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THE VALUE DISTRIBUTION OF ALGEBROID SOLUTION OF A TYPE OF COMPLEX HIGHER-ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we mainly use Nevanlinna theory of the value distribution of meromorphic functions or algebroid functions and the property $(w-a)^{(n)} = w^{(n)}$ of higher-order derivative of functions to investigate the problem of the value distribution of algebroid solutions of some algebraic differential equations. A result of the deficient values of a type of higher-order algebraic differential equations is obtained on the condition that the equations exist admissible algebroid solutions and meet some proper conditions, and the result of paper [3] is generalized.

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1. Introduction

We assume that the readers are familiar with the standard notions of the Nevanlinna theory of the value distribution of meromorphic functions, such as the characteristic function $T(r, f)$, proximity function $m(r, f)$, counting function $N(r, f)$, the first and second main theorems (see, e.g., [1-3]).

An analytic function $w(z)$ with v branches is an algebroid function if the function $w(z)$ satisfies an equation of the form

$$\psi(z, w) = A_v(z)w^v + A_{v-1}(z)w^{v-1} + \cdots + A_0(z) = 0,$$

where $A_j(z)\ (j = 0, 1, ..., v)$ are regular functions in z with no common zeros and $A_v(z) \neq 0$. Especially, when $v = 1$, $w(z)$ is a meromorphic function; when $A_j(z)\ (j = 0, 1, ..., v)$ are polynomials, $w(z)$ is an algebraic function. In general, we consider the case that at least one of $A_j(z)$ is transcendental function.

Recently, many authors studied a large number of algebraic differential equations, and they have obtained some good results (see, e.g., [3-10]). In paper [3], He and Xiao investigated the value distribution of admissible algebroid solutions of a type of higher-order algebraic differential equation of the form

$$\Omega(z, w) = R(z, w), \tag{1}$$

where $\Omega(z, w) = \sum_{(i)} a_{(i)} w^{i_0}(w')^{i_1} \cdots (w^{(n)})^{i_n}$ is a differential polynomial with meromorphic coefficients. (i) is a finite set of multi-indices, $i = (i_0, i_1, ..., i_n)$. $R(z, w) = \dfrac{\sum_{i=0}^{p} a_i(z)w^i}{\sum_{j=0}^{q} b_j(z)w^j}$ is an irreducible rational function in $w(z)$ with the meromorphic coefficients $a_i(z)$ and $b_j(z)$.

Also, they obtained the following result:

Theorem A [3]. *Let* $w(z)$ *be an admissible algebroid solution of* (1) *with* v *branches and* $p > q + \lambda$. *Then* $\delta(w, \infty) = 0$ *and* $\Theta(w, \infty) \le 1 - \frac{p - (q + \lambda)}{\mu\nu}$.

For any $\alpha \in C$, *there exists an* α *satisfying* $P(z, \alpha) \neq 0$,

if $\alpha \neq 0$, *then we have* $\delta(w, \alpha) = 0$ *and*

$$\Theta(w, \alpha) \le 1 - \frac{\min\{i_1 + i_2 + \cdots + i_n\}}{\mu\nu};$$

if $\alpha = 0$, *then we have* $\Theta(w, 0) \le 1 - \frac{\min\{i_0 + i_1 + \cdots + i_n\}}{\mu\nu}$.

For differential polynomial $\Omega(z, w) = \sum_{(i)} a_{(i)} w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}$, we denote

$$\lambda = \max\left\{\sum_{\alpha=0}^{n} i_\alpha\right\}, \quad \mu = \max\left\{\sum_{\alpha=0}^{n} \alpha i_\alpha\right\}, \quad \Delta = \max\left\{\sum_{\alpha=0}^{n} (\alpha + 1) i_\alpha\right\}.$$

In addition, we have

$$\delta(w, a) = \varliminf_{r \to \infty} \frac{m\left(r, \frac{1}{w - a}\right)}{T(r, w)}, \quad \Theta(w, a) = 1 - \varlimsup_{r \to \infty} \frac{\overline{N}(r, a)}{T(r, w)},$$

$$S(r) = \sum_{(i)} T(r, a_{(i)}) + \sum_{k=0}^{p} T(r, a_k) + \sum_{j=0}^{q} T(r, b_j).$$

Definition. Let $w(z)$ be an algebroid solution of differential equation (1) with *v* branches. If $w(z)$ satisfies the following condition: $S(r) = o\{T(r, w)\}$, outside a possible exception set E with a finite linear measure, we say that $w(z)$ is an *admissible algebroid solution* of (1).

