G', \chi', g', \mu')$ *be two group data. Then* $H_{\mathcal{D}}$ *and* $H_{\mathcal{D}'}$ *are isomorphic as Hopf algebras if and only if there is a group isomorphism* $f : G \to G'$ *such that* $f(g) = g'$, $\chi = \chi' \circ f$ *and* $\beta\mu'(g'^{n} - 1) = \mu(g'^{n} - 1)$ *for some non-zero* $\beta \in \Bbbk$, *where n is the order of* $\chi(g)$.

In the case when the characteristic of $\Bbbk$ is $p > 0$, the classification of finite dimensional pointed Hopf algebras of rank one was given by Scherotzke in [16]. The classification of infinite or finite dimensional pointed Hopf algebras of rank one over an arbitrary field $\Bbbk$ was obtained in [21].

In the following, we recall the definition and basic results of a quasitriangular Hopf algebra.

Definition 2.5. Let H be a finite dimensional Hopf algebra over the field $\Bbbk$ and $R = \sum_i a_i \otimes b_i$ be an invertible element of $H \otimes H$. If H satisfies

$$\Delta^{cop}(a) = R\,\Delta(a)R^{-1} \text{ for any } a \in H, \tag{2.4}$$

where $\Delta^{cop} = T \circ \Delta$ and T is the flip map, then H is called a *quasi-cocommutative Hopf algebra*.

If H is a cocommutative Hopf algebra, then H is quasi-cocommutative with the trivial $R = 1 \otimes 1$.

Definition 2.6. The pair (H, R) is said to be a *quasitriangular Hopf algebra* if H is quasi-cocommutative and satisfies the following conditions:

$$(\Delta \otimes id)(R) = R_{13}R_{23}, \tag{2.5}$$

$$(id \otimes \Delta)(R) = R_{13}R_{12}, \tag{2.6}$$

where $R_{13} = \sum_i a_i \otimes 1 \otimes b_i$, $R_{23} = \sum_i 1 \otimes a_i \otimes b_i$ and $R_{12} = \sum_i a_i \otimes b_i \otimes 1$.

The element R above is called a *universal* R*-matrix* of H. A quasitriangular Hopf algebra (H, R) is said to be *triangular* if $R^{-1} = R_{21}$, where $R_{21} = \sum_i b_i \otimes a_i$. Let $u = \sum_i S(b_i)a_i$. Then u is called a *Drinfeld element* of (H, R). It is known that u is invertible and $S^2(a) = uau^{-1}$ for any $a \in H$, see, e.g., [6]. Moreover, (H, R) is triangular if and only if u is a group-like element of H.

If (H, R) is quasitriangular, then $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, known as the quantum Yang-Baxter equation in statistical mechanics. The invertible element R has the additional properties $(S \otimes id)(R) = (id \otimes S^{-1})(R) = R^{-1}$, $(S \otimes S)(R) = R$ and $(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1$. The Radford's S^4 formula has a special form: for any $a \in H$, $S^4(a) = hah^{-1}$, where $h = u(S(u))^{-1}$ is a group-like element of H (cf. [6]).

3. Quasitriangular Structures

In this section, we work over an algebraically closed field $\Bbbk$ of characteristic 0. We investigate the quasitriangular structures on Hopf algebra $H_{\mathcal{D}}$ associated to the group datum $\mathcal{D} = (G, \chi, g, \mu)$. It turns out that if the order of $\chi(g)$ is $n \geq 3$, then the Hopf algebra $H_{\mathcal{D}}$ has no quasitriangular structures. If the order of $\chi(g)$ is $n = 2$, then a sufficient and necessary condition for $H_{\mathcal{D}}$ to be quasitriangular is given. As an application, we determine all quasitriangular structures on $H_{\mathcal{D}}$ such that the group G in the group datum $\mathcal{D}$ is cyclic.

Proposition 3.1. *Let* $(\Bbbk G, R)$ *be a quasitriangular Hopf algebra with the universal R-matrix R. Then* $H_{\mathcal{D}}$ *is quasitriangular with the same universal R-matrix R of* $\Bbbk G$ *if and only if* $(\chi \otimes id)(R) = g$ *and* $(id \otimes \chi)(R) = g^{-1}$.

