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SHARP POINTWISE ESTIMATES FOR FUNCTIONS IN THE SOBOLEV SPACES $H^s(\mathbb{R}^n)$

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Abstract

We provide the optimal value of the constant $K(n, m)$ in the Gagliardo-Nirenberg supnorm inequality

$$\| u \|_{L^\infty(\mathbb{R}^n)} \le K(n, m) \| u \|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2m}} \| D^m u \|_{L^2(\mathbb{R}^n)}^{\frac{n}{2m}}, \quad m > n/2,$$

and its generalizations to the Sobolev spaces $H^s(\mathbb{R}^n)$ of arbitrary order $s > n/2$ as well.

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1. Introduction

In recent decades there has been a growing interest in determining the sharpest form of many important inequalities in analysis, see e.g. [1-5, 8, 10] and references therein. A noticeable miss is the fundamental Gagliardo-Nirenberg supnorm inequality

$$\| u \|_{L^\infty(\mathbb{R}^n)} \leq K(n, m) \| u \|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2m}} \| D^m u \|_{L^2(\mathbb{R}^n)}^{\frac{n}{2m}} \tag{1.1}$$

for functions $u \in H^m(\mathbb{R}^n)$ when $m > n/2$ (see [7, 9]), as well as some of its generalizations. Here, as usual, $H^m(\mathbb{R}^n)$ is the Sobolev space of functions $u \in L^2(\mathbb{R}^n)$ with all derivatives of order up to m in $L^2(\mathbb{R}^n)$, which is a Banach space under its natural norm defined by

$$\| u \|_{H^m(\mathbb{R}^n)} = \{\| u \|_{L^2(\mathbb{R}^n)}^2 + \| D^m u \|_{L^2(\mathbb{R}^n)}^2\}^{1/2}, \tag{1.2a}$$

where

$$\| D^m u \|_{L^2(\mathbb{R}^n)} = \left\{ \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \| D_{i_1} D_{i_2} \ldots D_{i_m} u \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/2} \tag{1.2b}$$

(with D_i denoting the weak derivative with respect to the variable x_i). In this brief note, we will review some basic results in order to derive the optimal (i.e., minimal) value for the constant $K(n, m)$ in (1.1) above. It will be seen in Section 2 that it turns out to be

$$K(n, m) = \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma\left(\frac{n}{2m}\right)}{\sigma\left(\frac{n}{2m}\right)} \right\}^{-\frac{1}{2}} \left\{ \frac{n}{2m - n} \right\}^{-\frac{n}{4m}} \left\{ \frac{2m}{2m - n} \right\}^{\frac{1}{2}}, \tag{1.3}$$

where $\sigma(r) := r\pi$ and $\Gamma(\cdot)$ is the Gamma function (for its definition, see e.g. [6, p. 7]). For example, we get, with $m = 2$ and $n = 1, 2, 3,$ the sharp pointwise estimates

$$\| u \|_{L^\infty(\mathbb{R})} \leq \frac{\sqrt[4]{2}}{\sqrt[8]{27}} \| u \|_{L^2(\mathbb{R})}^{\frac{3}{4}} \| D^2 u \|_{L^2(\mathbb{R})}^{\frac{1}{4}}, \tag{1.4a}$$

$$\| u \|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2} \| u \|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \| D^2 u \|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \tag{1.4b}$$

$$\| u \|_{L^\infty(\mathbb{R}^3)} \leq \frac{\sqrt[8]{12}}{\sqrt{6\pi}} \| u \|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \| D^2 u \|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}}, \tag{1.4c}$$

and so forth. (In [10], it is obtained that

$$\| u \|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{\sqrt{2\pi}} \| Du \|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \| D^2 u \|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}},$$

which is also shown to be optimal.) Also, setting

$$\| u \|_{\dot{H}^s(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} | \xi |^{2s} | \hat{u}(\xi) |^2 d\xi \right\}^{\frac{1}{2}} \tag{1.5}$$

for real $s > 0$,[1] where $\hat{u}(\cdot)$ denotes the Fourier transform of $u(\cdot)$, that is,