In this paper, with the aid of $(w-a)^{(n)} = w^{(n)}$, we will mainly investigate the problem of the value distribution of algebroid solution on higher-order nonlinear differential equation of the following form:

$$\frac{\Omega(z, w)}{(w-a)^s} = R(z, w), \tag{2}$$

where $\Omega(z, w)$, $R(z, w)$ are same as what we have denoted in (1).

We obtain the following result:

Theorem 1. *Let* $w(z)$ *be an admissible algebroid solution of* (2) *with* v *branches*, $p + s > q + \lambda$ *and* $s < \min\{i_1 + i_2 + \cdots + i_n\}$. *Then*

(a) $\delta(w, \infty) = 0$ *and* $\Theta(w, \infty) \le 1 - \dfrac{p + s - (q + \lambda)}{\mu\nu}$.

(b) *For any* $\alpha \in C$, *there exists an* α *satisfying* $P(z, \alpha) \neq 0$. *Then* $\delta(w, \alpha) = 0$,

if $\alpha = a$, $\alpha \neq 0$, *then we have* $\Theta(w, \alpha) \le 1 - \dfrac{\min\{i_1 + i_2 + \cdots + i_n\} - s}{\mu\nu}$;

if $\alpha = a$, $\alpha = 0$, *then we have* $\Theta(w, 0) \le 1 - \dfrac{\min\{i_0 + i_1 + \cdots + i_n\} - s}{\mu\nu}$;

if $\alpha \neq a$, $\alpha \neq 0$, *then we have* $\Theta(w, \alpha) \le 1 - \dfrac{\min\{i_1 + i_2 + \cdots + i_n\} - 2s}{\mu\nu}$;

if $\alpha \neq a$, $\alpha = 0$, *then we have* $\Theta(w, 0) \le 1 - \dfrac{\min\{i_0 + i_1 + \cdots + i_n\} - 2s}{\mu\nu}$.

Note. (1) Compared with Theorem A, equation (2) we investigate is more general.

(2) As $s = 0$, we can find Theorem A is a special case of Theorem 1.

2. Several Lemmas

In order to prove our result, we need the following lemmas.

Lemma 1 [4]. *Let* $w(z)$ *be an admissible algebroid solution and* $P(z, w) = \sum_{i=0}^{p} a_i(z) w^i$, $Q(z, w) = \sum_{j=0}^{q} b_j(z) w^j$. *Then*

$$m(r, P(z, w)) \geq pm(r, w) - S(r, w),$$

$$m(r, Q(z, w)) \leq qm(r, w) + S(r, w).$$

Lemma 2 [3]. *Suppose that* $P(z, w) = \sum_{i=0}^{p} a_i(z) w^i$, $w(z)$ *is an algebroid function solution. Then*

$$T(r, P(z, w)) = pT(r, w) + O\left\{\sum_{i=0}^{p} T(r, a_k)\right\}.$$

Lemma 3. *Suppose that* $w(z)$ *is an algebroid function with v branches. Then* $m\left(r, \frac{w^{(\alpha)}}{w-a}\right) = S(r, w)$.

Proof. The procedure is similar to the proof of $m\left(r, \frac{w^{(\alpha)}}{w}\right) = S(r, w)$ in paper [3].