Proof. Note that $H_{\mathcal{D}}$ is generated as an algebra by $h \in G$ and y, then $H_{\mathcal{D}}$ is quasitriangular with the same R-matrix of $\Bbbk G$ if and only if the equality

$$\Delta^{cop}(y)R = R\Delta(y)$$

holds since the other equalities are automatically satisfied with the assumption that $(\Bbbk G, R)$ is quasitriangular. Suppose $R = \sum_i a_i \otimes b_i \in \Bbbk G \otimes \Bbbk G$. Then the equalities (2.5) and (2.6) can be written explicitly as follows:

$$\sum_i (a_i)_1 \otimes (a_i)_2 \otimes b_i = \sum_{i,j} a_i \otimes a_j \otimes b_i b_j, \tag{3.1}$$

$$\sum_i a_i \otimes (b_i)_1 \otimes (b_i)_2 = \sum_{i,j} a_i a_j \otimes b_j \otimes b_i. \tag{3.2}$$

Note that $\Delta^{cop}(y)R = (g \otimes y)R + (y \otimes 1)R$ and $R\Delta(y) = R(y \otimes g) + R(1 \otimes y)$. Then $\Delta^{cop}(y)R = R\Delta(y)$ if and only if $(g \otimes y)R = R(1 \otimes y)$ and $(y \otimes 1)R = R(y \otimes g)$. By the observation of equality (3.2), we have that

$$\begin{aligned}
(g \otimes y)R &= (g \otimes y)\left(\sum_i a_i \otimes b_i\right) = \sum_i ga_i \otimes yb_i \\
&= \sum_i ga_i \otimes \sigma(b_i)y = \sum_i ga_i \otimes \chi((b_i)_1)(b_i)_2 y \\
&= \sum_{i,j} ga_i a_j \otimes \chi(b_j)b_i y = R\left(g\sum_j a_j \chi(b_j) \otimes 1\right)(1 \otimes y).
\end{aligned}$$

Thus, $(g \otimes y)R = R(1 \otimes y)$ if and only if $(id \otimes \chi)(R) = g^{-1}$. The same argument as above shows that $(y \otimes 1)R = R(y \otimes g)$ if and only if $(\chi \otimes id)(R) = g$, as desired. □

Remark 3.2. The equality $(\chi \otimes id)(R) = g$ is not equivalent to $(id \otimes \chi)(R) = g^{-1}$ in general, see the equalities (3.3) and (3.4) below.

However, if $(\Bbbk G, R)$ is triangular, then it is easy to verify that $(\chi \otimes id)(R) = g$ if and only if $(id \otimes \chi)(R) = g^{-1}$ using the equality $R^{-1} = R_{21}$.

Example 3.3. Let G be a cyclic group of order m generated by the element g. Then all the universal R-matrices of $\Bbbk G$ are given by

$$R_d = \frac{1}{m} \sum_{i, j=0}^{m-1} \omega^{-ij} g^j \otimes g^{di},$$

where $\omega \in \Bbbk$ is a primitive mth root of unity and $0 \leq d \leq m-1$, see [19, Lemma 2.3]. Let χ be a $\Bbbk$-linear character of G determined by $\chi(g) = \omega$. Suppose the Hopf algebra $H_{\mathcal{D}}$ associated to the group datum $\mathcal{D} = (G, \chi^k, h, \mu)$ is quasitriangular with the universal R-matrix R_d for some $0 \leq k \leq m-1$ and $h \in G$. Note that

$$(\chi^k \otimes id)(R_d) = \frac{1}{m} \sum_{i, j=0}^{m-1} \omega^{(k-i)j} g^{di} = g^{dk}, \tag{3.3}$$

$$(id \otimes \chi^k)(R_d) = \frac{1}{m} \sum_{i, j=0}^{m-1} \omega^{(dk-j)i} g^j = g^r, \tag{3.4}$$

where $0 \leq r \leq m-1$ satisfies that $m \mid dk - r$. Then $h = g^{dk} = g^{-r}$. This implies that $m \mid dk + r$. Together with $m \mid dk - r$, we obtain that $m \mid 2r$. If m is odd, then $r = 0$ and hence $h = 1$, which is contradiction to the fact that $h \neq 1$ (see Remark 2.2). If m is even, then $r = \frac{m}{2}$. In this case, $h = g^{\frac{m}{2}}$ and $dk = \frac{m}{2}$. We conclude that the Hopf algebra $H_{\mathcal{D}}$ associated to the group datum $\mathcal{D} = (G, \chi^k, h, \mu)$ is quasitriangular with the universal R-matrix R_d if and only if m is even, $h = g^{\frac{m}{2}}$ and $dk = \frac{m}{2}$.

