JP Journal of Algebra, Number Theory and Applications
© 2016 Pushpa Publishing House, Allahabad, India
Published Online: February 2016
http://dx.doi.org/10.17654/NT038020151
Volume 38, Number 2, 2016, Pages 151-184 **ISSN: 0972-5555**

ON SOME GENERALIZATIONS OF MEAN VALUE THEOREMS FOR ARITHMETIC FUNCTIONS OF TWO VARIABLES

Noboru Ushiroya

National Institute of Technology
Wakayama College
Gobo, Wakayama, Japan
e-mail: ushiroya@wakayama-nct.ac.jp

Abstract

Let $f : \mathbb{N}^2 \mapsto \mathbb{C}$ be an arithmetic function of two variables. We study the existence of the limit:

$$\lim_{x\to\infty} \frac{1}{x^2 (\log x)^{k-1}} \sum_{n_1, n_2 \le x} f(n_1, n_2),$$

where k is a fixed positive integer. Moreover, we express this limit as an infinite product over all prime numbers in the case that f is a multiplicative function of two variables. This study is a generalization of Cohen-van der Corput's results to the case of two variables.

1. Introduction

Let μ denote the Möbius function and let $\mu_k = \underbrace{\mu * \mu * \cdots * \mu}_{k}$ be the

Received: December 7, 2015; Accepted: December 30, 2015
2010 Mathematics Subject Classification: 11N37.
Keywords and phrases: asymptotic results, arithmetic functions.
Communicated by K. K. Azad

k-folded Dirichlet convolution of μ, that is,

$$\mu_k(n) = \sum\nolimits_{d_1 d_2 \cdots d_k = n} \mu(d_1)\mu(d_2)\cdots\mu(d_k)$$

for every n. Cohen [2] proved that if $f : \mathbb{N} \mapsto \mathbb{C}$ is an arithmetic function satisfying $\sum_{n=1}^{\infty} |(f * \mu_k)(n)|/n < \infty$, then

$$\lim_{x\to\infty} \frac{1}{x(\log x)^{k-1}} \sum_{n\le x} f(n) = \frac{1}{(k-1)!} \sum_{n=1}^{\infty} \frac{(f * \mu_k)(n)}{n}. \tag{1.1}$$

van der Corput [12] proved that if $f : \mathbb{N} \mapsto \mathbb{C}$ is a multiplicative function satisfying $\prod_{p\in\mathcal{P}} \left(\sum_{\nu=0}^{\infty} |(f * \mu_k)(p^{\nu})|/p^{\nu} \right) < \infty$, where $\mathcal{P}$ is the set of prime numbers, then

$$\lim_{x\to\infty} \frac{1}{x(\log x)^{k-1}} \sum_{n\le x} f(n) = \frac{1}{(k-1)!} \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)^k \left(\sum_{\nu=0}^{\infty} \frac{f(p^{\nu})}{p^{\nu}} \right). \tag{1.2}$$

We would like to generalize these results to the case in which f is an arithmetic function of two variables and obtain several interesting examples.

Let $\gcd(n_1, n_2)$ denote the greatest common divisor of n_1 and n_2, $\sigma(n)$ the sum of divisors of n, and $\varphi(n)$ Euler's totient function. Cohen [3] proved that

$$\sum_{n_1, n_2 \le x} \sigma(\gcd(n_1, n_2)) = x^2 \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} \right) + O(x^{\frac{3}{2}} \log x), \tag{1.3}$$

$$\sum_{n_1, n_2 \le x} \varphi(\gcd(n_1, n_2)) = \frac{x^2}{\zeta^2(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{2\zeta'(2)}{\zeta(2)} \right)$$

$$+ O(x^{\frac{3}{2}} \log x), \tag{1.4}$$

where $\zeta(n)$ is the Riemann zeta function.

Next we consider two functions s and c, where $s(n_1, n_2) = \sum_{d_1 | n_1, d_2 | n_2} \gcd(d_1, d_2)$ and $c(n_1, n_2) = \sum_{d_1 | n_1, d_2 | n_2} \varphi(\gcd(d_1, d_2))$. Nowak and Tóth [4] proved that

$$\sum_{n_1, n_2 \leq x} s(n_1, n_2) = \frac{2}{\pi^2} x^2 (\log^3 x + a_1 \log^2 x + a_2 \log x + a_3) + (x^{\frac{1117}{701}+\varepsilon}), \tag{1.5}$$

$$\sum_{n_1, n_2 \leq x} c(n_1, n_2) = \frac{12}{\pi^4} x^2 (\log^3 x + b_1 \log^2 x + b_2 \log x + b_3) + (x^{\frac{1117}{701}+\varepsilon}), \tag{1.6}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are explicit constants.

We would like to obtain these leading coefficients in (1.3) ~ (1.6) by a systematic method. We will calculate those leading coefficients in Examples 3, 4, 7 and 8 in Section 5. Although we cannot obtain remainder terms by our theorems, our method for obtaining leading terms is very simple and is applicable to many arithmetic functions of two variables.

2. Some Results

Let $\tilde{\mu}(n_1, n_2)$ denote the Dirichlet inverse of the gcd function, that is, $\tilde{\mu}$ is the function which satisfies $(\tilde{\mu} * \gcd)(n_1, n_2) = \delta(n_1, n_2)$ for every $n_1, n_2 \in \mathbb{N}$, where $\delta(n_1, n_2) = 1$ or 0 according to whether $n_1 = n_2 = 1$ or not. Let $x \wedge y$ denote $\min(x, y)$. We first establish the following theorem.

Theorem 1. *Let f be an arithmetic function of two variables satisfying*

$$\sum_{n_1, n_2 = 1}^{\infty} \frac{|(f * \tilde{\mu})(n_1, n_2)|}{n_1 n_2} < \infty. \tag{2.1}$$

Then we have

$$\lim_{x, y \to \infty} \frac{1}{xy \log x \wedge y} \sum_{n_1 \le x, n_2 \le y} f(n_1, n_2) = \frac{1}{\zeta(2)} \sum_{n_1, n_2 = 1}^{\infty} \frac{(f * \tilde{\mu})(n_1, n_2)}{n_1 n_2}. \tag{2.2}$$

The proof of Theorem 1 will be given in the next section. To proceed to the next theorem, we need some notations. Let

$$\tau_k(n_1, n_2) = (\underbrace{1 * 1 * \cdots * 1}_{k})(n_1, n_2)$$

stand for the k-folded Dirichlet convolution of the function **1**, where $\mathbf{1}(n_1, n_2) = 1$ for every $n_1, n_2 \in \mathbb{N}$. Let $\mu_k = \tau_k^{-1}$ denote the Dirichlet inverse of τ_k. Note that $\mu_1(n_1, n_2) = \mu(n_1)\mu(n_2)$. Similarly, let

$$\tilde{\tau}_1(n_1, n_2) = \gcd(n_1, n_2),$$

$$\tilde{\tau}_k(n_1, n_2) = (\underbrace{\mathbf{1} * \mathbf{1} * \cdots * \mathbf{1}}_{k-1} * \gcd)(n_1, n_2) \text{ if } k \ge 2.$$

We also denote $\tilde{\mu}_k = \tilde{\tau}_k^{-1}$ the Dirichlet inverse of $\tilde{\tau}_k$. Note that $\tilde{\mu}_1 = \tilde{\mu} = \gcd^{-1}$ and $\tilde{\mu}_k = \mu_{k-1} * \tilde{\mu}$ if $k \ge 2$. The next theorem is an extension of Cohen's theorem (1.1) to the case in which f is an arithmetic function of two variables.

Theorem 2. *Let f be an arithmetic function of two variables and let $k \in \mathbb{N}$.*

(i) *Suppose*

$$\sum_{n_1, n_2 = 1}^{\infty} \frac{|(f * \mu_k)(n_1, n_2)|}{n_1 n_2} < \infty. \tag{2.3}$$

Then we have

$$\lim_{x, y \to \infty} \frac{1}{xy(\log x \log y)^{k-1}} \sum_{n_1 \le x, n_2 \le y} f(n_1, n_2)$$

$$= C_k \sum_{n_1, n_2 = 1}^{\infty} \frac{(f * \mu_k)(n_1, n_2)}{n_1 n_2}, \tag{2.4}$$

where $C_k = \frac{1}{((k-1)!)^2}$.

(ii) *Suppose*

$$\sum_{n_1, n_2 = 1}^{\infty} \frac{|(f * \tilde{\mu}_k)(n_1, n_2)|}{n_1 n_2} < \infty. \tag{2.5}$$

Then we have

$$\lim_{x \to \infty} \frac{1}{x^2 (\log x)^{2k-1}} \sum_{n_1, n_2 \le x} f(n_1, n_2) = \tilde{C}_k \sum_{n_1, n_2 = 1}^{\infty} \frac{(f * \tilde{\mu}_k)(n_1, n_2)}{n_1 n_2}, \tag{2.6}$$

where $\tilde{C}_k = \frac{1}{\zeta(2)} \frac{1}{((k-1)!)^2 (2k-1)}$.