3. The Proof of Theorem 1

Proof. We first prove (a). Let $w(z)$ be an admissible algebroid solution of (2) with *v* branches. We rewrite equation (2) as follows:

$$\frac{\Omega(z, w)}{(w-a)^s} Q(z, w) = P(z, w). \tag{3}$$

By Lemma 1, we get

$$m(r, P(z, w)) \geq pm(r, w) - s(r, w), \tag{4}$$

$$m(r, Q(z, w)) \leq qm(r, w) + s(r, w). \tag{5}$$

By applying the property of the positive logarithm to $(w - a)$, we have

$$\log^+ | w - a | \leq \log^+ | w | + \log^+ | a | + \log 2.$$

Hence,

$$m(r, w - a) \leq m(r, w) + O(1). \tag{6}$$

Using the fact that $\dfrac{w^{(n)}}{w - a} = \dfrac{(w - a)^{(n)}}{w - a}$, we have

$$\frac{\Omega(z, w)}{(w - a)^s} = \frac{\sum_{(i)} a_{(i)} w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}}{(w - a)^s}$$

$$= \sum_{(i)} a_{(i)} w^{i_0} \left(\frac{(w - a)'}{w - a} \right)^{i_1} \cdots \left(\frac{(w - a)^{(n)}}{w - a} \right)^{i_n} (w - a)^{(i_1 + i_2 + \cdots + i_n) - s}. \tag{7}$$

Combining (7) with Lemma 3 and (6), we get

$$m\left(r, \frac{\Omega(z, w)}{(w - a)^s} \right) \leq (\lambda - s) m(r, w) + S(r, w), \tag{8}$$

it follows from (3), (4), (5) and (8) that

$$(p + s - \lambda - q) m(r, w) \leq S(r, w). \tag{9}$$

Then, according to the definition of admissible solutions, both sides of (9) are divided by $T(r, w)$ and take the lower limit, this yields

$$(p + s - \lambda - q) \delta(w, \infty) = 0.$$

Noting that $p + s - \lambda - q > 0$, we have

$$\delta(w, \infty) = 0.$$

We proceed to rewrite equation (2) as follows:

$$\Omega(z, w)Q(z, w) = (w - a)^s P(z, w) = P_1(z), \text{ where } \deg_w P_1(z) = p + s.$$

Thus, we have

$$T\left(r, P(z, w)(w - a)^s \frac{1}{(w - a)^s}\right)$$

$$\leq T(r, P(z, w)(w - a)^s) + T\left(r, \frac{1}{(w - a)^s}\right)$$

$$\leq T(r, P(z, w)(w - a)^s) + T(r, (w - a)^s) + O(1),$$

i.e.,

$$T(r, P(z, w)(w - a)^s) \geq T(r, P(z, w)) - sT(r, w) + O(1). \tag{10}$$

Since

$$T(r, \Omega(z, w)Q(z, w)) \leq (\lambda + q)T(r, w) + \mu\nu\overline{N}(r, w) + S(r, w), \tag{11}$$

it follows from (10) and (11) that

$$T(r, P(z, w)) - sT(r, w) + O(1) \leq (\lambda + q)T(r, w) + \mu\nu\overline{N}(r, w) + S(r, w).$$

By Lemma 2, we get

$$[p - (q + \lambda + s)]T(r, w) \leq \mu\nu\overline{N}(r, w) + S(r, w). \tag{12}$$

According to the definition of admissible solutions, both sides of (12) are divided by $T(r, w)$ and take the upper limit, we obtain

$$\Theta(w, \infty) \leq 1 - \frac{p - (q + \lambda + s)}{\mu\nu}.$$

To prove (b) in Theorem 1, set

$$w = \frac{1}{u} + \alpha.$$

Then

$$w - a = \frac{1}{u} + \alpha - a = \frac{1 + (\alpha - a)u}{u}. \tag{13}$$

Substituting (13) into the right hand side of equation (2), we have

$$u^{(q-p)} \frac{P(z, \alpha)u^p + \cdots + A_p(z)}{Q(z, \alpha)u^q + \cdots + B_q(z)} = \frac{P(z, w)}{Q(z, w)}.$$

