Far East Journal of Mathematical Education

Published Online: April 2016
http://dx.doi.org/10.17654/ME016020237
Volume 16, Number 2, 2016, Pages 237-242 **ISSN: 0973-5631**

HOW TO FIT THE PARAMETERS: AN INTERESTING EXAMPLE

Kazuyuki Fujii[1] **and Hiroshi Oike**[2]

[1]International College of Arts and Sciences
Yokohama City University
Yokohama, 236-0027
Japan
e-mail: fujii@yokohama-cu.ac.jp

[2]Takado 85-5, Yamagata, 990-2464
Japan
e-mail: oike@tea.ocn.ne.jp

Abstract

In this paper, we revisit how to fit the parameters by using a simple mathematical model coming from Biology. Although, the Metropolis-Hastings method is standard, we show another approach based on algebraic equations.

To master multiple techniques is very good for undergraduates.

In this paper, we revisit a problem of how to fit the parameters in the model of [1] which comes from an experiment to measure the synthetic amount of Protein A in Escherichia coli and we present another approach based on algebraic equations. See for example [2] concerning what Protein A is ([3, 4]).

Received: January 4, 2016; Accepted: February 1, 2016

Keywords and phrases: escherichia coli, mathematical model, fitting of parameters, algebraic equation.

Communicated by K. K. Azad

We treat the experiment in [1] which measured the synthetic amount of Protein A in Escherichia coli. Let $A = A(t)$ be the synthetic amount of Protein A. If the synthetic rate of Protein A is constant in cells, then the differential equation becomes

$$\frac{dA}{dt} = \kappa - \gamma A, \tag{1}$$

where κ is the synthetic rate of Protein A (a constant) and γ is the rate of cell division (a constant). The second term of the right hand side indicates a dilution or degradation of Protein A arising from increase of Escherichia coli.

The equation is easily solved by the variation of constants (see for example [5]) and given by

$$A(t) = \frac{\kappa}{\gamma} + \left(P_0 - \frac{\kappa}{\gamma}\right)e^{-\gamma t}, \tag{2}$$

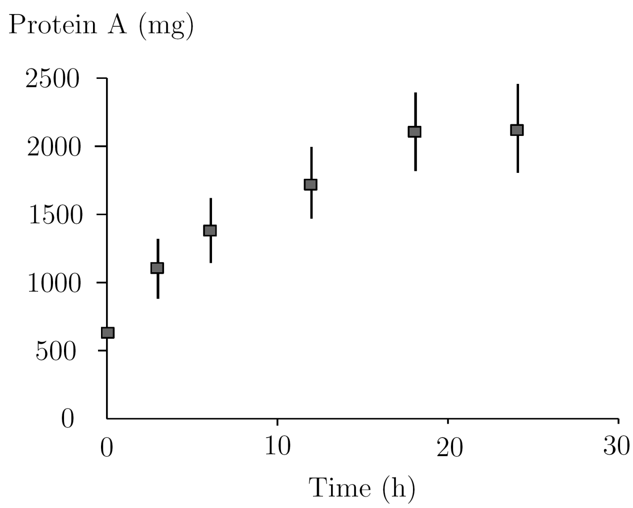
where $A(0) = P_0$ is the initial condition.

In general, a solution of the differential equation with two parameters is completely determined with two data. However, such data are not exact in almost all experiments. Data in an experiment are described with mean and standard deviation from Statistics.

We list the data in [1]:

Time	Protein	s.d.
0	540	0
3	1100	110
6	1400	140
12	1700	170
18	2100	210
24	2150	215

where for example 1100 means the mean of Protein A at three hours later and 110 means the standard deviation (s.d. for short) in Statistics. See the following figure:



How do we fit the parameters $\{\kappa, \gamma\}$ of (2) to data above? This is an important problem.

Method I. Usually, the Metropolis-Hastings method is one of standard ones. See for example [6] concerning what the method is. The result in [1] is

$$\kappa = 300,\ \gamma = 0.14. \tag{3}$$

We do not reproduce the result in the paper (it is desirable for undergraduates to follow this).

Method II. We show our method and result based on algebraic equations.