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx, \quad \xi \in \mathbb{R}^n, \tag{1.6}$$

and letting $H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \| u \|_{\dot{H}^s(\mathbb{R}^n)} < \infty\}$ be the Sobolev space of order s, we get the following generalization of (1.1), (1.3) above. If $s > n/2$, then

$$\| u \|_{L^\infty(\mathbb{R}^n)} \leq K(n, s) \| u \|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}} \| u \|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}} \tag{1.7a}$$

for all $u \in H^s(\mathbb{R}^n)$, with the optimal value of the constant $K(n, s)$ being

[1] Note that (1.5) corresponds to (1.2b) when $s = m$ (m integral), that is: $\| u \|_{\dot{H}^m(\mathbb{R}^n)} = \|D^m u\|_{L^2(\mathbb{R}^n)}$.

given by

$$K(n,\ s) = \{4\pi\}^{-\frac{n}{4}}\left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^{-\frac{1}{2}}\left\{\frac{\sin\sigma\left(\frac{n}{2m}\right)}{\sigma\left(\frac{n}{2m}\right)}\right\}^{-\frac{1}{2}}\left\{\frac{n}{2m-n}\right\}^{-\frac{n}{4m}}\left\{\frac{2m}{2m-n}\right\}^{\frac{1}{2}} \tag{1.7b}$$

for any $s > n/2$, where, as before, $\sigma(r) := r\pi$. The proof of (1.1), (1.3), (1.7) and of the sharpness of the values for K given in (1.3), (1.7b) above is provided in the sequel; in addition, a second classical estimate for $\| u \|_{L^\infty(\mathbb{R}^n)}$ when $u \in H^s(\mathbb{R}^n)$, $s > n/2$, is also reexamined here (see (2.4) below), so as to be similarly presented in its sharpest form.

2. Proof of (1.1), (1.3), (1.7) and Other Optimal Supnorm Results in $H^s(\mathbb{R}^n)$

To obtain the results stated in Section 1, we first review the following basic lemma. We recall that $\hat{u}$ denotes the Fourier transform of u, cf. (1.6) above.

Lemma 2.1. *Let* $u \in L^2(\mathbb{R}^n)$. *If* $\hat{u} \in L^1(\mathbb{R}^n)$, *then* $u \in L^\infty(\mathbb{R}^n)$ *and*

$$\| u \|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-n/2} \| \hat{u} \|_{L^1(\mathbb{R}^n)}. \tag{2.1}$$

Moreover, equality holds in (2.1) *if* $\hat{u}$ *is real-valued and of constant sign* (*say, nonnegative*).

Proof. Clearly, (2.1) is valid if $u \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of smooth, rapidly decreasing functions at infinity ([6, p. 4]), since we have, in this case, the representation (see e.g. [6, p. 16])

$$u_m(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}_m(\xi)d\xi, \quad \forall x \in \mathbb{R}^n. \tag{2.2}$$

For general $u \in L^2(\mathbb{R}^n)$ with $\hat{u} \in L^1(\mathbb{R}^n)$, let $\{\hat{u}_m\}$ be a sequence of Schwartz approximants to $\hat{u} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\| \hat{u}_m - \hat{u} \|_{L^1(\mathbb{R}^n)} \to 0$ and $\| \hat{u}_m - \hat{u} \|_{L^2(\mathbb{R}^n)} \to 0$ as $m \to \infty$, and let $u_m \in \mathcal{S}(\mathbb{R}^n)$ be the Fourier inverse of $\hat{u}_m$, for each m. Applying (2.1) to $\{u_m\}$, we see that $\{u_m\}$ is Cauchy in $L^\infty(\mathbb{R}^n)$, so that $\| u_m - v \|_{L^\infty(\mathbb{R}^n)} \to 0$ for some $v \in L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$. Since we have $\| u_m - u \|_{L^2(\mathbb{R}^n)} \to 0$, it follows that $u = v$. This shows that $u \in L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ and, letting $m \to \infty$ in (2.2), we also have

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \quad \forall x \in \mathbb{R}^n, \tag{2.2$'$}$$

from which (2.1) immediately follows. In particular, in the event that $\hat{u}(\xi) \geq 0$ for all ξ, we get from (2.2′) that $u(0) = (2\pi)^{-n/2} \| \hat{u} \|_{L^1(\mathbb{R}^n)}$, so that (2.1) becomes an identity in this case. □

