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ITERATION METHODS TO COMPUTE THE SEPARABLE CONVEX MINIMIZATION PROBLEMS

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Abstract

In this paper, the iteration method to compute the multi-block separable convex minimization problem with linear constraints. The condition of the problem is derived, and global convergences of the algorithm are proved.

1. Introduction

In this paper, we consider the following problems.

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Problem 1. Give matrices $A \in R^{p\times n}$, $B \in R^{n\times p}$, $C \in R^{p\times n}$, $D \in R^{n\times p}$ and $E \in R^{p\times p}$, find $X \in R^{n\times n}$, $Y \in R^{n\times n}$ and $Z \in R^{p\times p}$ such that

$$\min \| Z \|^2 \text{ s.t. } AXB + CYD + Z = E. \tag{1.1}$$

The iteration method to compute the matrix equation has become popular, and a series of good results have been obtained [1-4]. For example, Peng et al. [5] proposed iteration method to solve the solution of matrix equation $AX = B$ with constraint $CXD \geq E$. Peng et al. [6] used the Conjugate Gradient (CG) method to solve the symmetric solutions and optimal approximation solution of the system of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$. Peng [7, 8] and Huang et al. [9] used iteration algorithm to compute the least squares symmetric solution, skew-symmetric solution and constraint solution of matrix equation $AXB = C$. Sheng and Chen [10] proposed iteration method to solve the symmetric and skew-symmetric solutions of linear matrix equation $AXB + CYD = E$. Moreover, Han et al. [11, 12] used Douglas-Rachford splitting method (DRSM) to solve convex optimization problems with linear constraints. Luis [13] proposed a splitting method of DRSM for solving equilibrium problems involving the sum of two bifunctions satisfying standard conditions.

In this paper, we consider the convex minimization problem with linear constrains where the objective function is separable 3 individual convex functions. First, we give some basic notations and properties which are useful for further discussion. Then we derive the iterative algorithms for (1.1). And we study the global convergence of the algorithms. We will give an example for illustrating the effectiveness of the algorithms proposed.

Throughout this paper, the following notations are used. The symbol $R^{m\times n}$ denotes the set of $m \times n$ real matrices. R^n denotes the set of real n-vectors. A^T and $\| A \|$ denote the transpose, the Frobenius norm of the matrix A or 2-norm of the matrix A. Define the inner product in space $R^{m\times n}$ by $\langle A, B \rangle = trace(A^T B)$ for all $A, B \in R^{m\times n}$. For the matrices $A = (a_{ij}) \in$

$R^{m\times n}$, $B = (b_{ij}) \in R^{p\times q}$, $A \otimes B$ represents the Kronecker production of the matrices A and B, defined as $A \otimes B = (a_{ij}B) \in R^{mp\times nq}$. Obviously, $R^{m\times n}$ is a Hilbert inner product space, then the associated norm is the Frobenius norm denote by $\| A \|$.

2. Preliminaries

In this section, we first give some basic concepts and well known results that will useful for further discussion.

Lemma 2.1 [14]. *Let* Ω *be a nonempty closed convex subset of* R^n *and* P_Ω *be the projection operator onto a convex set* Ω, *i.e.*,

$$P_\Omega[X] := \operatorname{argmin}\{\| X - Y \| | Y \in \Omega\}.$$

Then we have the following inequality:

$$(u - P_\Omega[u])^T (v - P_\Omega[u]) \le 0, \ \forall u \in R^n, \ \forall v \in \Omega. \tag{2.1}$$

Lemma 2.2 [15]. *A set-valued map T from* R^n *to* 2^{R^n} *is said to be*

(1) *Monotone if*

$$(u - v)^T (u^* - v^*) \ge 0, \ \forall u, v \in R^n, \ u^* \in T(u), \ v^* \in T(v).$$

(2) *Strongly monotone if there exists a constant* $\eta > 0$ *such that*

$$(u - v)^T (u^* - v^*) \ge \eta \| u - v \|^2, \ \forall u, v \in R^n, \ u^* \in T(u), \ v^* \in T(v).$$