For undergraduates a detailed proof is given. For simplicity we set

$$\alpha = \frac{\kappa}{\gamma}$$

in the following. From the list of data above we would like to use long range data like

$$\begin{cases} 1100 = A(3) = \alpha + (540 - \alpha)e^{-3\gamma}, \\ 2150 = A(24) = \alpha + (540 - \alpha)e^{-24\gamma}. \end{cases} \tag{4}$$

This gives

$$e^{-3\gamma} = \frac{1100-\alpha}{540-\alpha},\ e^{-24\gamma} = \frac{2150-\alpha}{540-\alpha}.$$

From the equation

$$e^{-24\gamma} = (e^{-3\gamma})^8$$

we have

$$\frac{2150-\alpha}{540-\alpha} = \left(\frac{1100-\alpha}{540-\alpha}\right)^8$$

and

$$(2150-\alpha)(540-\alpha)^7 = (1100-\alpha)^8.$$

Since

$$(2150-\alpha)(540-\alpha)^7 - (1100-\alpha)^8 = 2870\alpha^7 - 19629400\alpha^6 + \cdots,$$

we set

$$f(\alpha) = \frac{(2150-\alpha)(540-\alpha)^7 - (1100-\alpha)^8}{2870}. \tag{5}$$

From here we use MATHEMATICA for calculation (because manual calculation is almost impossible). The expansion of $f(\alpha)$ becomes

$$\begin{aligned} f(\alpha) = \alpha^7 &- \frac{280420}{41}\alpha^6 + \frac{797986000}{41}\alpha^5 - \frac{1252310920000}{41}\alpha^4 \\ &+ \frac{1183225197280000}{41}\alpha^3 - \frac{676528764822400000}{41}\alpha^2 \\ &+ \frac{217188344755648000000}{41}\alpha - \frac{30211455971219200000000}{41}. \end{aligned} \tag{6}$$

This is complicated enough! The approximate solutions of $f(\alpha) = 0$ are given by

$$742.9675 - 512.7367i, \quad 742.9675 + 512.7367i,$$

$$780.0031 - 206.9822i, \quad 780.0031 + 206.9822i,$$

$$789.5461 - 60.1318i, \quad 789.5461 + 60.1318i,$$

$$2214.4788.$$

Here we have rounded off a real number to five decimal places like $3.14159\cdots = 3.1416.$ The real solution is just a one and we set

$$\alpha_{sol} = 2214 \tag{7}$$

($\alpha = \alpha_{sol}$ in the following for simplicity). Namely, we have

$$\alpha = \frac{\kappa}{\gamma} = 2214.$$

On the other hand, from (4)

$$e^{-3\gamma} = \frac{1100 - \alpha}{540 - \alpha} = \frac{1114}{1674}$$

so we obtain

$$\gamma = \frac{1}{3}\log\frac{1674}{1114} = 0.1358 \approx 0.136. \tag{8}$$

Therefore, the equation above gives

$$\kappa = \alpha\gamma = 2214 \times 0.136 \approx 301. \tag{9}$$

As a result we have

$$\kappa = 301, \ \gamma = 0.136. \tag{10}$$

Comparing (3) with (10) we reach the following:

Conclusion. The results obtained by the two methods are almost identical.

In the paper we studied the simple mathematical model coming from Biology or Systems Biology. Phenomena in Physics, Chemistry, Biology, etc. are in general very complicated, so to construct simple and deep (if possible) mathematical models for them is very important.

Undergraduates should master several techniques through mathematical models. The books [7, 8] are highly recommended.

References

[1] H. Takahashi, Let's try to touch with mathematical model (in Japanese), Journal of the Society for Biotechnology 91(2) (2013), 109.

[2] Wikipedia contributors, Protein A, ID: 670960874, Wikipedia, The Free Encyclopedia.

[3] D. J. Wilkinson, Stochastic Modelling for Systems Biology, Chapman and Hall/CRC, 2011.

[4] Uri Alon, An Introduction to Systems Biology: Design Principles of Biological Circuits, Chapman and Hall/CRC, 2007.

[5] S. Furuya, Introduction to Differential Equations (in Japanese), Saiensu Publishers, Tokyo, 1996.

[6] Wikipedia contributors, Metropolis-Hastings algorithm, ID: 694568942, Wikipedia, The Free Encyclopedia.

[7] D. N. Burghes and M. S. Borrie, Modelling with Differential Equations, Ellis Horwood Limited, 1990.

[8] K. Fujii et al., Treasure Box of Mathematical Sciences (in Japanese), Yuseisha Publishers, Tokyo, 2010.