Lemma 3.4. *Let $H_{\mathcal{D}}$ be the Hopf algebra associated to the group datum*

$\mathcal{D} = (G, \chi, h, \mu)$ *and n be the order of* $\chi(g)$. *If* $H_{\mathcal{D}}$ *is quasi-cocommutative with an invertible element* $R \in H_{\mathcal{D}} \otimes H_{\mathcal{D}}$, *then R has the form* $R = R^{(0)} + \sum_{i=1}^{n-1} R^{(i)}(y^i \otimes y^{n-i})$, *where* $R^{(i)} \in \Bbbk G \otimes \Bbbk G$ *for* $0 \leq i \leq n-1$.

Proof. Let R be an invertible element of $H_{\mathcal{D}} \otimes H_{\mathcal{D}}$. Then R can be written as the following form:

$$R = \sum_{i=0}^{n-1} \sum_{h \in G} hy^i \otimes Y_h^{(i)},$$

where for each h and i, $Y_h^{(i)} \in H_{\mathcal{D}}$. Observe that $\Delta^{cop}(g)R = R\Delta(g)$. It follows that

$$\sum_{i=0}^{n-1} \sum_{h \in G} ghy^i \otimes gY_h^{(i)} = \sum_{i=0}^{n-1} \sum_{h \in G} ghy^i \otimes q^i Y_h^{(i)} g.$$

Hence, $gY_h^{(i)} = q^i Y_h^{(i)} g$ for any $0 \leq i \leq n-1$ and $h \in G$. We assume that $Y_h^{(i)} = \sum_{j=0}^{n-1} \sum_{t \in G} \mu_{j,t} ty^j$. It follows from $gY_h^{(i)} = q^i Y_h^{(i)} g$ that $\mu_{j,t} gty^j = \mu_{j,t} q^{i+j} gty^j$. Thus, $\mu_{j,t} = \mu_{j,t} q^{i+j}$ for any $t \in G$ and $0 \leq j \leq n-1$. We conclude that $\mu_{j,t} = 0$ if $i + j$ is not divisible by n. This argument shows that $Y_h^{(0)} = \sum_{t \in G} \mu_{0,t} t$ and $Y_h^{(i)} = \left(\sum_{t \in G} \mu_{n-i,t} t \right) y^{n-i}$. Thus, the element R can be simplified as follows:

$$R = R^{(0)} + \sum_{i=1}^{n-1} R^{(i)}(y^i \otimes y^{n-i}),$$

where $R^{(i)} \in \Bbbk G \otimes \Bbbk G$ for $0 \leq i \leq n-1$, as desired. □

Proposition 3.5. *Let* $H_{\mathcal{D}}$ *be the Hopf algebra associated to the group datum* $\mathcal{D} = (G, \chi, g, \mu)$. *If* $(H_{\mathcal{D}}, R)$ *is quasitriangular and the order of* $\chi(g)$ *is* $n \geq 3$, *then the universal R-matrix R of* $H_{\mathcal{D}}$ *is contained in* $\Bbbk G \otimes \Bbbk G$.

Proof. By Lemma 3.4, the universal R-matrix of $H_{\mathcal{D}}$ has the form

$$R = R^{(0)} + \sum_{i=1}^{n-1} R^{(i)}(y^i \otimes y^{n-i}),$$

where $R^{(i)} \in \Bbbk G \otimes \Bbbk G$ for $0 \le i \le n-1$. This enables me to compute the terms of both sides of equation (2.5). On the one hand, observe from [17, equation (1)] that

$$\Delta(y^i) = \sum_{k=0}^{i} \binom{i}{k}_q y^{i-k} \otimes g^{i-k} y^k.$$

Then

$$\begin{aligned}
(\Delta \otimes id)(R) &= (\Delta \otimes id)(R^{(0)}) + \sum_{i=1}^{n-1} (\Delta \otimes id)(R^{(i)})(\Delta(y^i) \otimes y^{n-i}) \\
&= (\Delta \otimes id)(R^{(0)}) + \sum_{i=2}^{n-1} (\Delta \otimes id)(R^{(i)}) \\
&\quad \times \sum_{k=0}^{i} \binom{i}{k}_q (y^{i-k} \otimes g^{i-k} y^k \otimes y^{n-i}) \\
&\quad + (\Delta \otimes id)(R^{(1)})(y \otimes g \otimes y^{n-1} + 1 \otimes y \otimes y^{n-1}) \\
&= (\Delta \otimes id)(R^{(0)}) + \sum_{i=2}^{n-1} (\Delta \otimes id)(R^{(i)}) \\
&\quad \times \sum_{k=1}^{i-1} \binom{i}{k}_q (y^{i-k} \otimes g^{i-k} y^k \otimes y^{n-i}) \\
&\quad + \sum_{i=1}^{n-1} (\Delta \otimes id)(R^{(i)})(y^i \otimes g^i \otimes y^{n-i} + 1 \otimes y^i \otimes y^{n-i}).
\end{aligned}$$

On the other hand, we have that $y^j a = \sigma^j(a) y^j$ for any $a \in \Bbbk G$ (see (2.3)).