We observe that the hypotheses of Lemma 2.1 are satisfied for $u \in H^s(\mathbb{R}^n)$ if $s > n/2$. An important consequence of this fact is the fundamental embedding property revisited next, where the norm in $H^s(\mathbb{R}^n)$ is set to be (in accordance with (1.2a) above):

$$\begin{aligned} \| u \|_{H^s(\mathbb{R}^n)} &= \{ \| u \|^2_{L^2(\mathbb{R}^n)} + \| u \|^2_{\dot{H}^s(\mathbb{R}^n)} \}^{1/2} \\ &= \left\{ \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi \right\}^{1/2}. \end{aligned} \tag{2.3}$$

Theorem 2.1. *Let* $s > n/2$. *If* $u \in H^s(\mathbb{R}^n)$, *then* $u \in L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ *and*

$$\| u \|_{L^\infty(\mathbb{R}^n)} \leq \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma\left(\frac{n}{2s}\right)}{\sigma\left(\frac{n}{2s}\right)} \right\}^{-\frac{1}{2}} \cdot \| u \|_{H^s(\mathbb{R}^n)}, \tag{2.4}$$

where $\sigma(r) = r\pi$. *Moreover, equality holds in* (2.4) *when* $\hat{u}(\xi) = c/(1 + |\xi|^{2s})$,

$\forall \xi \in \mathbb{R}^n$ *for some constant* $c \geq 0$, *so that the constant given in* (2.4) *above is optimal*.

Proof. Let $u \in H^s(\mathbb{R}^n)$, with $s > n/2$. By (2.1) and Cauchy-Schwarz's inequality, we have

$$\| u \|_{L^\infty(\mathbb{R}^n)} \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1 + |\xi|^{2s})^{-1/2} (1 + |\xi|^{2s})^{1/2} |\hat{u}(\xi)| d\xi \tag{2.5a}$$

$$\leq (2\pi)^{-n/2} \left\{ \int_{\mathbb{R}^n} (1 + |\xi|^{2s})^{-1} d\xi \right\}^{1/2}$$

$$\cdot \left\{ \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi \right\}^{1/2} \tag{2.5b}$$

$$= \{4\pi\}^{-n/4} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-1/2} \left\{ \frac{\sin \sigma\left(\frac{n}{2s}\right)}{\sigma\left(\frac{n}{2s}\right)} \right\}^{-1/2} \cdot \| u \|_{H^s(\mathbb{R}^n)}$$

by (2.3), since, using polar coordinates and the change of variable $t = (1 + r^{2s})^{-1}$, we obtain

$$\int_{\mathbb{R}^n} (1 + |\xi|^{2s})^{-1} d\xi = \omega_n \int_0^\infty (1 + r^{2s})^{-1} r^{n-1} dr$$

$$= \frac{\omega_n}{2s} \int_0^1 t^{-\frac{n}{2s}} (1 - t)^{\frac{n}{2s} - 1} dt = \frac{\omega_n}{n} \frac{\sigma\left(\frac{n}{2s}\right)}{\sin \sigma\left(\frac{n}{2s}\right)},$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit ball in $\mathbb{R}^n$ (see [6, p. 8]), and $\sigma(r) = r\pi$. This shows (2.4). Finally, if $\hat{u}(\xi) = c/(1 + |\xi|^{2s})$ for all ξ, for some $c \geq 0$ constant, then equality holds in both steps (2.5a) and (2.5b) above, so that (2.4) is an identity in this case, as claimed. □

We are now in very good standing to obtain (1.1), (1.7a) with their sharpest constants.

Theorem 2.2. *Let* $s > n/2$. *If* $u \in H^s(\mathbb{R}^n)$, *then*

$$\| u \|_{L^\infty(\mathbb{R}^n)} \leq K(n, s) \| u \|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}} \| u \|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}} \tag{2.6}$$

with $K(n, s)$ *defined in* (1.7b). *Moreover, equality holds in* (2.6) *if* $\hat{u}(\xi) = c/(1 + |\xi|^{2s})$, $\forall \xi \in \mathbb{R}^n$, $c \geq 0$ *constant, so that the numerical value of* K *given in* (1.7b) *is optimal*.