To simplify the coming discussion, we denote the following notations:

$$\Omega := \Omega_0 \times \Omega_1 \times \Omega_2 \times \Omega_3 = R^{p\times n} \times R^{n\times p} \times R^{p\times p} \times R^{n\times n},$$

$$H^* := (Z^*, X^*, Y^*), \ H^k := (Z^k, X^k, Y^k),$$

$$\overline{U}^k := (Z^k, \overline{M}^k, \beta_k), \overline{V}^k := (X^k, \overline{M}^k, \beta_k),$$

$$\overline{N}^k := (Y^k, \overline{M}^k, \beta_k), \overline{W}^k := (Z^k, X^k, Y^k, \overline{M}^k, \beta_k),$$

$$f(\omega) = AXB + CYD + Z - E,$$

$$L(\omega, M) := L(Z, X, Y, M) = \frac{1}{2}\| Z \|^2 - \langle M, f_1(\omega)\rangle,$$

$$E_1(Z^k, \overline{M}^k, \beta_k) = E_1(\overline{U}^k), E_2(X^k, \overline{M}^k, \beta_k) = E_1(\overline{V}^k),$$

$$E_3(Y^k, \overline{M}^k, \beta_k) = E_1(\overline{N}^k).$$

3. Iteration Method to Solve Problem 1

In this section, we first give the DRSM Algorithm of Problem 1. We present the global convergence result of Algorithm 1.

Algorithm 1. A distributed Douglas-Rachford splitting method (DDRSM)

1. Given matrices $A, C \in R^{p\times n}$, $B, D \in R^{n\times p}$ and $E \in R^{p\times p}$. Set $\gamma \in (0, 2)$ and $\beta \in \left(0, \frac{1}{c}\right)$ with $c := \max\{1, \| B^T \otimes A \|_2, \| D^T \otimes C \|_2\}$. Choose initial points $(Z^0, X^0, Y^0, M^0) \in R^{n\times n}$.

2. For $k = 1, 2, \ldots$ do

3. Find $W^{k+1} \in R^{n\times n}$ such that:

$$\overline{M}^k = M^k - \beta(AX^kB + CY^kD + Z^k - E); \tag{3.1a}$$

$$Z^{k+1} = \underset{Z\in R^{p\times p}}{\arg\min}\| Z \|^2 + \frac{1}{2\beta}\| Z - [Z^k + \beta Z^k - \gamma\alpha_k E_1(\overline{U}^k)] \|^2; \tag{3.1b}$$

$$X^{k+1} = X^k - \gamma\alpha_k E_2(\overline{V}^k); \tag{3.1c}$$

$$Y^{k+1} = Y^k - \gamma\alpha_k E_3(\overline{N}^k); \tag{3.1d}$$

$$\begin{aligned} M^{k+1} = M^k - \gamma\alpha_k\{M^k - \overline{M}^k - \beta_k[E_1(\overline{U}^k) \\ + AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D]\}, \end{aligned} \tag{3.1e}$$

where

$$\alpha_k := \frac{\varphi(\overline{W}^k)}{\psi(\overline{W}^k)}; \tag{3.1f}$$

$$\varphi(\overline{W}^k) := \| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2 + \| M^k - \overline{M}^k \|^2$$

$$- \beta\langle M^k - \overline{M}^k, E_1(\overline{U}^k) + AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D\rangle; \tag{3.1g}$$

$$\psi(\overline{W}^k) := \| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2$$

$$+ \| M^k - \overline{M}^k - \beta_k [E_1(\overline{U}^k) + AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D] \|^2; \tag{3.1h}$$

4. End for.

Lemma 3.1. *For the step size* α_k *given by* (3.1f), *there exists a constant* $\alpha_{\min} > 0$ *such that*

$$\alpha_k > \alpha_{\min}, \text{ for all } k > 0. \tag{3.2}$$

Proof. For any two matrices $A', A'' \in R^{n\times n}$, we have the inequality

$$2\| A'^T A'' \|^2 \le \varsigma \| A' \|^2 + \frac{1}{\varsigma} \| A'' \|^2, \quad \forall \varsigma > 0.$$