Remark. In part (ii), we do not deal with:

$$\lim_{x, y \to \infty} (xy(\log x \log y)^{k-1} \log x \wedge y)^{-1} \sum_{n_1 \le x, n_2 \le y} f(n_1, n_2)$$

since it is too complicated and we cannot obtain a simple formula.

The proof of Theorem 2 will also be given in the next section.

3. Proof of Theorem 1 and Theorem 2

The following lemma is well known (cf. Cohen [2]) and will be needed later.

Lemma 1. *For fixed* $\alpha \ge 0$ *and all x, we have*

$$\sum_{n \le x} \frac{\log^{\alpha} n}{n} = \frac{\log^{\alpha+1} x}{\alpha + 1} + O(1). \tag{3.1}$$

It is also well known that $\sum_{n_1, n_2 \le x} \gcd(n_1, n_2) = x^2 \log x / \zeta(2) + cx^2 + o(x^2)$, where c is a suitable constant (cf. Cesàro [1]). We would like to modify this formula as follows.

Lemma 2.

$$\lim_{x, y \to \infty} \frac{1}{xy \log x \wedge y} \sum_{n_1 \le x, n_2 \le y} \gcd(n_1, n_2) = \frac{1}{\zeta(2)}. \tag{3.2}$$

Proof. Let

$$A(x, y) = \#\{(n_1, n_2) : 1 \le n_1 \le x, 1 \le n_2 \le y, \gcd(n_1, n_2) = 1\}$$

$$= \sum_{n_1 \le x, n_2 \le y} \mu^2((\gcd(n_1, n_2))^2).$$

Applying Theorem 7 in Ushiroya [11] to the function $\mu^2((\gcd(n_1, n_2))^2)$ we have

$$\lim_{x, y \to \infty} \frac{1}{xy} A(x, y) = \frac{1}{\zeta(2)}.$$

From this we have

$$\sum_{n_1 \le x, n_2 \le y} \gcd(n_1, n_2)$$

$$= \sum_{1 \le d \le x \wedge y} d\, \#\{(n_1, n_2); 1 \le n_1 \le x, 1 \le n_2 \le y, \gcd(n_1, n_2) = d\}$$

$$= \sum_{1 \le d \le x \wedge y} d\, \#\left\{(n_1', n_2'); 1 \le n_1' \le \frac{x}{d}, 1 \le n_2' \le \frac{y}{d}, \gcd(n_1', n_2') = 1\right\}$$

$$= \sum_{1 \le d \le x \wedge y} dA\left(\frac{x}{d}, \frac{y}{d}\right) = \sum_{1 \le d \le x \wedge y} d\left(\frac{1}{\zeta(2)} \frac{x}{d} \frac{y}{d} + o\left(\frac{x}{d} \frac{y}{d}\right)\right)$$

$$= \frac{1}{\zeta(2)} xy \log x \wedge y + o(xy \log x \wedge y),$$

which implies (3.2). □

Lemma 3. *Let* $a(n_1, n_2)$ *be an arithmetic function of two variables satisfying* $\sum_{n_1, n_2=1}^{\infty} |a(n_1, n_2)| < \infty$. *Then we have*

$$\lim_{x, y\to\infty} \frac{1}{\log x \wedge y} \sum_{n_1 \leq x, n_2 \leq y} a(n_1, n_2) \log \frac{x}{n_1} \wedge \frac{y}{n_2} = \sum_{n_1, n_2=1}^{\infty} a(n_1, n_2). \tag{3.3}$$

Proof. We put $M = \sum_{n_1, n_2=1}^{\infty} a(n_1, n_2)$. Then for any $\varepsilon > 0$, there exists $N > 0$ such that $\left| \sum_{n_1, n_2 < N} a(n_1, n_2) - M \right| < \varepsilon$. If we take x and y sufficiently large such that $x \wedge y > N$, then we have

$$\begin{aligned} &\sum_{n_1 \leq x, n_2 \leq y} a(n_1, n_2) \log \frac{x}{n_1} \wedge \frac{y}{n_2} \\ &= \sum_{n_1, n_2 < N} a(n_1, n_2) \left(\log \frac{x}{n_1} \wedge \frac{y}{n_2} - \log x \wedge y \right) \\ &\quad + \log x \wedge y \sum_{n_1, n_2 < N} a(n_1, n_2) \\ &\quad + \sum_{\substack{n_1 \leq x, n_2 \leq y \\ n_1 \wedge n_2 \geq N}} a(n_1, n_2) \log \frac{x}{n_1} \wedge \frac{y}{n_2} =: I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 \ll \left(\sup_{n_1, n_2 < N} \log \left| \frac{\frac{x}{n_1} \wedge \frac{y}{n_2}}{x \wedge y} \right| \right) \sum_{n_1, n_2 < N} a(n_1, n_2) \ll \log N,$$

$$| I_2 - M \log x \wedge y | < \varepsilon \log x \wedge y,$$

and

$$I_3 \ll \log x \wedge y \sum_{\substack{n_1 \leq x, n_2 \leq y \\ n_1 \wedge n_2 \geq N}} a(n_1, n_2) \ll \varepsilon \log x \wedge y.$$

Therefore, we have

$$\limsup_{x,\, y\to\infty}\left|\frac{1}{\log x \wedge y}\sum_{n_1\le x,\, n_2\le y} a(n_1,\, n_2)\log\frac{x}{n_1}\wedge\frac{y}{n_2} - M\right| \ll 2\varepsilon.$$

Since ε is arbitrary, (3.3) holds. □

Now we can prove Theorem 1.

Proof of Theorem 1. We put $g = f * \tilde{\mu}$. Noting that $\tilde{\mu} * \tilde{\tau}_1 = \delta$ we have

$$\begin{aligned}\sum_{n_1\le x,\, n_2\le y} f(n_1,\, n_2) &= \sum_{n_1\le x,\, n_2\le y} (f * \tilde{\mu} * \tilde{\tau}_1)(n_1,\, n_2)\\ &= \sum_{n_1\le x,\, n_2\le y} (g * \tilde{\tau}_1)(n_1,\, n_2)\\ &= \sum_{d_1\delta_1\le x,\, d_2\delta_2\le y} g(d_1,\, d_2)\tilde{\tau}(\delta_1,\, \delta_2)\\ &= \sum_{n_1\le x,\, n_2\le y} g(n_1,\, n_2)\sum_{\delta_1\le\frac{x}{n_1},\, \delta_2\le\frac{y}{n_2}} \tilde{\tau}_1(\delta_1,\, \delta_2).\end{aligned}$$

From Lemma 2 we see that this equals

$$\sum_{n_1\le x,\, n_2\le y} g(n_1,\, n_2)\left\{\frac{1}{\zeta(2)}\frac{x}{n_1}\frac{y}{n_2}\log\left(\frac{x}{n_1}\wedge\frac{y}{n_2}\right) + o\left(\frac{x}{n_1}\frac{y}{n_2}\log\left(\frac{x}{n_1}\wedge\frac{y}{n_2}\right)\right)\right\}.$$

Applying Lemma 3 to the function $a(n_1, n_2) = g(n_1, n_2)/n_1n_2$, we see that the above equals

$$\frac{xy\log x\wedge y}{\zeta(2)}\sum_{n_1,\, n_2=1}^{\infty}\frac{g(n_1,\, n_2)}{n_1n_2} + o(xy\log x\wedge y),$$

which implies (2.2). Thus, the proof of Theorem 1 is now complete. □

Next we prove several lemmas needed later.

Lemma 4. $\sum_{n_1 \le x,\, n_2 \le y} \log n_1 \wedge n_2 = xy \log x \wedge y + o(xy \log x \wedge y).$

Proof. Without loss of generality, we may assume that $y \le x$. Let $[x]$ denote the greatest integer that is less than or equal to x. Using the well known formula $\sum_{1 \le n \le x} \log n = x \log x - x + O(\log x)$, we have

$$\sum_{n_1 \le x,\, n_2 \le y} \log n_1 \wedge n_2 = \sum_{n_2 \le y} \left(\sum_{n_1 = 1}^{n_2} \log n_1 + \sum_{n_1 = n_2 + 1}^{[x]} \log n_2 \right)$$

$$= \sum_{n_2 \le y} (n_2 \log n_2 - n_2 + O(\log n_2) + ([x] - n_2) \log n_2)$$

$$= \sum_{n_2 \le y} ([x] \log n_2 - n_2 + O(\log n_2))$$

$$= [x](y \log y - y + O(\log y)) + O(y^2)$$

$$= xy \log x \wedge y + o(xy \log x \wedge y).$$

Lemma 5. $\sum_{n_1, n_2 \le x} \frac{\log n_1 \wedge n_2}{n_1 n_2} = \frac{1}{3} (\log x)^3 + o((\log x)^3).$