It follows that

$$
\begin{aligned}
R_{13}R_{23} &= \left(R_{13}^{(0)} + \sum_{i=1}^{n-1} R_{13}^{(i)}(y^i \otimes 1 \otimes y^{n-i}) \right) \\
&\quad \times \left(R_{23}^{(0)} + \sum_{i=1}^{n-1} R_{23}^{(i)}(1 \otimes y^i \otimes y^{n-i}) \right) \\
&= R_{13}^{(0)}R_{23}^{(0)} + \sum_{i=1}^{n-1} R_{13}^{(0)}R_{23}^{(i)}(1 \otimes y^i \otimes y^{n-i}) \\
&\quad + \sum_{i=1}^{n-1} R_{13}^{(i)}(y^i \otimes 1 \otimes y^{n-i}) R_{23}^{(0)} \\
&\quad + \sum_{i=1}^{n-1} R_{13}^{(i)}(y^i \otimes 1 \otimes y^{n-i}) \sum_{j=1}^{n-1} R_{23}^{(j)}(1 \otimes y^j \otimes y^{n-j}) \\
&= R_{13}^{(0)}R_{23}^{(0)} + \sum_{i=1}^{n-1} R_{13}^{(0)}R_{23}^{(i)}(1 \otimes y^i \otimes y^{n-i}) \\
&\quad + \sum_{i=1}^{n-1} R_{13}^{(i)}(\sigma^i \otimes id \otimes \sigma^{n-i})(R_{23}^{(0)})(y^i \otimes 1 \otimes y^{n-i}) \\
&\quad + \sum_{i,\, j=1}^{n-1} R_{13}^{(i)}(\sigma^i \otimes id \otimes \sigma^{n-i})(R_{23}^{(j)})(y^i \otimes y^j \otimes y^{2n-i-j}).
\end{aligned}
$$

By comparing the elements of the forms $\Bbbk G \otimes \Bbbk G \otimes \Bbbk G$, $\Bbbk G \otimes \Bbbk Gy^i \otimes \Bbbk Gy^{n-i}$ and $\Bbbk Gy^i \otimes \Bbbk G \otimes \Bbbk Gy^{n-i}$, respectively, in both sides of the equality (2.5), we obtain that

$$(\Delta \otimes id)(R^{(0)}) = R_{13}^{(0)}R_{23}^{(0)},$$

$$\sum_{i=1}^{n-1}(\Delta \otimes id)(R^{(i)})(1 \otimes y^i \otimes y^{n-i}) = \sum_{i=1}^{n-1} R_{13}^{(0)}R_{23}^{(i)}(1 \otimes y^i \otimes y^{n-i}),$$

$$\sum_{i=1}^{n-1}(\Delta \otimes id)(R^{(i)})(y^i \otimes g^i \otimes y^{n-i})$$

$$= \sum_{i=1}^{n-1} R_{13}^{(i)}(\sigma^i \otimes id \otimes \sigma^{n-i})(R_{23}^{(0)})(y^i \otimes 1 \otimes y^{n-i}).$$

Omitting the above equations from the equality (2.5), we obtain that

$$\sum_{i=2}^{n-1}(\Delta \otimes id)(R^{(i)})\sum_{k=1}^{i-1}\binom{i}{k}_q (y^{i-k} \otimes g^{i-k}y^k \otimes y^{n-i}) \tag{3.5}$$

$$= \sum_{i,\, j=1}^{n-1} R_{13}^{(i)}(\sigma^i \otimes id \otimes \sigma^{n-i})(R_{23}^{(j)})(y^i \otimes y^j \otimes y^{2n-i-j}). \tag{3.6}$$