Since $w^{(j)} = \frac{P_j(u, u', \ldots, u^{(j)})}{u^{j+1}}$ $(j = 1, 2, \ldots, n)$, where $P_j(u, u', \ldots, u^{(j)})$ is a homogeneous in u of degree j, we obtain the general term of $\frac{\Omega(z, w)}{(w-a)^s}$ as follows:

$$\frac{a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}}{(w-a)^s} = \frac{a_{(i)}(z) \left(\frac{\alpha u + 1}{u}\right)^{i_0} \cdots \left(\frac{P_n(u, u', \ldots, u^{(n)})}{u^{n+1}}\right)^{i_n}}{\left(\frac{u(\alpha - a) + 1}{u}\right)^s}. \tag{14}$$

Next, we discuss (14) with the following cases:

Case (1) Suppose first that $\alpha = a$.

If $\alpha \neq 0$, then we have

$$\frac{a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}}{(w-a)^s}$$

$$= a_{(i)}(z) \left(\frac{\alpha u + 1}{u}\right)^{i_0} \cdots \left(\frac{P_j(u, u', \ldots, u^{(n)})}{u^{n+1}}\right)^{i_n} u^s$$

$$= P_{(j)}(z;\, u,\, u',\, \ldots,\, u^{(n)})u^{-(i_0 + 2i_1 + \cdots + (n+1)i_n) + s},$$

it shows at once that the left hand side of equation (2) becomes $\Omega_1(z;\, u,\, u',\, \ldots,\, u^{(n)})u^{-\Delta + s}$.

Again, for differential polynomial $\Omega(z,\, w)$, we have

$$\Delta = \max\left\{\sum_{\alpha=0}^{n}(\alpha + 1)i_\alpha\right\}, \quad \lambda = \max\left\{\sum_{\alpha=0}^{n} i_\alpha\right\},$$

then

$$0 \le \min(\delta_i - \lambda_i) \le \delta - \lambda.$$

Since $\delta_i - \lambda_i = i_1 + 2i_2 + \cdots + ni_n \ge i_1 + i_2 + \cdots + i_n$,

$$\delta - \lambda = l \ge \min\{i_1 + i_2 + \cdots + i_n\}, \text{ when } \alpha \ne 0.$$

If $\alpha = 0$, then we have

$$\frac{a_{(i)}(z)w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}}{(w-a)^s} = a_{(i)}(z)\left(\frac{1}{u}\right)^{i_0}\cdots\left(\frac{P_n(u,\, u',\, \ldots,\, u^{(n)})}{u^{n+1}}\right)^{i_n} u^s$$

$$= H_{(i)}(z;\, u,\, u',\, \ldots,\, u^{(n)})u^{-(i_0 + 2i_1 + \cdots + (n+1)i_n) + s}.$$

Then we get the left hand side of equation (2) as following form:

$$\Omega_2(z;\, u,\, u',\, \ldots,\, u^{(n)})u^{-\Delta - i_0 + s}.$$

By a similar deduction, when $\alpha = 0$, we get

$$\delta - \lambda = l \ge \min\{i_0 + i_1 + \cdots + i_n\}.$$

Combining with the argument above, we find that equation (2) becomes

$$\sum_{(i)} b_{(i)}(z)u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n} = \sum_{k=0}^{q+l+\lambda-s} \hat{a}_k(z)u^k \Bigg/ \sum_{j=0}^{q} \hat{b}_j(z)u^j, \qquad (15)$$

where $\hat{a}_{q+\lambda+l-s}(z) = P(z, \alpha) \neq 0$, $l \geq \min\{i_1 + i_2 + \cdots + i_n\}$, as $\alpha \neq 0$ and $l \geq \min\{i_0 + i_1 + \cdots + i_n\}$, as $\alpha = 0$.

Now, we note equation (15) as $\hat{\Omega}(z; u, u', ..., u^{(n)}) = \dfrac{P_1(z, u)}{Q(z, u)}$.

Then we have

$$m(r, P_1(z, u)) = m(r, Q(z, u))\hat{\Omega}(z, u) \leq m(r, Q(z, u)) + m(r, \hat{\Omega}(z, u)).$$

Using Lemma 1, we deduce that $p_1 m(r, u) \leq (q + \lambda)m(r, u) + S(r, u)$.