Proof. Using Lemma 1 we have

$$\sum_{n_1, n_2 \le x} \frac{\log n_1 \wedge n_2}{n_1 n_2} = \sum_{n_2 \le x} \left(\sum_{n_1 \le n_2} \frac{\log n_1}{n_1 n_2} + \sum_{n_2 < n_1 \le x} \frac{\log n_2}{n_1 n_2} \right)$$

$$= \sum_{n_2 \le x} \left(\frac{(\log n_2)^2 + O(1)}{2 n_2} + \frac{\log n_2 (\log x - \log n_2 + O(1))}{n_2} \right)$$

$$= \frac{1}{6} (\log x)^3 + \frac{1}{2} (\log x)^3 - \frac{1}{3} (\log x)^3 + o((\log x)^3)$$

$$= \frac{1}{3} (\log x)^3 + o((\log x)^3). \qquad \square$$

Lemma 6. *For fixed* $\alpha, \beta \geq 0$ *and all* x, *we have*

$$\sum_{n_1, n_2 \leq x} \frac{(\log n_1)^{\alpha} (\log n_2)^{\beta} \log \frac{x}{n_1} \wedge \frac{x}{n_2}}{n_1 n_2} = \frac{(\log x)^{\alpha+\beta+3}}{(\alpha+1)(\beta+1)(\alpha+\beta+3)} + o((\log x)^{\alpha+\beta+3}). \tag{3.4}$$

Proof. Using Lemma 1 we see that the left side of (3.4) equals

$$\sum_{n_2 \leq x} \left(\sum_{n_1 \leq n_2} \frac{(\log n_1)^{\alpha} (\log n_2)^{\beta} \log \frac{x}{n_2}}{n_1 n_2} + \sum_{n_2 < n_1 \leq x} \frac{(\log n_1)^{\alpha} (\log n_2)^{\beta} \log \frac{x}{n_1}}{n_1 n_2} \right)$$

$$= \sum_{n_2 \leq x} \left(\sum_{n_1 \leq n_2} \frac{(\log n_1)^{\alpha} (\log n_2)^{\beta} (\log x - \log n_2)}{n_1 n_2} + \sum_{n_2 < n_1 \leq x} \frac{(\log n_1)^{\alpha} (\log n_2)^{\beta} (\log x - \log n_1)}{n_1 n_2} \right)$$

$$= \sum_{n_2 \leq x} \left(\frac{((\log n_2)^{\alpha+1} + O(1))(\log n_2)^{\beta} (\log x - \log n_2)}{(\alpha+1) n_2} + \frac{((\log x)^{\alpha+1} - (\log n_2)^{\alpha+1} + O(1))(\log n_2)^{\beta} \log x}{(\alpha+1) n_2} - \frac{((\log x)^{\alpha+2} - (\log n_2)^{\alpha+2} + O(1))(\log n_2)^{\beta}}{(\alpha+2) n_2} \right)$$

$$= \sum_{n_2 \leq x} \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+2} \right) \frac{(\log x)^{\alpha+2} (\log n_2)^{\beta} - (\log n_2)^{\alpha+\beta+2}}{n_2} + o((\log x)^{\alpha+\beta+3})$$

$$= \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{(\log x)^{\alpha+\beta+3}}{\beta+1} - \frac{(\log x)^{\alpha+\beta+3}}{\alpha+\beta+3}\right) + o((\log x)^{\alpha+\beta+3})$$

$$= \frac{(\log x)^{\alpha+\beta+3}}{(\alpha+1)(\beta+1)(\alpha+\beta+3)} + o((\log x)^{\alpha+\beta+3}).$$

This proves Lemma 6. □

The next lemma gives a partial summation formula in the case of a function of two variables.

Lemma 7. *Let* $a(n_1, n_2)$ *be an arithmetic function of two variables and let* $M(x, y) = \sum_{n_1 \le x, n_2 \le y} a(n_1, n_2)$. *Then we have*

$$\sum_{n_1 \le x, n_2 \le y} \frac{a(n_1, n_2)}{n_1 n_2} = \sum_{\substack{n_1 \le x \\ n_2 \le y}} \frac{M(n_1, n_2)}{n_1(n_1+1)n_2(n_2+1)} + \sum_{n_1 \le x} \frac{M(n_1, y)}{n_1(n_1+1)([y]+1)}$$

$$+ \sum_{n_2 \le y} \frac{M(x, n_2)}{n_2(n_2+1)([x]+1)} + \frac{M(x, y)}{([x]+1)([y]+1)}, \quad (3.5)$$

where $[x]$ *is the greatest integer that is less than or equal to x.*

Proof. We put $M(x, y) = 0$ if $x < 1$ or $y < 1$ for convenience. Then we see that the left side of (3.5) equals

$$\sum_{n_1 \le x, n_2 \le y} \frac{M(n_1, n_2) - M(n_1 - 1, n_2) - M(n_1, n_2 - 1) + M(n_1 - 1, n_2 - 1)}{n_1 n_2}$$

$$= \sum_{n_1 \le x, n_2 \le y} M(n_1, n_2)\left\{\frac{1}{n_1 n_2} - \frac{1}{(n_1+1)n_2}\right.$$

$$\left. - \frac{1}{n_1(n_2+1)} + \frac{1}{(n_1+1)(n_2+1)}\right\}$$

$$+ \sum_{n_2 \le y} \frac{M(x, n_2)}{([x]+1)n_2} + \sum_{n_1 \le x} \frac{M(n_1, y)}{n_1([y]+1)} - \sum_{n_1 \le x} \frac{M(n_1, y)}{(n_1+1)([y]+1)}$$

$$- \sum_{n_2 \le y} \frac{M(x, n_2)}{([x]+1)(n_2+1)} + \frac{M(x, y)}{([x+1]+1)([y]+1)}$$

$$= \sum_{n_1 \le x, n_2 \le y} M(n_1, n_2) \frac{1}{n_1(n_1+1)n_2(n_2+1)}$$

$$+ \sum_{n_2 \le y} \frac{M(x, n_2)}{([x]+1)} \left(\frac{1}{n_2} - \frac{1}{n_2+1} \right)$$

$$+ \sum_{n_1 \le x} \frac{M(n_1, y)}{([y]+1)} \left(\frac{1}{n_1} - \frac{1}{n_1+1} \right) + \frac{M(x, y)}{([x]+1)([y]+1)},$$

which equals the right side of (3.5). □

The next lemma is an extension of Proposition 5 in van der Corput [12] to the case of arithmetical functions of two variables.

Lemma 8. *Let a, b be arithmetical functions of two variables and let $c = a * b$. For $\alpha, \beta \ge 0$, we assume that*

$$\lim_{x, y \to \infty} \frac{1}{xy(\log x)^{\alpha}(\log y)^{\beta}} \sum_{n_1 \le x, n_2 \le y} a(n_1, n_2) = A,$$

where A is a constant.

(i) *If* $\lim_{x, y \to \infty} \frac{1}{xy} \sum_{n_1 \le x, n_2 \le y} b(n_1, n_2) = B$, *where B is a constant, then*

$$\lim_{x \to \infty} \frac{1}{xy(\log x)^{\alpha+1}(\log y)^{\beta+1}} \sum_{n_1 \le x, n_2 \le y} c(n_1, n_2) = \frac{AB}{(\alpha+1)(\beta+1)}. \quad (3.6)$$

(ii) *If* $\lim_{x, y \to \infty} \frac{1}{xy \log x \wedge y} \sum_{n_1 \le x, n_2 \le y} b(n_1, n_2) = B$, *where B is a constant, then*

$$\lim_{x \to \infty} \frac{1}{x^2(\log x)^{\alpha+\beta+3}} \sum_{n_1, n_2 \le x} c(n_1, n_2) = \frac{AB}{(\alpha+1)(\beta+1)(\alpha+\beta+3)}. \quad (3.7)$$

Proof. We first prove (i). We have

$$\sum_{n_1 \le x,\, n_2 \le y} c(n_1, n_2) = \sum_{n_1 \le x,\, n_2 \le y} (a * b)(n_1, n_2)$$

$$= \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le y} a(\ell_1, \ell_2) b(m_1, m_2)$$

$$= \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le y} a(\ell_1, \ell_2)(b(m_1, m_2) - B)$$

$$+ \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le y} (a(\ell_1, \ell_2) - A(\log \ell_1)^{\alpha} (\log \ell_2)^{\beta})$$

$$+ AB \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le y} (\log \ell_1)^{\alpha} (\log \ell_2)^{\beta} =: I_1 + I_2 + I_3,$$

where, by Lemma 7 and Lemma 1,

$$I_1 = \sum_{\ell_1 \le x,\, \ell_2 \le y} a(\ell_1, \ell_2) \sum_{\substack{m_1 \le x/\ell_1 \\ m_2 \le y/\ell_2}} (b(m_1, m_2) - B)$$

$$= \sum_{\ell_1 \le x,\, \ell_2 \le y} a(\ell_1, \ell_2)\, o\left(\frac{xy}{\ell_1 \ell_2}\right)$$

$$= o\left(xy \sum_{\ell_1 \le x,\, \ell_2 \le y} \frac{A \ell_1 \ell_2 (\log \ell_1)^{\alpha} (\log \ell_2)^{\beta}}{\ell_1(\ell_1 + 1)\ell_2(\ell_2 + 1)} \right)$$