Hence, for any $\varsigma > 0$ we have

$$\beta\langle M^k - \overline{M}^k, E_1(\overline{U}^k) + AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D\rangle$$

$$= \beta\langle M^k - \overline{M}^k, E_1(\overline{U}^k)\rangle + \beta\langle M^k - \overline{M}^k, AE_2(\overline{V}^k)B\rangle$$

$$+ \beta\langle M^k - \overline{M}^k, CE_3(\overline{N}^k)D\rangle$$

$$\le \frac{3\varsigma}{2} \| M^k - \overline{M}^k \|^2 + \frac{\beta^2}{2\varsigma} (\| E_1(\overline{U}^k) \|^2 + \| B^T \otimes A \|^2 \| E_2(\overline{V}^k) \|^2$$

$$+ \| D^T \otimes C \|^2 \| E_3(\overline{N}^k) \|^2)$$

$$\le \frac{3\varsigma}{2} \| M^k - \overline{M}^k \|^2 + \frac{\beta^2 c^2}{2\varsigma} (\| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2), \tag{3.3}$$

where $c := \max\{1, \| B^T \otimes A \|_2, \| D^T \otimes C \|_2\}$. Consequently, let $\varsigma := \beta c$, inserting (3.3) into (3.1g), we obtain

$$\varphi(\overline{W}^k) \geq \left(1 - \frac{3\varsigma}{2}\right)\| M^k - \overline{M}^k \|^2$$

$$+ \left(1 - \frac{\beta^2 c^2}{2\varsigma}\right)(\| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2)$$

$$\geq \left(1 - \frac{\beta c}{2}\right)(\| M^k - \overline{M}^k \|^2 + \| E_1(\overline{U}^k) \|^2$$

$$+ \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2), \tag{3.4}$$

where $\beta \in \left(0, \frac{1}{c}\right)$, it is clear that $\varphi(\overline{W}^k) \geq 0$.

On the other hand, use the Cauchy-Schwarz inequality on equation (3.1h), we obtain

$$\psi(\overline{W}^k) := \| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2 + 2\| M^k - \overline{M}^k \|^2$$

$$+ 4\| \beta E_1(\overline{U}^k) \|^2 + 4\| \beta A E_2(\overline{V}^k) B \|^2 + 4\| \beta C E_3(\overline{N}^k) D \|^2$$

$$\leq c'(\| M^k - \overline{M}^k \|^2 + \| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2), \tag{3.5}$$

where $c' := \max\{2, 1 + 4\beta, 1 + 4\beta\| A \|^2\| B \|^2, 1 + 4\beta\| C \|^2\| D \|^2\}$.

Combining (3.4) and (3.5), we can obtain the opinion (3.2), where $\alpha_{\min} := \left(1 - \frac{\beta_k c}{2}\right) \Big/ c'$.

Lemma 3.2. *Suppose that* $Q^* = (Z^*, X^*, Y^*, M^*)$ *is an arbitrary solution of* (1.1), *the sequence* $\{Q^k = (Z^k, X^k, Y^k, M^k)\}$ *generated by Algorithm* 1 *satisfies*

$$\left\langle \begin{pmatrix} H_{\beta_k}^k - H_{\beta_k}^* \\ M^k - M^* \end{pmatrix}, \begin{pmatrix} E_1(\overline{U}^k) \\ E_2(\overline{V}^k) \\ E_3(\overline{N}^k) \\ M^k - \overline{M}^k - \beta_k[E_1(\overline{U}^k) + AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D] \end{pmatrix} \right\rangle$$

$\geq \varphi(\overline{W}^k), \ \forall k > 0.$

Proof. Since $Q^* = (Z^*, X^*, Y^*, M^*) \in R^*$, $J_1^* \in \partial(Z^*)$, $J_2^* \in \partial(X^*)$ and $J_3^* \in \partial(Y^*)$, it follows from (1.1) that

$$\langle Z - Z^*, \nabla_Z L(\omega^*, M^*)\rangle \geq 0, \quad \forall Z' \in R^{n\times n},$$

$$\langle X - X^*, \nabla_X L(\omega^*, M^*)\rangle \geq 0, \quad \forall X' \in R^{n\times n}, \tag{3.6}$$

and

$$\langle Y - Y^*, \nabla_Y L(\omega^*, M^*)\rangle \geq 0, \quad \forall Y' \in R^{n\times n}.$$