Since $l > s$, we use again the definition of admissible solution, both sides of the equation are divided by $T(r, u)$ and take the lower limit, this gives

$$\delta(u, \infty) = 0.$$

Furthermore, we have

$$T(r, P_1(z, w)) = T(r, Q(z, u)\hat{\Omega}(z, u)) \leq T(r, Q(z, u)) + T(r, \hat{\Omega}(z, u)).$$

By Lemma 2, we get

$$p_1 T(r, u) \leq (p + \lambda)T(r, u) + \mu\nu\overline{N}(z, u) + S(z, u).$$

Similarly, both sides of the equation are divided by $T(r, u)$ and take the upper limit, we obtain

$$\Theta(w, \alpha) = \Theta(u, \infty) \leq 1 - \frac{\min\{i_1 + i_2 + \cdots + i_n\}}{\mu\nu}, \text{ when } \alpha = a \neq 0,$$

$$\Theta(w, 0) = \Theta(u, \infty) \leq 1 - \frac{\min\{i_0 + i_1 + \cdots + i_n\}}{\mu\nu}, \text{ when } \alpha = a = 0.$$

Case (2) Suppose $\alpha \neq a$, we have

$$\frac{a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}}{(w - a)^s} = \frac{P_{(i)}(z; u, u', ..., u^{(n)}) u^{-(i_0 + \cdots + (n+1) i_n) + s}}{(u + (\alpha - a)^{-1})^s}.$$

So the left hand side of equation (2) becomes

$$\frac{\widetilde{\Omega}(z; u, u', ..., u^{(n)})}{(u + (\alpha - a)^{-1})^s} u^{-\Delta + s}.$$

Hence, we make a transformation of equation (2) as follows:

$$\sum_{(i)} \frac{c_{(i)}(z) u^{i_0} (u')^{i_1} \cdots (u^{(n)})^{i_n}}{(u + (\alpha - a)^{-1})^s} = \frac{\sum_{k=0}^{q+\lambda+l-s} c_k(z) u^k}{\sum_{j=0}^{q} d_j(z) u^j}, \tag{16}$$

where $c_{q+\lambda+l-s}(z) = P(z, \alpha) \neq 0$, $l \geq \min\{i_1 + i_2 + \cdots + i_n\}$, as $\alpha \neq 0$ and $l \geq \min\{i_0 + i_1 + \cdots + i_n\}$, as $\alpha = 0$.

Applying (a) in Theorem 1 to (16), we get $\delta(w, \alpha) = \delta(u, \infty) = 0$,

$$\Theta(w, \alpha) = \Theta(u, \infty) \leq 1 - \frac{\min\{i_1 + i_2 + \cdots + i_n\} - 2s}{\mu\nu}, \text{ when } \alpha \neq 0,$$

$$\Theta(w, 0) = \Theta(u, \infty) \leq 1 - \frac{\min\{i_0 + i_1 + \cdots + i_n\} - 2s}{\mu\nu}, \text{ when } \alpha = 0.$$

In summary,

if $\alpha = a$, $\alpha \neq 0$, then $\Theta(w, \alpha) \leq 1 - \dfrac{\min\{i_1 + i_2 + \cdots + i_n\} - s}{\mu\nu}$;

if $\alpha = a$, $\alpha = 0$, then $\Theta(w, 0) \leq 1 - \dfrac{\min\{i_0 + i_1 + \cdots + i_n\} - s}{\mu\nu}$;

if $\alpha \neq a$, $\alpha \neq 0$, then $\Theta(w, \alpha) \leq 1 - \dfrac{\min\{i_1 + i_2 + \cdots + i_n\} - 2s}{\mu\nu}$;

if $\alpha \neq a$, $\alpha = 0$, then $\Theta(w, 0) \leq 1 - \dfrac{\min\{i_0 + i_1 + \cdots + i_n\} - 2s}{\mu\nu}$.

This is the proof of Theorem 1.

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