$$= o(xy(\log x)^{\alpha+1} (\log y)^{\beta+1}),$$

$$I_2 = B \sum_{\ell_1 \le x,\, \ell_2 \le y} (a(\ell_1, \ell_2) - A(\log \ell_1)^{\alpha} (\log \ell_2)^{\beta}) \sum_{m_1 \le x/\ell_1,\, m_2 \le y/\ell_2} 1$$

$$= B \sum_{\ell_1 \le x,\, \ell_2 \le y} \frac{xy}{\ell_1 \ell_2} (a(\ell_1, \ell_2) - A(\log \ell_1)^{\alpha} (\log \ell_2)^{\beta})$$

$$= B \sum_{\ell_1 \le x,\, \ell_2 \le y} xy \frac{o(\ell_1 \ell_2 (\log \ell_1)^{\alpha} (\log \ell_2)^{\beta})}{\ell_1(\ell_1+1)\ell_2(\ell_2+1)}$$

$$= o(xy(\log x)^{\alpha+1}(\log y)^{\beta+1}),$$

and

$$I_3 = AB \sum_{\ell_1 \le x,\, \ell_2 \le y} (\log \ell_1)^{\alpha} (\log \ell_2)^{\beta} \sum_{m_1 \le x/\ell_1,\, m_2 \le y/\ell_2} 1$$

$$= AB \sum_{\ell_1 \le x,\, \ell_2 \le y} \frac{xy}{\ell_1 \ell_2} (\log \ell_1)^{\alpha} (\log \ell_2)^{\beta}$$

$$= \frac{AB}{(\alpha+1)(\beta+1)} xy(\log x)^{\alpha+1}(\log y)^{\beta+1} + o(xy(\log x)^{\alpha+1}(\log y)^{\beta+1}).$$

Therefore, (3.6) holds. This proves (i).

Next we prove (ii). Similarly we have

$$\sum_{n_1, n_2 \le x} c(n_1, n_2)$$

$$= \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le x} a(\ell_1, \ell_2) b(m_1, m_2)$$

$$= \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le x} a(\ell_1, \ell_2)(b(m_1, m_2) - B \log m_1 \wedge m_2)$$

$$+ B \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le x} (a(\ell_1, \ell_2) - A(\log \ell_1)^{\alpha}(\log \ell_2)^{\beta}) \log m_1 \wedge m_2$$

$$+ AB \sum_{\ell_1 m_1 \le x,\, \ell_2 m_2 \le x} (\log \ell_1)^{\alpha}(\log \ell_2)^{\beta} \log m_1 \wedge m_2 =: J_1 + J_2 + J_3.$$

First, we have

$$J_1 = \sum_{\ell_1, \ell_2 \le x} a(\ell_1, \ell_2) \sum_{m_1 \le x/\ell_1, m_2 \le x/\ell_2} (b(m_1, m_2) - B \log m_1 \wedge m_2)$$

$$= \sum_{\ell_1, \ell_2 \le x} a(\ell_1, \ell_2) o\left(\frac{x}{\ell_1}\frac{x}{\ell_2} \log \frac{x}{\ell_1} \wedge \frac{x}{\ell_2}\right).$$

Since $\log \frac{x}{k_1} \wedge \frac{x}{k_2} \le \log x,$ we have by Lemma 7 and Lemma 1

$$J_1 \ll o(x^2 \log x) \sum_{\ell_1, \ell_2 \le x} \frac{|a(\ell_1, \ell_2)|}{\ell_1 \ell_2} = o(x^2 (\log x)^{\alpha+\beta+3}).$$

Second, we have by Lemma 5

$$J_2 = B \sum_{m_1, m_2 \le x} \log m_1 \wedge m_2 \sum_{\ell_1 \le x/m_1, \ell_2 \le x/m_2} (a(\ell_1, \ell_2) - A(\log \ell_1)^{\alpha} (\log \ell_2)^{\beta})$$

$$= B \sum_{m_1, m_2 \le x} (\log m_1 \wedge m_2) o\left(\frac{x}{m_1}\frac{x}{m_2}\left(\log \frac{x}{m_1}\right)^{\alpha}\left(\log \frac{x}{m_2}\right)^{\beta}\right)$$

$$= o\left(x^2 (\log x)^{\alpha+\beta} \sum_{m_1, m_2 \le x} \frac{\log m_1 \wedge m_2}{m_1 m_2}\right)$$

$$= o\left(x^2 (\log x)^{\alpha+\beta} \cdot \frac{1}{3} (\log x)^3\right)$$

$$= o(x^2 (\log x)^{\alpha+\beta+3}).$$

Third, we have by Lemma 4 and Lemma 6

$$J_3 = AB \sum_{\ell_1, \ell_2 \le x} (\log \ell_1)^{\alpha} (\log \ell_2)^{\beta} \sum_{m_1 \le x/\ell_1, m_2 \le x/\ell_2} \log m_1 \wedge m_2$$

$$= AB \sum_{\ell_1, \ell_2 \le x} (\log \ell_1)^{\alpha} (\log \ell_2)^{\beta}$$

$$\times \left(\frac{x}{\ell_1}\frac{x}{\ell_2} \log \frac{x}{\ell_1} \wedge \frac{x}{\ell_2} + o\left(\frac{x}{\ell_1}\frac{x}{\ell_2} \log \frac{x}{\ell_1} \wedge \frac{x}{\ell_2}\right)\right)$$

$$= AB \frac{x^2 (\log x)^{\alpha+\beta+3}}{(\alpha+1)(\beta+1)(\alpha+\beta+3)} + o(x^2 (\log x)^{\alpha+\beta+3}).$$

From these estimates we have

$$\sum_{n_1, n_2 \le x} c(n_1, n_2) = AB \frac{x^2 (\log x)^{\alpha+\beta+3}}{(\alpha+1)(\beta+1)(\alpha+\beta+3)} + o(x^2 (\log x)^{\alpha+\beta+3}).$$

Thus, the proof of Lemma 8 is now complete. □

Now we can prove Theorem 2.

Proof of Theorem 2. We first prove (i). We proceed by induction on k. If $k = 1$, then (2.4) holds by Theorem 1 in Ushiroya [10]. Let $k \ge 2$ and suppose that (2.4) holds for $k - 1$ instead of k. We put $g = f * \mu_k$ and $h = g * \tau_{k-1}$. Since

$$\sum_{n_1, n_2 = 1}^{\infty} \frac{|g(n_1, n_2)|}{n_1 n_2} = \sum_{n_1, n_2 = 1}^{\infty} \frac{|h * \mu_{k-1}(n_1, n_2)|}{n_1 n_2} < \infty$$

holds by the induction hypothesis, we obtain

$$\lim_{x, y \to \infty} \frac{1}{xy(\log x \log y)^{k-2}} \sum_{n_1 \le x, n_2 \le y} h(n_1, n_2) = C_{k-1} \sum_{n_1, n_2 = 1}^{\infty} \frac{g(n_1, n_2)}{n_1 n_2}.$$

Since $f = h * \mathbf{1}$, we have by taking $a = h$, $b = \mathbf{1}$ and $\alpha = \beta = k - 2$ in Lemma 8(i)

$$\lim_{x, y \to \infty} \frac{1}{xy(\log x \log y)^{k-1}} \sum_{n_1 \le x, n_2 \le y} f(n_1, n_2)$$

$$= \frac{1}{(k-1)^2} C_{k-1} \sum_{n_1, n_2 = 1}^{\infty} \frac{g(n_1, n_2)}{n_1 n_2} = C_k \sum_{n_1, n_2 = 1}^{\infty} \frac{g(n_1, n_2)}{n_1 n_2}.$$

This proves (i).

Next we prove (ii). Similarly we proceed by induction on k. If $k = 1$, then (2.6) holds by Theorem 1. Let $k \ge 2$ and suppose that (2.6) holds for $k - 1$ instead of k. We put $g = f * \tilde{\mu}_k$ and $h = g * \tau_{k-1}$. Since

$$\sum_{n_1, n_2 = 1}^{\infty} \frac{|g(n_1, n_2)|}{n_1 n_2} = \sum_{n_1, n_2 = 1}^{\infty} \frac{|h * \mu_{k-1}(n_1, n_2)|}{n_1 n_2} < \infty,$$

we have by Theorem 2(i)

$$\lim_{x, y \to \infty} \frac{1}{xy(\log x \log y)^{k-2}} \sum_{n_1 \leq x, n_2 \leq y} h(n_1, n_2) = C_{k-1} \sum_{n_1, n_2 = 1}^{\infty} \frac{g(n_1, n_2)}{n_1 n_2}.$$

Since $f = h * \tilde{\tau}_1$, we have by taking $a = h$, $b = \tilde{\tau}_1$ and $\alpha = \beta = k - 2$ in Lemma 8(ii)

$$\lim_{x \to \infty} \frac{1}{x^2 (\log x)^{2k-1}} \sum_{n_1, n_2 \leq x} f(n_1, n_2)$$

$$= \frac{1}{(k-1)^2 (2k-1)} \frac{C_{k-1}}{\zeta(2)} \sum_{n_1, n_2 = 1}^{\infty} \frac{g(n_1, n_2)}{n_1 n_2} = \tilde{C}_k \sum_{n_1, n_2 = 1}^{\infty} \frac{g(n_1, n_2)}{n_1 n_2}.$$

Thus, the proof of Theorem 2 is now complete. □

4. Multiplicative Case

We say that f is a multiplicative function of two variables if f satisfies

$$f(m_1 n_1, m_2 n_2) = f(m_1, m_2) f(n_1, n_2)$$

for any $m_1, m_2, n_1, n_2 \in \mathbb{N}$ satisfying $\gcd(m_1 m_2, n_1 n_2) = 1$. It is well known that if f and g are multiplicative functions of two variables, then $f * g$ also becomes a multiplicative function of two variables. The next theorem is an extension of van der Corput's theorem (1.2) to the case in which f is a multiplicative function of two variables.