Then, setting $Z := P_{\Omega_3}[Z^k - \beta_k \nabla_Z L(\omega^k, \overline{M}^k)] = Z^k - E_1(\overline{U}^k)$ in (3.6), we have

$$\langle Z^k - E_1(\overline{U}^k) - Z^*, \nabla_Z (\omega^*, M^*)\rangle \geq 0. \tag{3.7}$$

On the other hand, setting $\Omega := \Omega_3, U := Z^k - \beta_k \nabla_Z L(\omega^k, \overline{M}^k)$, and $V := Z^*$ in (2.1), we obtain

$$\langle E_1(\overline{U}^k) - \beta_k \nabla_Z L(\omega^k, \overline{M}^k), Z^k - E_1(\overline{U}^k) - Z^*\rangle \geq 0. \tag{3.8}$$

By summing (3.7) and (3.8), we obtain

$$\langle Z^k - E_1(\overline{U}^k) - Z^*, E_1(\overline{U}^k) + \beta_k(\nabla_Z L(\omega^*, M^*) - \nabla_Z L(\omega^k, \overline{M}^k))\rangle \geq 0.$$

By collating, we arrive at

$$\begin{aligned} &\langle Z^k + \beta_k \nabla_Z L(\omega^k, \overline{M}^k) - Z^* - \beta_k \nabla_Z L(\omega^*, M^*), E_1(\overline{U}^k)\rangle \\ &\geq \| E_1(\overline{U}^k) \|^2 - \beta_k \langle \overline{M}^k - M^*, Z^k - E_1(\overline{U}^k) - Z^*\rangle. \end{aligned} \tag{3.9}$$

In a similar, we can prove that

$$\langle X^k + \beta_k \nabla_X L(\omega^k, \overline{M}^k) - X^* - \beta_k \nabla_X L(\omega^*, M^*), E_2(\overline{V}^k)\rangle$$

$$\geq \| E_2(\overline{V}^k) \|^2 - \beta_k \langle A^T(\overline{M}^k - M^*)B^T, X^k - E_2(\overline{V}^k) - X^*\rangle$$

$$= \| E_2(\overline{V}^k) \|^2 - \beta_k \langle \overline{M}^k - M^*, A[X^k - E_2(\overline{V}^k) - X^*]B\rangle, \qquad (3.10)$$

$$\langle Y^k + \beta_k J_3^k - Y^* - \beta_k J_3^*, E_3(\overline{N}^k)\rangle$$

$$\geq \| E_3(\overline{N}^k) \|^2 - \beta_k \langle \overline{M}^k - M^*, C[Y^k - E_3(\overline{N}^k) - Y^*]D\rangle. \qquad (3.11)$$

Adding (3.9), (3.10) and (3.11) and finishing terms, we get

$$\left\langle (H_{\beta_k}^k - H_{\beta_k}^*), \begin{pmatrix} E_1(\overline{U}^k) \\ E_2(\overline{V}^k) \\ E_3(\overline{N}^k) \end{pmatrix} \right\rangle$$

$$\geq \| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2$$

$$- \beta_k tr((\overline{M}^k - M^*)^T \{[X^k - E_1(\overline{U}^k) - Z^*]$$

$$+ A[X^k - E_2(\overline{V}^k) - X^*]B + C[Y^k - E_3(\overline{N}^k) - Y^*]D\})$$

$$= \| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2$$

$$- \langle \overline{M}^k - M^*, M^k - \overline{M}^k - \beta_k [E_1(\overline{U}^k)$$

$$+ AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D]\rangle$$

$$= \| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2 + \| M^k - \overline{M}^k \|^2$$

$$- \beta_k \langle M^k - \overline{M}^k, E_1(\overline{U}^k) + AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D\rangle$$

$$- \langle M^k - M^*, M^k - \overline{M}^k - \beta_k [E_1(\overline{U}^k)$$

$$+ AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D]\rangle.$$

Recall the definition $\varphi(\overline{W}^k)$ in (3.1g), the iterative scheme (3.1a) and the fact $AX^*B + CY^*D + Z^* = E$, the assertion of this lemma then follows immediately.

