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COLLOCATION METHOD WITH CUBIC B-SPLINES FOR SOLVING THE GENERALIZED REGULARIZED LONG WAVE EQUATION

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Abstract

In this paper, collocation algorithms using cubic B-splines for solving the generalized regularized long wave (GRLW) equation are presented. The proposed algorithms are based on Crank-Nicolson formulation and central-finite difference approximation. The nonlinear term in each case is computed during executing the algorithm without linearization. The stability analysis using Von-Neumann technique

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shows the schemes are unconditionally stable. To test accuracy the error norms L_2, L_∞ are computed. Also, conservation quantities are evaluated which are found to be small. These results show that the technique introduced here is accurate and easy to apply.

1. Introduction

In general, the finite element methods provide more accurate and efficient numerical solutions than the finite difference methods; one of the popular finite element methods is called the *collocation method* which has two great advantages, since this procedure does not involve integrations and the resulting matrix equation is banded with a small band width. In our study, we choose the cubic B-splines or (bell-shaped splines) as the basis for the space of the solution, because the cubic splines have an additional advantage in which the resulting matrix system is always tridiagonal so we can be solved easily by using any algorithm, which is easy to program and also economical. The requirements of the continuity up to the second degree of the derivatives are evaluated directly. Also the formula for cubic B-splines is given in [1] which is

$$B_j(x) = \frac{1}{h^3}\begin{cases} (x - x_{j-2}), & x_{j-2} \le x \le x_{j-1}, \\ h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 - 3h(x - x_{j-1})^3, & x_{j-1} \le x \le x_j, \\ h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^2 - 3h(x_{j+1} - x)^3, & x_j \le x \le x_{j+1}, \\ (x_{j+1} - x)^3, & x_{j+1} \le x \le x_{j+2}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Indeed the RLW and MRLW equations are special cases of the generalized long wave (GRLW) equation. This equation is very important in physics media since it describes phenomena with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ion-acoustic and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals. The solutions of this equation are kinds of solitary waves named solitons whose shape are not affected by a collision. Historically, we find that the RLW equation is solved numerically by various form of finite element methods as Dag et al. applied cubic B-splines for numerical solution of the RLW equation [1]. The MRLW equation is solved numerically by various form of finite element methods as Khalifa et al. used collocation methods with quadratic B-splines and cubic B-splines [2]. The GRLW equation is studied by few authors, Mokhtari and Mohammadi used Sinc-collocation [3], Kaya used a numerical simulation of solitary wave solutions [4], El-Danaf et al. used Adomian decomposition method (ADM) [5], Roshan used a Petrov-Galerkin method [6], Mohammadi and Mokhtari used the basis of a reproducing kernel space [7], EL-Danaf et al. used finite difference methods [8], Roshan studied the GRLW equation by A Petrov-Galerkin method [9] and Zhang used finite difference method for a Cauchy problem [10]. In this paper, we design a new technique for solving the GRLW equation, a collocation method with cubic B-spline. The nonlinear term in each case is computed during executing the algorithm without linearization. And we show the numerical results for GRLW equation. The interaction of solitary waves and other properties of the GRLW equation are also studied in the three methods. The development of the Maxwellian initial condition into solitary waves is also shown and we show that the number of solitons which are generated from the Maxwellian initial condition can be determined.

2. Collocation Method using Cubic B-spline for Solving the GRLW Equation

2.1. Governing equation and analysis of the method

Consider the GRLW equation

$$u_t + u_x + \varepsilon u^p u_x - \mu u_{xxt} = 0, \tag{2}$$

where $\varepsilon = p(p+1)$ and subscripts x and t denote differentiation, is considered with the boundary and initial conditions $u \to 0$ as $x \to \pm\infty$. In this work, periodic boundary conditions on the region $a \leq x \leq b$ are assumed in the form:

$$u(a, t) = u(b, t) = 0,$$

$$u_x(a, t) = u_x(b, t) = 0, \quad t \geq 0, \tag{3}$$

$$u(x, 0) = f(x), \tag{4}$$

and then the analytical solution of equation (2) takes the form [9]

$$u(x, t) = \sqrt[p]{\frac{(p+2)c}{2p} \operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{c}{\mu(c+1)}}(x - (c+1)t - x_0)\right)}, \tag{5}$$

where x_0 is an arbitrary constant. Actually it is not always available to get an analytic solution for nonlinear partial differential equations, so we try to provide numerical methods to solve such problems.