Theorem 3. *Let f be a multiplicative function of two variables and let $k \in \mathbb{N}$.*

(i) *Suppose*

$$\sum_{p \in \mathcal{P}} \sum_{\substack{\nu_1, \nu_2 \geq 0 \\ \nu_1 + \nu_2 \geq 1}} \frac{|(f * \mu_k)(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1 + \nu_2}} < \infty. \tag{4.1}$$

Then we have

$$\lim_{x,\, y\to\infty} \frac{1}{xy(\log x \log y)^{k-1}} \sum_{\substack{n_1\le x \\ n_2\le y}} f(n_1,\, n_2)$$

$$= C_k \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right)^{2k} \left(\sum_{\nu_1,\, \nu_2\ge 0} \frac{f(p^{\nu_1},\, p^{\nu_2})}{p^{\nu_1+\nu_2}} \right), \tag{4.2}$$

where $C_k = \dfrac{1}{((k-1)!)^2}$.

(ii) *Suppose*

$$\sum_{p\in\mathcal{P}} \sum_{\substack{\nu_1,\, \nu_2\ge 0 \\ \nu_1+\nu_2\ge 1}} \frac{|(f * \tilde{\mu}_k)(p^{\nu_1},\, p^{\nu_2})|}{p^{\nu_1+\nu_2}} < \infty. \tag{4.3}$$

Then we have

$$\lim_{x\to\infty} \frac{1}{x^2(\log x)^{2k-1}} \sum_{n_1\le x} f(n_1,\, n_2)$$

$$= \tilde{C}_k' \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right)^{2k+1} \left(\sum_{\nu_1,\, \nu_2\ge 0} \frac{f(p^{\nu_1},\, p^{\nu_2})}{p^{\nu_1+\nu_2}} \right), \tag{4.4}$$

where $\tilde{C}_k' = \zeta(2)\tilde{C}_k = \dfrac{1}{((k-1)!)^2(2k-1)}$.

Remark. In part (ii), we do not deal with:

$$\lim_{x,\, y\to\infty} (xy(\log x \log y)^{k-1} \log x \wedge y)^{-1} \sum_{n_1\le x,\, n_2\le y} f(n_1,\, n_2)$$

since it is too complicated and we cannot obtain a simple formula.

Before we prove Theorem 3, we give lemmas needed later.

Lemma 9 (Sándor and Crstici [5], p. 107)**.** *For* $k \in \mathbb{N}$ *and* $p \in \mathcal{P}$, *we have*

$$\mu_k(p^{\nu_1}, p^{\nu_2}) = \begin{cases} (-1)^{\nu_1+\nu_2}\binom{k}{\nu_1}\binom{k}{\nu_2}, & \text{if } \nu_1, \nu_2 \le k, \\ 0, & \text{otherwise,} \end{cases}$$

where $\binom{k}{\nu}$ *is a binomial coefficient.*

Lemma 10. *For* $p \in \mathcal{P}$ *we have*

$$\tilde{\mu}(p^{\nu_1}, p^{\nu_2}) = \begin{cases} -1, & \text{if } \nu_1 + \nu_2 = 1, \\ 2 - p, & \text{if } \nu_1 = \nu_2 = 1, \\ p - 1, & \text{if } |\nu_1 - \nu_2| = 1 \text{ and } \nu_1, \nu_2 \ge 1, \\ 2 - 2p, & \text{if } \nu_1 = \nu_2 \ge 2, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let f be the multiplicative function defined by the same formulas as the above. Then, by an elementary calculation, it is easy to see that $(f * \gcd)(p^a, p^b) = \delta(p^a, p^b)$ holds for every $a, b \ge 0$. By the uniqueness of the Dirichlet inverse of the gcd function, we have $f = \tilde{\mu}$. □

Now we can prove Theorem 3.

Proof of Theorem 3. We first prove (i). Since the function: $(n_1, n_2) \mapsto \frac{(f * \mu_k)(n_1, n_2)}{n_1 n_2}$ is multiplicative, we have

$$\sum_{n_1 \le x, n_2 \le y} \frac{|(f * \mu_k)(n_1, n_2)|}{n_1 n_2}$$

$$\le \prod_{p \in \mathcal{P}} \left(\sum_{\nu_1, \nu_2 \ge 0} \frac{1}{p^{\nu_1+\nu_2}} |(f * \mu_k)(p^{\nu_1}, p^{\nu_2})| \right)$$

$$= \prod_{p \in \mathcal{P}} \left(1 + \sum_{\nu_1 + \nu_2 \geq 1} \frac{1}{p^{\nu_1 + \nu_2}} |(f * \mu_k)(p^{\nu_1}, p^{\nu_2})| \right)$$

$$\leq \exp \left(\sum_{p} \left(\sum_{\nu_1 + \nu_2 \geq 1} \frac{1}{p^{\nu_1 + \nu_2}} |(f * \mu_k)(p^{\nu_1}, p^{\nu_2})| \right) \right) < \infty,$$

where we have used the well known inequality $1 + x \leq \exp(x)$ for $x \geq 0$. Therefore, (2.4) holds by Theorem 2(i). On the other hand, using Lemma 9 we have

$$\sum_{\nu_1, \nu_2 \geq 0} \frac{(f * \mu_k)(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1 + \nu_2}}$$

$$= \sum_{a_1, a_2, b_1, b_2 = 0}^{\infty} \frac{f(p^{a_1}, p^{a_2}) \mu_k(p^{b_1}, p^{b_2})}{p^{a_1 + b_1 + a_2 + b_2}}$$

$$= \sum_{a_1, a_2 = 0}^{\infty} \frac{f(p^{a_1}, p^{a_2})}{p^{a_1 + a_2}} \sum_{b_1, b_2 = 0}^{k} \frac{(-1)^{b_1 + b_2} \binom{k}{b_1} \binom{k}{b_2}}{p^{b_1 + b_2}}$$

$$= \sum_{a_1, a_2 = 0}^{\infty} \frac{f(p^{a_1}, p^{a_2})}{p^{a_1 + a_2}} \left(1 - \frac{1}{p} \right)^{2k}.$$

Hence, the right side of (2.4) is equal to

$$C_k \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p} \right)^{2k} \left(\sum_{\nu_1, \nu_2 \geq 0} \frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1 + \nu_2}} \right).$$

This proves (i).

Next we prove (ii). Similarly we have

$$\sum_{m_1, m_2 \leq x} \frac{|(f * \tilde{\mu}_k)(m_1, m_2)|}{m_1 m_2}$$

$$\leq \prod_{p \in \mathcal{P}} \left(\sum_{\nu_1, \nu_2 \geq 0} \frac{1}{p^{\nu_1 + \nu_2}} | (f * \tilde{\mu}_k)(p^{\nu_1}, p^{\nu_2}) | \right)$$

$$\leq \prod_{p \in \mathcal{P}} \left(1 + \sum_{\nu_1 + \nu_2 \geq 1} \frac{1}{p^{\nu_1 + \nu_2}} | (f * \tilde{\mu}_k)(p^{\nu_1}, p^{\nu_2}) | \right)$$

$$\leq \exp \left(\sum_{p \in \mathcal{P}} \left(\sum_{\nu_1 + \nu_2 \geq 1} \frac{1}{p^{\nu_1 + \nu_2}} | (f * \tilde{\mu}_k)(p^{\nu_1}, p^{\nu_2}) | \right) \right) < \infty.$$

Therefore, (2.6) holds by Theorem 2(ii). On the other hand, we have

$$\sum_{\nu_1, \nu_2 \geq 0} \frac{(f * \tilde{\mu}_k)(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1 + \nu_2}} = \sum_{a_1, a_2 = 0}^{\infty} \frac{f(p^{a_1}, p^{a_2})}{p^{a_1 + a_2}} \sum_{b_1, b_2 = 0}^{\infty} \frac{\tilde{\mu}_k(p^{b_1}, p^{b_2})}{p^{b_1 + b_2}}.$$