Note that the expression (3.5) has no terms of the form $\Bbbk Gy^i \otimes \Bbbk Gy^j \otimes H_{\mathcal{D}}$ such that $i + j \geq n$. It follows that such forms in (3.6) are zero, namely,

$$\sum_{i+j\geq n} R_{13}^{(i)}(\sigma^i \otimes id \otimes \sigma^{n-i})(R_{23}^{(j)})(y^i \otimes y^j \otimes y^{2n-i-j}) = 0.$$

In view of this, the expression (3.6) now can be reduced to be

$$\sum_{2\leq i+j\leq n-1} R_{13}^{(i)}(\sigma^i \otimes id \otimes \sigma^{n-i})(R_{23}^{(j)})(y^i \otimes y^j \otimes y^{2n-i-j})$$

$$= \sum_{2\leq i+j\leq n-1} R_{13}^{(i)}(\sigma^i \otimes id \otimes \sigma^{n-i})(R_{23}^{(j)})(y^i \otimes y^j \otimes \mu(g^n - 1)y^{n-i-j})$$

$$= \sum_{s=2}^{n-1}\sum_{j=1}^{s-1} R_{13}^{(s-j)}(\sigma^{s-j} \otimes id \otimes \sigma^{n-s+j})$$

$$\times (R_{23}^{(j)})(y^{s-j} \otimes y^j \otimes \mu(g^n - 1)y^{n-s})$$

$$= \sum_{i=2}^{n-1}\sum_{k=1}^{i-1} R_{13}^{(i-k)}(\sigma^{i-k} \otimes id \otimes \sigma^{n-i+k})$$

$$\times (R_{23}^{(k)})(y^{i-k} \otimes y^{k} \otimes \mu(g^{n} - 1)y^{n-i}).$$

As a consequence, the equality (3.5) = (3.6) becomes

$$\sum_{i=2}^{n-1}(\Delta \otimes id)(R^{(i)})\sum_{k=1}^{i-1}\binom{i}{k}_{q}(y^{i-k} \otimes g^{i-k}y^{k} \otimes y^{n-i})$$

$$= \sum_{i=2}^{n-1}\sum_{k=1}^{i-1} R_{13}^{(i-k)}(\sigma^{i-k} \otimes id \otimes \sigma^{n-i+k})(R_{23}^{(k)})(y^{i-k} \otimes y^{k} \otimes \mu(g^{n} - 1)y^{n-i}).$$

Comparing each terms of this equality, we have that

$$(\Delta \otimes id)(R^{(i)})\binom{i}{k}_{q}(1 \otimes g^{i-k} \otimes 1) \tag{3.7}$$

$$= R_{13}^{(i-k)}(\sigma^{i-k} \otimes id \otimes \sigma^{n-i+k})(R_{23}^{(k)})(1 \otimes 1 \otimes \mu(g^{n} - 1)) \tag{3.8}$$

for $2 \leq i \leq n-1$ and $1 \leq k \leq i-1$. Applying $\varepsilon \otimes id \otimes \varepsilon$ to both (3.7) and (3.8), we obtain that $(\varepsilon \otimes id \otimes \varepsilon)(\Delta \otimes id)(R^{(i)}) = 0$ since $\varepsilon(g^{n} - 1) = 0$. Note that $(\varepsilon \otimes id \otimes \varepsilon)(\Delta \otimes id)(R^{(i)}) = (id \otimes \varepsilon)(R^{(i)})$. It deduces that

$$(id \otimes \varepsilon)(R^{(i)}) = 0 \text{ for } 2 \leq i \leq n-1. \tag{3.9}$$

Similarly, by comparing the terms of the form $\Bbbk Gy^{i} \otimes \Bbbk G \otimes \Bbbk Gy^{n-i}$ in the equality $(id \otimes \Delta)(R) = R_{13}R_{12}$, we obtain the following equation:

$$(id \otimes \Delta)(R^{(i)}) = R_{13}^{(i)}(\sigma^{i} \otimes id \otimes \sigma^{n-i})(R_{12}^{(0)}). \tag{3.10}$$

Applying $id \otimes id \otimes \varepsilon$ to both sides of (3.10), we obtain that $R^{(i)} = (id \otimes \varepsilon)(R^{(i)})(\sigma^{i} \otimes id)(R^{(0)})$. The observation of (3.9) shows that $R^{(i)} = 0$

for $2 \leq i \leq n-1$. Thus, the universal R-matrix of $H_{\mathcal{D}}$ is reduced to be

$$R = R^0 + R^{(1)}(y \otimes y^{n-1}).$$

Note that $n \geq 3$. It is straightforward to verify that the following expressions

$$(id \otimes \Delta)(R^{(1)})\binom{n-1}{k}_q (y \otimes y^{n-1-k} \otimes g^{n-1-k} y^k)$$

for $1 \leq k \leq n-2$ are summands of $(id \otimes \Delta)(R)$, while they are not appeared in $R_{13}R_{12}$. Hence, $(id \otimes \Delta)(R^{(1)}) = 0$ and thus $(R^{(1)}) = 0$ since the map $id \otimes \Delta$ is injective. We complete the proof. □