Theorem 3.1. *Suppose that the parameter* $\beta \in \left(0, \frac{1}{c}\right)$ *and* Q^* *is an arbitrary solution of* (2.1), *the sequence* $\{Q^k\}$ *generated by Algorithm* 1 *satisfies*

$$\left\| \begin{pmatrix} H_{\beta_{k+1}}^{k+1} - H_{\beta_{k+1}}^{*} \\ M^{k+1} - M^* \end{pmatrix} \right\|^2 \leq \left\| \begin{pmatrix} H_{\beta_k}^{k} - H_{\beta_k}^{*} \\ M^{k} - M^* \end{pmatrix} \right\|^2 - \gamma(2-\gamma)\alpha_{\min}\varphi(\overline{W}^k). \quad (3.12)$$

Proof. It follows from (3.1c)-(3.2e) and Lemma 3.2 that

$$\left\| \begin{pmatrix} H_{\beta_{k+1}}^{k+1} - H_{\beta_{k+1}}^{*} \\ M^{k+1} - M^* \end{pmatrix} \right\|^2$$

$$= \left\| \begin{pmatrix} H_{\beta_k}^{k} - H_{\beta_k}^{*} \\ M^{k} - M^* \end{pmatrix} \right\|^2 + \gamma^2\alpha_k^2(\| E_1(\overline{U}^k) \|^2 + \| E_2(\overline{V}^k) \|^2 + \| E_3(\overline{N}^k) \|^2)$$

$$+ \gamma^2\alpha_k^2\| M^k - \overline{M}^k - \beta_k(E_1(\overline{U}^k) + AE_2(\overline{V}^k)B + CE_3(\overline{N}^k)D) \|^2$$

$$- 2\gamma\alpha_k \left\langle \begin{pmatrix} H_{\beta_k}^{k} - H_{\beta_k}^{*} \\ M^{k} - M^* \end{pmatrix}, \begin{pmatrix} E_1(\overline{U}^k) \\ E_2(\overline{V}^k) \\ E_3(\overline{N}^k) \\ M^k - \overline{M}^k - \beta_k[E_1(\overline{U}^k) + AE_2(\overline{V}^k)B \\ + CE_3(\overline{N}^k)D] \end{pmatrix} \right\rangle$$

$$\leq \left\| \begin{pmatrix} H_{\beta_k}^{k} - H_{\beta_k}^{*} \\ M^{k} - M^* \end{pmatrix} \right\|^2 + \gamma^2\alpha_k^2\psi(\overline{W}^k) - 2\gamma\alpha_k\varphi(\overline{W}^k)$$

$$= \left\| \begin{pmatrix} H_{\beta_k}^k - H_{\beta_k}^* \\ M^k - M^* \end{pmatrix} \right\|^2 - \gamma(2-\gamma)\alpha_k \varphi(\overline{W}^k).$$

The proof is completed.

Theorem 3.2. *Suppose that the parameter* $\beta \in \left(0, \frac{1}{c}\right)$. *The sequence* $\{Q^k\}$ *generated by Algorithm* 1 *converges to a solution of* (1.1).