If $k \geq 2$, then noting that $\tilde{\mu}_k = \mu_{k-1} * \tilde{\mu}$ we have

$$\sum_{b_1, b_2 = 0}^{\infty} \frac{\tilde{\mu}_k(p^{b_1}, p^{b_2})}{p^{b_1 + b_2}} = \sum_{c_1, c_2, d_1, d_2 = 0}^{\infty} \frac{\mu_{k-1}(p^{c_1}, p^{c_2})}{p^{c_1 + c_2}} \frac{\tilde{\mu}(p^{d_1}, p^{d_2})}{p^{d_1 + d_2}}$$

$$= \sum_{c_1, c_2 = 0}^{k} \frac{(-1)^{c_1 + c_2}}{p^{c_1 + c_2}} \binom{k-1}{c_1} \binom{k-1}{c_2} \sum_{d_1, d_2 = 0}^{\infty} \frac{\tilde{\mu}(p^{d_1}, p^{d_2})}{p^{d_1 + d_2}}$$

$$= \left(1 - \frac{1}{p}\right)^{2(k-1)} \sum_{d_1, d_2 = 0}^{\infty} \frac{\tilde{\mu}(p^{d_1}, p^{d_2})}{p^{d_1 + d_2}}.$$

Using the relation $\tilde{\mu} * \gcd = \delta$ we have

$$\left(\sum_{d_1, d_2 = 0}^{\infty} \frac{\tilde{\mu}(p^{d_1}, p^{d_2})}{p^{d_1 + d_2}} \right) \left(\sum_{d_1, d_2 = 0}^{\infty} \frac{\gcd(p^{d_1}, p^{d_2})}{p^{d_1 + d_2}} \right) = 1,$$

where, by an elementary calculation, we can easily derive

$$\sum_{d_1, d_2=0}^{\infty} \frac{\gcd(p^{d_1}, p^{d_2})}{p^{d_1+d_2}} = \sum_{d_1, d_2=0}^{\infty} \frac{p^{d_1 \wedge d_2}}{p^{d_1+d_2}} = \frac{1-\frac{1}{p^2}}{\left(1-\frac{1}{p}\right)^3}.$$

Therefore, we have obtained the following two formulas:

$$\sum_{b_1, b_2=0}^{\infty} \frac{\tilde{\mu}(p^{b_1}, p^{b_2})}{p^{b_1+b_2}} = \frac{\left(1-\frac{1}{p}\right)^3}{1-\frac{1}{p^2}}, \tag{4.5}$$

$$\sum_{b_1, b_2=0}^{\infty} \frac{\tilde{\mu}_k(p^{b_1}, p^{b_2})}{p^{b_1+b_2}} = \left(1-\frac{1}{p}\right)^{2(k-1)} \frac{\left(1-\frac{1}{p}\right)^3}{1-\frac{1}{p^2}} = \frac{\left(1-\frac{1}{p}\right)^{2k+1}}{1-\frac{1}{p^2}} \text{ if } k \geq 2.$$

Hence, we see that, for every $k \in \mathbb{N}$, the right side of (2.6) equals

$$\tilde{C}_k \prod_{p \in \mathcal{P}} \left(\sum_{\nu_1, \nu_2 \geq 0} \frac{(f * \tilde{\mu}_k)(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} \right)$$

$$= \tilde{C}_k \prod_{p \in \mathcal{P}} \left(\sum_{a_1, a_2=0}^{\infty} \frac{f(p^{a_1}, p^{a_2})}{p^{a_1+a_2}} \right) \frac{\left(1-\frac{1}{p}\right)^{2k+1}}{1-\frac{1}{p^2}}$$

$$= \tilde{C}'_k \prod_{p \in \mathcal{P}} \left(1-\frac{1}{p}\right)^{2k+1} \left(\sum_{\nu_1, \nu_2=0}^{\infty} \frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} \right),$$

where $\tilde{C}'_k = \zeta(2)\tilde{C}_k$. Thus, the proof of Theorem 3 is now complete. □

It is well known (Schwarz and Spilker [6]) that if $f : \mathbb{N} \mapsto \mathbb{C}$ is a multiplicative function satisfying

$$\sum\nolimits_{p \in \mathcal{P}} \left(|f(p) - 1|/p + \sum\nolimits_{\nu \geq 2} f(p^{\nu})/p^{\nu} \right) < \infty,$$

then the mean value $M(f) = \lim_{x\to\infty} x^{-1}\sum_{n\le x} f(n)$ exists and equals $\prod_{p\in\mathcal{P}}(1-1/p)\left(\sum_{\nu\ge 2} f(p^{\nu})/p^{\nu}\right)$. The following theorem is a generalization of this result.

Theorem 4. *Let f be a multiplicative function of two variables and let* $k \in \mathbb{N}$.

(i) *Suppose*

$$\sum_{p\in\mathcal{P}}\left(\frac{|f(p, 1)-k| + |f(1, p)-k|}{p} + \sum_{\nu_1+\nu_2\ge 2}\frac{|f(p^{\nu_1+\nu_2})|}{p^{\nu_1+\nu_2}}\right) < \infty. \quad (4.6)$$

Then we have

$$\lim_{x, y\to\infty}\frac{1}{xy(\log x\log y)^{k-1}}\sum_{\substack{n_1\le x\\ n_2\le y}} f(n_1, n_2)$$

$$= C_k\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{2k}\left(\sum_{\nu_1, \nu_2\ge 0}\frac{f(p^{\nu_1+\nu_2})}{p^{\nu_1+\nu_2}}\right), \quad (4.7)$$

where $C_k = \dfrac{1}{((k-1)!)^2}$.

(ii) *Suppose*

$$\sum_{p\in\mathcal{P}}\left(\frac{|f(p, 1)-k| + |f(1, p)-k|}{p} + \frac{|f(p, p)-p|}{p^2}\right.$$

$$\left. + \sum_{\substack{\nu_1+\nu_2\ge 2\\ (\nu_1, \nu_2)\ne(1,1)}}\frac{|f(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1+\nu_2}}\right) < \infty. \quad (4.8)$$

Then we have

$$\lim_{x\to\infty}\frac{1}{x^2(\log x)^{2k-1}}\sum_{n_1, n_2\le x} f(n_1, n_2)$$

$$= \tilde{C}_k' \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{2k+1}\left(\sum_{\nu_1, \nu_2\ge 2}\frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}}\right), \tag{4.9}$$

where $\tilde{C}_k' = \dfrac{1}{((k-1)!)^2(2k-1)}$.

Proof. We first prove (i). We would like to show that *f* satisfies (4.1). We have

$$\sum_{p\in\mathcal{P}}\sum_{\nu_1+\nu_2\ge 1}\frac{|(f*\mu_k)(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1+\nu_2}} =: I_1 + I_2,$$

where

$$I_1 = \sum_{p\in\mathcal{P}}\sum_{\nu_1+\nu_2=1}\frac{|(f*\mu_k)(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1+\nu_2}}$$

$$= \sum_{p\in\mathcal{P}}\frac{|(f*\mu_k)(p, 1)|+|(f*\mu_k)(1, p)|}{p}$$

$$= \sum_{p\in\mathcal{P}}\frac{|f(p, 1)-k|+|f(1, p)-k|}{p} < \infty,$$

and

$$I_2 = \sum_{p\in\mathcal{P}}\sum_{\nu_1+\nu_2\ge 2}\frac{|(f*\mu_k)(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1+\nu_2}}$$

$$= \sum_{p\in\mathcal{P}}\sum_{a_1+a_2+b_1+b_2\ge 2}\frac{|f(p^{a_1}, p^{a_2})\mu_k(p^{b_1}, p^{b_2})|}{p^{a_1+a_2+b_1+b_2}}$$

$$= \sum_{p \in \mathcal{P}} \left(\sum_{\substack{a_1 + a_2 = 0 \\ b_1 + b_2 \geq 2}} + \sum_{\substack{a_1 + a_2 = 1 \\ b_1 + b_2 \geq 1}} + \sum_{\substack{a_1 + a_2 \geq 2 \\ b_1 + b_2 \geq 0}} \right) \frac{| f(p^{a_1}, p^{a_2}) \mu_k (p^{b_1}, p^{b_2}) |}{p^{a_1 + a_2 + b_1 + b_2}}$$

$$\ll \sum_{p \in \mathcal{P}} \left(\sum_{b_1 + b_2 \geq 2} \frac{1}{p^{b_1 + b_2}} + \sum_{b_1 + b_2 \geq 1} \frac{| f(p, 1) | + | f(1, p) |}{p^{1 + b_1 + b_2}} \right.$$

$$\left. + \sum_{\substack{a_1 + a_2 \geq 2 \\ b_1 + b_2 \geq 0}} \frac{| f(p^{a_1}, p^{a_2}) |}{p^{a_1 + a_2 + b_1 + b_2}} \right) < \infty.$$

Therefore, f satisfies (4.1), and hence (4.7) (which is equal to (4.2)) holds by Theorem 3(i). This proves (i).