Proof. Let $s > n/2$, $u \in H^s(\mathbb{R}^n)$ fixed. Given $\lambda > 0$, setting $u_\lambda \in H^s(\mathbb{R}^n)$ by $u_\lambda(x) := u(\lambda x)$, we have, by (2.4), Theorem 2.1,

$$\| u \|_{L^\infty(\mathbb{R}^n)} = \| u_\lambda \|_{L^\infty(\mathbb{R}^n)} \leq \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma\left(\frac{n}{2s}\right)}{\sigma\left(\frac{n}{2s}\right)} \right\}^{-\frac{1}{2}}$$

$$\cdot \| u_\lambda \|_{H^s(\mathbb{R}^n)}$$

$$= \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma\left(\frac{n}{2s}\right)}{\sigma\left(\frac{n}{2s}\right)} \right\}^{-\frac{1}{2}}$$

$$\cdot \{ \lambda^{-n} \| u \|_{L^2(\mathbb{R}^n)}^2 + \lambda^{2s-n} \| u \|_{\dot{H}^s(\mathbb{R}^n)}^2 \}^{\frac{1}{2}}$$

for $\lambda > 0$ arbitrary. Choosing λ that minimizes the last term on the right of the above expression gives us (2.6), with $K(n, s)$ defined by (1.7b), as claimed. Now, to show that the value provided in (1.7b) is the best possible, we proceed as follows. First, we observe that, by Young's inequality,

$$\left\{\frac{n}{2s-n}\right\}^{-\frac{n}{4s}}\left\{\frac{2s}{2s-n}\right\}^{\frac{1}{2}}\| u \|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}}\| u \|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}}$$

$$\leq \{\| u \|_{L^2(\mathbb{R}^n)}^2 + \| u \|_{\dot{H}^s(\mathbb{R}^n)}^2\}^{\frac{1}{2}} = \| u \|_{H^s(\mathbb{R}^n)} \tag{2.7}$$

for all $u \in H^s(\mathbb{R}^n)$. Therefore, taking $w \in H^s(\mathbb{R}^n)$ defined by $\hat{w}(\xi) = c/(1+|\xi|^{2s})$, $c \geq 0$, we get, with $K(n, s)$ given in (1.7b):

$$\| w \|_{L^\infty(\mathbb{R}^n)} \leq K(n, s)\| w \|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}}\| w \|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}} \qquad \text{[by (2.6)]}$$

$$\leq \{4\pi\}^{-\frac{n}{4}}\left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^{-\frac{1}{2}}\left\{\frac{\sin \sigma\left(\frac{n}{2s}\right)}{\sigma\left(\frac{n}{2s}\right)}\right\}^{-\frac{1}{2}} \cdot \| w \|_{H^s(\mathbb{R}^n)}$$

[by (1.7b), (2.7)]

$$= \| w \|_{L^\infty(\mathbb{R}^n)}, \qquad \text{[by Theorem 2.1]}$$

thus showing that we have $\| w \|_{L^\infty(\mathbb{R}^n)} = K(n, s)\| w \|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}}\| w \|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}}$, as claimed. □

References

[1] M. Agueh, Gagliardo-Nirenberg inequalities involving the gradient L^2-norm, C. R. Math. Acad. Sci. Paris 346 (2008), 757-762.

[2] W. Beckner, Inequalities in Fourier analysis, Ann. Math. 102 (1975), 159-182.

[3] E. A. Carlen and M. Loss, Sharp constant in Nash's inequality, Inter. Math. Res. Notices 7 (1993), 213-215.

[4] D. Cordero-Erausquin, B. Nazaret and C. Villani, A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities, Adv. Math. 182 (2004), 307-332.

[5] M. Del Pino and J. Dolbeault, Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions, J. Math. Pures Appl. 81 (2002), 847-875.

[6] G. B. Folland, An Introduction to Partial Differential Equations, 2nd ed., Princeton University Press, Princeton, 1995.

[7] E. Gagliardo, Proprietà di alcune classi di funzioni in più variabili, Ricerche Mat. 7 (1958), 102-137.

[8] E. H. Lieb, Inequalities (Selecta), Springer, New York, 2003.

[9] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa 13 (1959), 115-162.

[10] W. Xie, A sharp pointwise bound for functions with L^2-Laplacians and zero boundary values of arbitrary three-dimensional domains, Indiana Univ. Math. J. 40 (1991), 1185-1192.