As an immediate consequence of Proposition 3.1 and Proposition 3.5, we have the following result:

Theorem 3.6. *If the order of* $\chi(g)$ *is* $n \geq 3$, *then the Hopf algebra* $H_{\mathcal{D}}$ *associated to the group datum* $\mathcal{D} = (G, \chi, g, \mu)$ *admits no quasitriangular structures*.

Proof. If $(H_{\mathcal{D}}, R)$ is quasitriangular and the order of $\chi(g)$ is $n \geq 3$, by Proposition 3.5, the universal R-matrix R of $H_{\mathcal{D}}$ is contained in $\Bbbk G \otimes \Bbbk G$, and therefore $(\Bbbk G, R)$ is quasitriangular. Let $R = \sum_i a_i \otimes b_i \in \Bbbk G \otimes \Bbbk G$. It follows from Proposition 3.1 that $\sum_i \chi(a_i) b_i = g$ and $\sum_i a_i \chi(b_i) = g^{-1}$. Note that the Drinfeld element of $(\Bbbk G, R)$ is $u = \sum_i S(b_i) a_i$. We obtain that

$$\chi(u) = \chi\left(\sum_i S(b_i) a_i\right) = \chi\left(S\left(\sum_i \chi(a_i) b_i\right)\right) = \chi(g^{-1}). \tag{3.11}$$

Similarly, it follows from $\sum_i a_i \chi(b_i) = g^{-1}$ and $S^2|_{\Bbbk G} = id$ that

$$\chi(S(u)) = \chi(g). \tag{3.12}$$

Note that $\Bbbk G$ is semisimple and cosemisimple over $\Bbbk$. Then $u = S(u)$ [10, Lemma 2.1.1]. It follows from (3.11) and (3.12) that $\chi(g) = \chi^{-1}(g)$. Thus, the order of $\chi(g)$ is $n = 2$, a contradiction. □

There exists a group datum $\mathcal{D} = (G, \chi, g, \mu)$ with the order of $\chi(g)$ being $n = 2$ such that the Hopf algebra $H_{\mathcal{D}}$ is quasitriangular. Sweedler's four dimensional Hopf algebra gives such an example [18, p. 174]. In the following, we shall describe the quasitriangular structures on $H_{\mathcal{D}}$ such that the order of $\chi(g)$ is $n = 2$. By Lemma 3.4, the universal R-matrix R of $H_{\mathcal{D}}$ has the following form

$$R = R^{(0)} + R^{(1)}(y \otimes y)$$

for $R^{(0)}$ and $R^{(1)}$ in $\Bbbk G \otimes \Bbbk G$. If $R^{(1)} = 0$, by the same argument as the proof of Proposition 3.1, we can see that $(H_{\mathcal{D}}, R)$ is quasitriangular if and only if $(\Bbbk G, R)$ is quasitriangular, $(\chi \otimes id)(R) = g$ and $(id \otimes \chi)(R) = g^{-1}$. In the following, we only need to determine the universal R-matrix of $H_{\mathcal{D}}$ such that $R^{(1)}$ in R is not 0.

Theorem 3.7. *Let $H_{\mathcal{D}}$ be the Hopf algebra associated to the group datum $\mathcal{D} = (G, \chi, g, \mu)$ with the order of $\chi(g)$ being $n = 2$. Then $(H_{\mathcal{D}}, R)$ is quasitriangular with the universal R-matrix $R = R^{(0)} + R^{(1)}(y \otimes y)$ and $R^{(1)} \neq 0$ if and only if the following hold*:

(1) $R^{(1)} = \lambda R^{(0)}(1 \otimes g^{-1})$, *where* $\lambda = (\varepsilon \otimes \varepsilon)(R^{(1)}) \neq 0$.

(2) *The group datum* $\mathcal{D} = (G, \chi, g, \mu)$ *is of nilpotent type*.

(3) $R^{(0)}$ *is a universal R-matrix of* $\Bbbk G$.

(4) *The order of* χ *and the order of g are both equal to* 2.