Next we prove (ii). If $k = 1,$ then it is easy to see that (4.8) implies (4.3) since $(f * \tilde{\mu})(p, 1) = f(p, 1) - 1,$ $(f * \tilde{\mu})(1, p) = f(1, p) - 1$ and $(f * \tilde{\mu})(p, p) = f(p, p) - f(p, 1) - f(1, p) + 2 - p$ hold by Lemma 10. Let $k \geq 2.$ We put $\tilde{f} = f * \tilde{\mu}.$ We show that $\tilde{f}$ satisfies (4.6) for $k - 1$ instead of k. We first see that

$$\sum_{p \in \mathcal{P}} \frac{| \tilde{f}(p, 1) - (k - 1) | + | \tilde{f}(1, p) - (k - 1) |}{p}$$

$$= \sum_{p \in \mathcal{P}} \frac{| f(p, 1) - k | + | f(1, p) - k |}{p} < \infty.$$

We also have

$$\sum_{p \in \mathcal{P}} \sum_{\nu_1 + \nu_2 \geq 2} \frac{| \tilde{f}(p^{\nu_1}, p^{\nu_2}) |}{p^{\nu_1 + \nu_2}} = \sum_{p \in \mathcal{P}} \left(\sum_{\nu_1 + \nu_2 = 2} + \sum_{\nu_1 + \nu_2 \geq 3} \right) \frac{| \tilde{f}(p^{\nu_1}, p^{\nu_2}) |}{p^{\nu_1 + \nu_2}}$$

$$=: J_1 + J_2,$$

where

$$J_1 = \sum_{p\in\mathcal{P}}\sum_{\nu_1+\nu_2=2}\frac{|\tilde{f}(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1+\nu_2}}$$

$$= \sum_{p\in\mathcal{P}}\frac{|\tilde{f}(p^2, 1)| + |\tilde{f}(p, p)| + |\tilde{f}(1, p^2)|}{p^2}.$$

Noting that $\tilde{f}(p^2, 1) = f(p^2, 1) - f(p, 1)$, $\tilde{f}(p, p) = f(p, p) - f(p, 1) - f(1, p) + 2 - p$ and $\tilde{f}(1, p^2) = f(1, p^2) - f(1, p)$ hold by Lemma 10, we have

$$J_1 \ll \sum_{p\in\mathcal{P}}\frac{|\tilde{f}(p^2, 1)| + |f(p, 1) - k| + |f(p, p) - p| + |f(1, p) - k| + |\tilde{f}(1, p^2)| + 1}{p^2},$$

which implies that $J_1 < \infty$.

As for J_2, since $|\tilde{\mu}(p^{\nu_1}, p^{\nu_2})| \ll 1 + p$ holds for every $\nu_1, \nu_2 \geq 0$ by Lemma 10, we have

$$J_2 = \sum_{p\in\mathcal{P}}\sum_{\nu_1+\nu_2\geq 3}\frac{|\tilde{f}(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1+\nu_2}}$$

$$= \sum_{p\in\mathcal{P}}\sum_{a_1+a_2+b_1+b_2\geq 3}\frac{|f(p^{a_1}, p^{a_2})\tilde{\mu}(p^{b_1}, p^{b_2})|}{p^{a_1+a_2+b_1+b_2}}$$

$$\ll \sum_{p\in\mathcal{P}}\left(\sum_{\nu_1+\nu_2\geq 2}\frac{1 + |f(p^{\nu_1}, p^{\nu_2})|}{p^{\nu_1+\nu_2}}\right) < \infty.$$

Therefore, $\tilde{f}$ satisfies (4.6) for $k-1$ instead of k. Hence, by Theorem 4(i), we have

$$\lim_{x, y\to\infty} \frac{1}{xy(\log x \log y)^{k-2}} \sum_{\substack{n_1 \le x \\ n_2 \le y}} \tilde{f}(n_1, n_2)$$

$$= C_{k-1} \prod_{p \in P} \left(1 - \frac{1}{p}\right)^{2(k-1)} \left(\sum_{\substack{\nu_1 \ge 0 \\ \nu_2 \ge 0}} \frac{\tilde{f}(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} \right).$$

Since $f = \tilde{f} * \tilde{\tau}_1$, we have by taking $a = \tilde{f}, b = \tilde{\tau}_1$ and $\alpha = \beta = k - 2$ in Lemma 8(ii)

$$\lim_{x\to\infty} \frac{1}{x^2(\log x)^{2k-1}} \sum_{n_1, n_2 \le x} f(n_1, n_2)$$

$$= \frac{1}{(k-1)^2(2k-1)} \frac{1}{\zeta(2)} C_{k-1} \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{2(k-1)} \left(\sum_{\nu_1, \nu_2 \ge 0} \frac{\tilde{f}(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} \right)$$

$$= \frac{1}{\zeta(2)} \tilde{C}_k' \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{2(k-1)} \left(\sum_{a_1, a_2, b_1, b_2 \ge 0} \frac{\tilde{f}(p^{a_1}, p^{a_2})}{p^{a_1+a_2}} \frac{\tilde{\mu}(p^{b_1}, p^{b_2})}{p^{b_1+b_2}} \right).$$

By (4.5) we see that the above equals

$$\frac{1}{\zeta(2)} \tilde{C}_k' \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{2(k-1)} \frac{\left(1 - \frac{1}{p}\right)^3}{1 - \frac{1}{p^2}} \left(\sum_{a_1, a_2 \ge 0} \frac{f(p^{a_1}, p^{a_2})}{p^{a_1+a_2}} \right)$$

$$= \tilde{C}_k' \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{2k+1} \left(\sum_{\nu_1, \nu_2 \ge 0} \frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} \right).$$

Thus, the proof of Theorem 4 is now complete. □

5. Examples

Let $\omega(n) = \sum_{p \mid n} 1$ be the counting function of the total number of prime factors of n taken without multiplicity. It is known that for a fixed positive integer k,

$$\lim_{x\to\infty} x^{-1}(\log x)^{1-k}\sum_{n\le x} k^{\omega(n)}$$

$$= ((k-1)!)^{-1}\prod_{p\in\mathcal{P}} (1-1/p)^{k-1}(1+(k-1)/p)$$

(cf. Tenenbaum and Wu [7] p. 25). The following example is an extension of this result to the case of a function of two variables.

Example 1. Let $k \in \mathbb{N}$ and let $f(n_1, n_2) = k^{\omega(n_1 n_2)}$. Then we have

$$\lim_{x,\, y\to\infty} \frac{1}{xy(\log x \log y)^{k-1}} \sum_{\substack{n_1\le x \\ n_2\le y}} f(n_1, n_2)$$

$$= C_k \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right)^{2(k-1)}\left(1+\frac{2(k-1)}{p}+\frac{1-k}{p^2}\right),$$

where $C_k = \dfrac{1}{((k-1)!)^2}$.

Proof. Since $f(p^{\nu_1}, p^{\nu_2}) = k$ if $\nu_1+\nu_2 \ge 1$, it is easy to see that f satisfies (4.6). Therefore, we can apply Theorem 4(i) to obtain

$$\lim_{x,\, y\to\infty} \frac{1}{xy(\log x \log y)^{k-1}} \sum_{n_1\le x,\, n_2\le y} f(n_1, n_2)$$

$$= C_k \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right)^{2k}\left(1+\sum_{\nu_1+\nu_2\ge 1}\frac{k}{p^{\nu_1+\nu_2}}\right)$$

$$= C_k \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right)^{2k}\left(1+\frac{k(2p-1)}{(p-1)^2}\right)$$

$$= C_k \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right)^{2(k-1)}\left(1+\frac{2(k-1)}{p}+\frac{1-k}{p^2}\right). \qquad \square$$

Example 2. Let $f(q, n) = |c_q(n)|$, where $c_q(n) = \mu(q/(q, n))\varphi(q)/\varphi(q/(q, n))$ is the Ramanujan sum. Then we have

$$\lim_{x\to\infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \le x} f(n_1, n_2) = \prod_{p\in\mathcal{P}} \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right).$$

Proof. It is easy to see that $f(p, 1) = f(1, p) = 1$, $f(p, p) = p - 1$, $f(p^\nu, 1) = 0$, $f(1, p^\nu) = 1$ if $\nu \ge 2$, and

$$f(p^{\nu_1}, p^{\nu_2}) = \begin{cases} \mu^2(p^{\nu_1-\nu_2})p^{\nu_2}, & \text{if } 1 \le \nu_2 < \nu_1, \\ p^{\nu_1}(1 - 1/p), & \text{if } 1 \le \nu_1 \le \nu_2. \end{cases}$$

From these relations, we see that f satisfies (4.8) for $k = 1$. After an elementary calculation we obtain

$$\sum_{\nu_1, \nu_2 \ge 0} \frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} = \frac{p+2}{p-1}.$$

Therefore, we have by (4.9)

$$\lim_{x\to\infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \le x} f(n_1, n_2)$$

$$= \tilde{C}_1' \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)^3 \frac{p+2}{p-1} = \prod_{p\in\mathcal{P}} \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right). \qquad \square$$

Next we obtain the leading coefficients in (1.3) and (1.4) using Theorem 4.