Proof. Suppose that U^* is a solution of (1.1). From Theorem 3.1 that

$$\left\| \begin{pmatrix} H_{\beta_{k+1}}^{k+1} - H_{\beta_{k+1}}^* \\ M^{k+1} - M^* \end{pmatrix} \right\|^2 \leq \left\| \begin{pmatrix} H_{\beta_k}^k - H_{\beta_k}^* \\ M^k - M^* \end{pmatrix} \right\|^2 \leq \cdots \leq \left\| \begin{pmatrix} H_{\beta_0}^0 - H_{\beta_0}^* \\ M^0 - M^* \end{pmatrix} \right\|^2, \quad (3.13)$$

which means that the sequence $\{Q^*\}$ is bounded. Thus, $\{U^*\}$ is also bounded, and

$$\lim_{k\to\infty} \left\| \begin{pmatrix} H_{\beta_k}^k - H_{\beta_k}^* \\ M^k - M^* \end{pmatrix} \right\|^2 \text{ exists.} \quad (3.14)$$

Rearranging terms of (3.12) we obtain

$$\gamma(2-\gamma)\alpha_{\min}\varphi(\overline{W}^k) \leq \left\| \begin{pmatrix} H_{\beta_k}^k - H_{\beta_k}^* \\ M^k - M^* \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} H_{\beta_{k+1}}^{k+1} - H_{\beta_{k+1}}^* \\ M^{k+1} - M^* \end{pmatrix} \right\|^2.$$

Summing both sides of the above inequality for all k yields

$$\sum_{k=0}^{\infty} \gamma(2-\gamma)\alpha_{\min}\varphi(\overline{W}^k) \leq \sum_{k=0}^{\infty} \left(\left\| \begin{pmatrix} H_{\beta_k}^k - H_{\beta_k}^* \\ M^k - M^* \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} H_{\beta_{k+1}}^{k+1} - H_{\beta_{k+1}}^* \\ M^{k+1} - M^* \end{pmatrix} \right\|^2 \right)$$

$$\leq \left\| \begin{pmatrix} H_{\beta_0}^0 - H_{\beta_0}^* \\ M^0 - M^* \end{pmatrix} \right\|^2$$

which, together with the assumption that $\gamma \in (0, 2)$ and the fact that $\alpha_{\min} > 0$, implies that $\lim_{k \to \infty} \varphi(\overline{W}^k) = 0$. Hence, it follows from (3.4) that

$$\lim_{k \to \infty} \| M^k - \overline{M}^k \|^2 = \lim_{k \to \infty} \| E_1(\overline{U}^k) \|^2 = \lim_{k \to \infty} \| E_2(\overline{V}^k) \|^2$$

$$= \lim_{k \to \infty} \| E_3(\overline{N}^k) \|^2 = 0$$

and furthermore, we have

$$\lim_{k \to \infty} \| E(Q^k, \beta) \|^2 = 0. \tag{3.15}$$

Since the sequence $\{Q^*\}$ is bounded, it has at least one cluster point. Let $Q^\infty = (Z^\infty, X^\infty, Y^\infty, M^\infty)$ be a cluster point with $\{Q^{k_j} = (Z^{k_j}, X^{k_j}, Y^{k_j}, M^{k_j})\}$ being the corresponding subsequence converging to it. Thus, taking limit along this subsequence in (3.15), we have

$$\| E(Q^\infty, \beta) \|^2 = \| E(\lim_{j \to \infty} Q^{k_j}, \beta) \|^2 = \lim_{j \to \infty} \| E(Q^{k_j}, \beta) \|^2.$$

Recall lemma, the above fact means that Q^∞ is a solution of (1.1).

Since $Q^* = (Z^*, X^*, Y^*, M^*)$ is an arbitrary solution of (1.1), we can set $(Z^*, X^*, Y^*, M^*) := (Z^\infty, X^\infty, Y^\infty, M^\infty)$ in the above analysis. Then, (3.13) and (3.14) implies that

$$\lim_{k \to \infty} \left\| \begin{pmatrix} H_{\beta_k}^k - H_{\beta_k}^\infty \\ M^k - M^\infty \end{pmatrix} \right\|^2 = \lim_{j \to \infty} \left\| \begin{pmatrix} H_{\beta_{k_j}}^{k_j} - H_{\beta_{k_j}}^\infty \\ M^{k_j} - M^\infty \end{pmatrix} \right\| = 0.$$

This proves that the full sequence $\{Q^*\}$ converges to $(Z^\infty, X^\infty, Y^\infty, M^\infty)$, a solution point of (1.1). This proof is completed.

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