(5) $(\chi \otimes id)(R^{(0)}) = (id \otimes \chi)(R^{(0)}) = g$.

Proof. It is direct to verify that $(\Delta \otimes id)(R) = R_{13}R_{23}$ if and only if the following four equalities hold:

$$(\Delta \otimes id)(R^{(0)}) = R_{13}^{(0)}R_{23}^{(0)}, \tag{3.13}$$

$$(\Delta \otimes id)(R^{(1)})(1 \otimes g \otimes 1) = R_{13}^{(1)}(\sigma \otimes id \otimes \sigma)(R_{23}^{(0)}), \tag{3.14}$$

$$(\Delta \otimes id)(R^{(1)}) = R_{13}^{(0)}R_{23}^{(1)}, \tag{3.15}$$

$$R_{13}^{(1)}(\sigma \otimes id \otimes \sigma)(R_{23}^{(1)})(1 \otimes 1 \otimes \mu(g^2 - 1)) = 0. \tag{3.16}$$

Similarly, $(id \otimes \Delta)(R) = R_{13}R_{12}$ holds if and only if the following hold:

$$(id \otimes \Delta)(R^{(0)}) = R_{13}^{(0)}R_{12}^{(0)}, \tag{3.17}$$

$$(id \otimes \Delta)(R^{(1)})(1 \otimes 1 \otimes g) = R_{13}^{(0)}R_{12}^{(1)}, \tag{3.18}$$

$$(id \otimes \Delta)(R^{(1)}) = R_{13}^{(1)}(\sigma \otimes id \otimes \sigma)(R_{12}^{(0)}), \tag{3.19}$$

$$R_{13}^{(1)}(\sigma \otimes id \otimes \sigma)(R_{12}^{(1)})(\mu(g^2 - 1) \otimes 1 \otimes 1) = 0. \tag{3.20}$$

Note that

$$(\varepsilon \otimes id)(R^{(0)}) = (\varepsilon \otimes id)(R - R^{(1)}(y \otimes y)) = (\varepsilon \otimes id)(R) = 1. \tag{3.21}$$

Then (3.13) and (3.21) show that $R^{(0)}$ is invertible, see [19, Lemma 2.2]. Denote by $\lambda = (\varepsilon \otimes \varepsilon)(R^{(1)})$. Applying $id \otimes \varepsilon \otimes \varepsilon$ and $\varepsilon \otimes \varepsilon \otimes id$ to the equalities (3.15) and (3.18), respectively, we obtain that $(id \otimes \varepsilon)(R^{(1)}) = \lambda$ and $(\varepsilon \otimes id)(R^{(1)}) = \lambda(1 \otimes g^{-1})$. Applying $id \otimes \varepsilon \otimes id$ to (3.15), we obtain that $R_{13}^{(1)} = R_{13}^{(0)}(1 \otimes (\varepsilon \otimes id)(R^{(1)})) = R_{13}^{(0)}(1 \otimes \lambda \otimes g^{-1})$. It follows that

$$R^{(1)} = \lambda R^{(0)}(1 \otimes g^{-1}). \tag{3.22}$$

This is exactly the condition (1). The equality (3.22) means that $R^{(1)}$ is invertible since it is not zero by assumption. Accordingly, the equality (3.16) implies that $\mu(g^2-1)=0$, and therefore the group datum $\mathcal{D}$ is of nilpotent type (i.e., $y^2=0$), so we obtain the condition (2). Now the universal R-matrix of $H_{\mathcal{D}}$ can be written as

$$R=R^{(0)}+R^{(1)}(y\otimes y)=R^{(0)}(1\otimes 1+\lambda y\otimes g^{-1}y).$$

Note that $\Delta^{cop}(h)R=R\Delta(h)$ if and only if $\Delta^{cop}(h)R^{(0)}=R^{(0)}\Delta(h)$ and $\Delta^{cop}(h)R^{(0)}(\lambda y\otimes g^{-1}y)=R^{(0)}(\lambda y\otimes g^{-1}y)\Delta(h)$ for any $h\in G$. This is equivalent to saying that