Example 3. Let $f(n_1, n_2) = \sigma(\gcd(n_1, n_2))$, where $\sigma(n) = \sum_{d\,|\,n} d$. Then we have

$$\lim_{x\to\infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \le x} f(n_1, n_2) = 1.$$

Proof. Since $f(p^{\nu_1}, p^{\nu_2}) = (p^{\nu_1 \wedge \nu_2 + 1} - 1)/(p-1)$ if $\nu_1, \nu_2 \geq 0,$ it is easy to see that *f* satisfies (4.8) for $k = 1.$ Therefore, we can apply Theorem 4(ii) for $k = 1.$ After an elementary calculation we obtain

$$\sum_{\nu_1, \nu_2 \geq 0} \frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1 + \nu_2}} = \frac{1}{\left(1 - \frac{1}{p}\right)^3}.$$

Therefore, we have by (4.9)

$$\lim_{x \to \infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \leq x} f(n_1, n_2) = \tilde{C}_1' \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^3 \frac{1}{\left(1 - \frac{1}{p}\right)^3} = 1. \qquad \square$$

Example 4. Let $f(n_1, n_2) = \varphi(\gcd(n_1, n_2)).$ Then we have

$$\lim_{x \to \infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \leq x} f(n_1, n_2) = \frac{1}{\zeta^2(2)}.$$

Proof. Since $f(p^{\nu_1}, p^{\nu_2}) = p^{\nu_1 \wedge \nu_2}(1 - 1/p)$ if $\nu_1, \nu_2 \geq 1,$ it is easy to see that *f* satisfies (4.8) for $k = 1.$ Therefore, we can apply Theorem 4(ii) for $k = 1.$ After an elementary calculation we obtain

$$\sum_{\nu_1, \nu_2 \geq 0} \frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1 + \nu_2}} = \frac{\left(1 + \frac{1}{p}\right)^2}{1 - \frac{1}{p}}.$$

Therefore, we have by (4.9)

$$\lim_{x \to \infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \leq x} f(n_1, n_2) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^3 \frac{\left(1 + \frac{1}{p}\right)^2}{1 - \frac{1}{p}}$$

$$= \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^2}\right)^2 = \frac{1}{\zeta^2(2)}.$$

The proof of the following example is similar. $\square$

Example 5. Let

$$f_1(n_1, n_2) = \gcd(n_1, n_2)\mu^2(\gcd(n_1, n_2)),$$

$$f_2(n_1, n_2) = \gcd(n_1, n_2)\mu^2(\operatorname{lcm}(n_1, n_2)).$$

Then we have

$$\lim_{x\to\infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \le x} f_1(n_1, n_2) = \frac{1}{\zeta^2(2)},$$

$$\lim_{x\to\infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \le x} f_2(n_1, n_2) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{3}{p}\right).$$

Example 6. Let $f(n_1, n_2) = \dfrac{\phi(n_1)\phi(n_2)}{\operatorname{lcm}(n_1, n_2)}$. Then we have

$$\lim_{x\to\infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \le x} f(n_1, n_2) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{3}{p} + \frac{1}{p^2}\right).$$

Proof. Since $f(p^{\nu}, 1) = f(1, p^{\nu}) = 1 - 1/p$ if $\nu \ge 1$ and

$$f(p^{\nu_1}, p^{\nu_2}) = (1 - 1/p)^2 p^{\nu_1 \wedge \nu_2}$$

if $\nu_1, \nu_2 \ge 1$, it is easy to see that f satisfies (4.8) for $k = 1$. Therefore, we can apply Theorem 4(ii) for $k = 1$. After an elementary calculation we obtain

$$\sum_{\nu_1, \nu_2 \ge 0} \frac{f(p^{\nu_1}, p^{\nu_2})}{p^{\nu_1+\nu_2}} = 1 + \frac{3}{p} + \frac{1}{p^2}.$$

Therefore, using (4.9) for $k = 1$, we have the desired result. □

Next we obtain the leading coefficients in (1.5) and (1.6).

Example 7. Let $s(n_1, n_2) = \sum_{d_1 | n_1, d_2 | n_2} \gcd(d_1, d_2)$. Then we have

$$\lim_{x\to\infty} \frac{1}{x^2 (\log x)^3} \sum_{n_1, n_2 \le x} s(n_1, n_2) = \frac{2}{\pi^2}.$$

Proof. Since $s = \gcd * \mathbf{1} = \tilde{\tau}_2$, we have $s * \tilde{\mu}_2 = \delta$. Therefore, (2.5) trivially holds for $k = 2$ and (2.6) gives

$$\lim_{x\to\infty} \frac{1}{x^2(\log x)^3} \sum_{n_1, n_2 \le x} s(n_1, n_2) = \tilde{C}_2' \sum_{n_1, n_2 \le x} \frac{\delta(n_1, n_2)}{n_1 n_2} = \frac{2}{\pi^2}. \qquad \square$$

Example 8. Let $c(n_1, n_2) = \sum_{d_1 | n_1, d_2 | n_2} \varphi(\gcd(d_1, d_2))$. Then we have

$$\lim_{x\to\infty} \frac{1}{x^2(\log x)^3} \sum_{n_1, n_2 \le x} c(n_1, n_2) = \frac{12}{\pi^4}.$$

Proof. We note that $c = \varphi(\gcd) * \mathbf{1}$. Since $\varphi(\gcd)$ satisfies (4.8) for $k = 1$ from the proof of Example 4, we see that $\varphi(\gcd)$ also satisfies (2.1) from the proofs of Theorem 4, Theorem 3 and Theorem 2. Therefore, we have by Theorem 1 and Example 4

$$\lim_{x, y\to\infty} \frac{1}{xy \log x \wedge y} \sum_{n_1 \le x, n_2 \le y} \varphi(\gcd(n_1, n_2)) = \frac{1}{\zeta^2(2)}.$$

Taking $a = \mathbf{1}$, $b = \varphi(\gcd)$ and $\alpha = \beta = 0$ in Lemma 8(ii), we have

$$\lim_{x\to\infty} \frac{1}{x^2(\log x)^3} \sum_{n_1, n_2 \le x} c(n_1, n_2) = \frac{1}{3} \frac{1}{\zeta^2(2)} = \frac{12}{\pi^4}. \qquad \square$$

Remark. According to Nowak and Tóth [4], it holds that $c(p, 1) = c(1, p) = 2$, $c(p, p) = p + 2$, $c(p^a, 1) = c(1, p^a) = a + 1$ if $a \ge 1$, and, moreover, $c(p^a, p^b) = 2(1 + p + p^a + \cdots + p^{a-1}) + (b - a + 1) p^a$ if $1 \le a \le b$. Using this explicit formulas we can directly show that c satisfies (4.8) for $k = 2$ and also can directly calculate (4.9). However, we did not prove in that way for simplicity.

Example 9. Let $A(n_1, n_2) = \sum_{d_1 | n_1, d_2 | n_2} \phi(d_1)\phi(d_2)/\mathrm{lcm}(d_1, d_2)$. Then we have

$$\lim_{x\to\infty} \frac{1}{x^2(\log x)^3} \sum_{n_1, n_2 \le x} A(n_1, n_2) = \frac{1}{3} \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{3}{p} + \frac{1}{p^2}\right). \quad (5.1)$$

Proof. Let $g(n_1, n_2) = \phi(n_1)\phi(n_2)/\mathrm{lcm}(n_1, n_2)$. Since $A = g * \mathbf{1}$, by a similar argument as in Example 8, we see that the left side of (5.1) equals

$$\frac{1}{3} \lim_{x\to\infty} \frac{1}{x^2 \log x} \sum_{n_1, n_2 \le x} g(n_1, n_2).$$

By Example 6, it is easy to see that the above equals the right side of (5.1). □

References

[1] E. Cesàro, Étude moyenne du plus grand commun diviseur de deux nombres, Annal. di Mat. Pura ed Appli. 13 (1885), 235-250.

[2] E. Cohen, Arithmetical Notes, I. On a theorem of van der Corput, Proc. Amer. Math. Soc. 12 (1961), 214-217.

[3] E. Cohen, Arithmetical functions of a greatest common divisor, I., Proc. Amer. Math. Soc. 11 (1960), 164-171.

[4] W. G. Nowak and L. Tóth, On the average number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_m$, Int. J. Number Theory 10 (2014), 363-374.

[5] J. Sándor and B. Crstici, Handbook of Number Theory II, Kluwer Academic Publishers, Dordrecht, 2004.

[6] W. Schwarz and J. Spilker, Arithmetical Functions, Cambridge Univ. Press, 1994.

[7] G. Tenenbaum and J. Wu, Exercices corriges de theorie analytique et proba-biliste des nombres, Soc. Math. France, 1996.

[8] L. Tóth, Multiplicative arithmetic functions of several variables: a survey, Mathematics without Boundaries, Surveys in Pure Mathematics, Th. M. Rassias and P. Pardalos, eds., Springer, 2014, pp. 483-514.

[9] L. Tóth, A survey of gcd-sum functions, J. Integer Sequences 13 (2010), 1-23.

[10] N. Ushiroya, On a mean value of a multiplicative function of two variables, Probability and Number Theory - Kanazawa 2005, Adv. Studies in Pure Math. 49, S. Akiyama, K. Matsumoto, L. Murata and H. Sugita, eds., 2007, pp. 507-515.

[11] N. Ushiroya, Mean-value theorems for multiplicative arithmetic functions of several variables, Integers 12 (2012), 989-1002.

[12] J. G. van der Corput, Sur quelques fonctions arithmetique elementaires, Proc. Roy. Acad. Sci. 42 (1939), 859-866.