$$\Delta^{cop}(h)R^{(0)}=R^{(0)}\Delta(h) \text{ and} \tag{3.23}$$

$$\chi^2(h)=1 \tag{3.24}$$

for any $h\in G$. It follows from (3.13), (3.17) and (3.23) that $(\Bbbk G, R)$ is quasitriangular. This is exactly the condition (3). Finally, note that $\Delta^{cop}(y)R=R\Delta(y)$ if and only if $\Delta^{cop}(y)R^{(0)}=R^{(0)}\Delta(y)$ since $y^2=0$. By Proposition 3.1, this is equivalent to $(\chi\otimes id)(R^{(0)})=g$ and $(id\otimes\chi)(R^{(0)})$ $=g^{-1}$. However, it is straightforward to check that $(id\otimes\sigma)(R^{(0)})=R^{(0)}$ $\cdot(id\otimes\chi)(R^{(0)})$ and $(\sigma\otimes id)(R^{(0)})=(\chi\otimes id)(R^{(0)})R^{(0)}$. By the observation of (3.14) and (3.19), we obtain that $(id\otimes\chi)(R^{(0)})=g$ and $(\chi\otimes id)(R^{(0)})$ $=g^{-1}$. It follows that

$$g^2=1 \text{ and} \tag{3.25}$$

$$(\chi\otimes id)(R^{(0)})=(id\otimes\chi)(R^{(0)})=g. \tag{3.26}$$

Then the equalities (3.24) and (3.25) give the condition (4) and the equality (3.26) gives the condition (5).

Conversely, if $R=R^{(0)}+R^{(1)}(y\otimes y)$ satisfies the conditions (1)-(5), it is straightforward to verify that R is the universal R-matrix of $H_{\mathcal{D}}$. □

As an example, we use Theorem 3.7 to determine all quasitriangular structures on a finite dimensional pointed Hopf algebra of rank one such that the group of group-like elements is a cyclic group.

Example 3.8. Let G be a cyclic group of order m generated by the element g. If m is odd, then the condition (4) in Theorem 3.7 does not hold since there is no non-trivial element of G of order 2. In the following, we assume that m is even. The universal R-matrices of $\Bbbk G$ are described as follows:

$$R_d = \frac{1}{m}\sum_{i,\, j=0}^{m-1} \omega^{-ij} g^j \otimes g^{di},$$

where $\omega \in \Bbbk$ is a primitive mth root of unity and $0 \le d \le m-1$, see Example 3.3. Let χ be a $\Bbbk$-linear character of G defined by $\chi(g) = \omega$. Then $g^{\frac{m}{2}}$ is the unique non-trivial element of G of order 2 and $\chi^{\frac{m}{2}}$ is the unique non-trivial character of G of order 2. Let $\mathcal{D} = (G, \chi^{\frac{m}{2}}, g^{\frac{m}{2}}, 0)$. Then $\mathcal{D}$ is a group datum of nilpotent type. Note that

$$(\chi^{\frac{m}{2}} \otimes id)(R_d) = \frac{1}{m}\sum_{i,\, j=0}^{m-1} \omega^{\left(\frac{m}{2}-i\right)j} g^{di} = g^{\frac{dm}{2}}, \tag{3.27}$$

$$(id \otimes \chi^{\frac{m}{2}})(R_d) = \frac{1}{m}\sum_{i,\, j=0}^{m-1} \omega^{\left(\frac{dm}{2}-j\right)i} g^{j} = \begin{cases} 1, & 2 \mid d, \\ g^{\frac{m}{2}}, & 2 \nmid d. \end{cases} \tag{3.28}$$

By Theorem 3.7, the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is quasitriangular with the universal R-matrix

$$R_\lambda = R_1 + \lambda R_1(1 \otimes g^{\frac{m}{2}})(y \otimes y),$$

where $R_1 = \frac{1}{m}\sum_{i,\, j=0}^{m-1} \omega^{-ij} g^j \otimes g^i$ and $\lambda \in \Bbbk$. It is obvious that $(H_{\mathcal{D}}, R)$

is triangular if and only if $\lambda = 0$. In particular, if $m = 2$, then we recover the universal R-matrices of Sweedler's four dimensional Hopf algebra [18, Example 2, p. 174].

Remark 3.9. It follows from Proposition 3.1 and Theorem 3.7 that the classification of triangular structures on $(H_{\mathcal{D}}, R)$ is reduced to the classification of triangular semisimple Hopf algebras $(\Bbbk G, R)$ satisfying some special conditions. The classification of triangular semisimple Hopf algebras was solved in [7, 8]. However, the key result about such classification is that every triangular semisimple Hopf algebra can be obtained by twisting a group algebra of a finite group [7, Theorem 2.1]. From this point of view, it seems not clear how to classify triangular structures on $(\Bbbk G, R)$ (satisfying some special conditions).

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