Now, partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x = b, \quad h = x_{j+1} - x_j = \frac{b-a}{N}, \quad j = 0, 1, \ldots, N.$$

Let $\{B_j\}_{j=-1}^{N+1}$ be the cubic B-splines at the knot points x_i; the set of splines form a basis of functions defined over $[a, b]$. A global approximate solution $U_N(x, t)$ is expressed in terms of the cubic B-splines and unknown time dependent parameters as

$$U_N(x, t) = \sum_{j=-1}^{N+1} c_j(t) B_j(x), \tag{6}$$

where $c_j(t)$ is the parameter in terms of the time t for $j = -1, 0, \ldots, N+1$, to be determined by the collocation boundary and initial conditions, and $B_j(x)$ cubic B-splines function.

The values of $B_j(x)$, first and second derivatives at the knots points are given in Table 1 as below:

Table 1. The values of Cubic B-spline and its first and second derivatives at the knots points

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
B_j	0	1	4	1	0
B'_j	0	$\frac{3}{h}$	0	$\frac{-3}{h}$	0
B''_j	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

Considering the equation (6) and the cubic B-splines $B_j(x)$ defined in Table 1, the required values of U_j and its first and the second derivatives, U'_j and U''_j, at nodal points x_j are identified in terms of c_j as

$$U_j = U(x_j) = c_{j-1} + 4c_j + c_{j+1},$$

$$U'_j = U'(x_j) = \frac{3}{h}(c_{j+1} - c_{j-1}),$$

$$U''_j = U''(x_j) = \frac{6}{h^2}(c_{j-1} - 2c_j + c_{j+1}), \tag{7}$$

now, let us rewrite the equation (2) in the form

$$\frac{\partial(u - \mu u_{xx})}{\partial t} + u_x + \varepsilon u^p u_x = 0, \tag{8}$$

if the time derivative is discretized using central-finite differences, we have $\frac{\partial}{\partial t}u = \frac{u^{n+1} - u^{n-1}}{2k}$, where $k = \Delta t = t_{n+1} - t_n$. Also if we consider, then equation (8) becomes as

$$u^{n+1} - \mu(u_{xx})^{n+1} - u^{n-1} + \mu(u_{xx})^{n-1} + 2k((u_x)^n + \varepsilon(u^p u_x)^n) = 0. \tag{9}$$

Introducing equation (7) into equation (9) yields

$$\left(1-\frac{6\mu}{h^2}\right)c_{j-1}^{n+1}+\left(4+\frac{12\mu}{h^2}\right)c_j^{n+1}+\left(1-\frac{6\mu}{h^2}\right)c_{j+1}^{n+1}$$

$$=(L_3)_j^n-\frac{6\mu}{h^2}(L_4)_j^n-2k[\varepsilon((L_1)_j^n)^p(L_2)_j^n+(L_2)_j^n], \tag{10}$$

where, we have

$$(L_1)_j^n=c_{j-1}^n+4c_j^n+c_{j+1}^n, \quad (L_2)_j^n=\frac{3}{h}(c_{j+1}^n-c_{j-1}^n),$$

$$(L_3)_j^n=c_{j-1}^{n-1}+4c_j^{n-1}+c_{j+1}^{n-1},$$

$$(L_4)_j^n=c_{j-1}^{n-1}-2c_j^{n-1}+c_{j+1}^{n-1}, \quad j=0,1,\ldots,N.$$

Also we can set

$$a=\left(1-\frac{6\mu}{h^2}\right), \quad b=\left(4+\frac{12\mu}{h^2}\right),$$

$$(F)_j^n=(L_3)_j^n-\frac{6\mu}{h^2}(L_4)_j^n-2k[\varepsilon((L_1)_j^n)^p(L_2)_j^n+(L_2)_j^n],\ j=0,1,\ldots,N.$$

So, the system (10) can be rewritten in the simple form as

$$ac_{j-1}^{n+1}+bc_j^{n+1}+ac_{j+1}^{n+1}=F_j^n, \quad j=0,1,\ldots,N. \tag{11}$$

It is clear that the system (11) consists of $(N+1)$ equations in the $(N+3)$ unknowns $(c_{-1}, c_0, \ldots, c_N, c_{N+1})^T$. Appling boundary conditions (3) and use the values in Table 1 we eliminate the unknowns c_{-1}, c_{N+1} from the system (12). The system (12) is reduced to $(N+1)\times(N+1)$ tridiagonal matrix system as fallows:

$$\begin{pmatrix} b-4a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & b & a & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & b & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b-4a \end{pmatrix}\begin{pmatrix} c_0^{n+1} \\ c_1^{n+1} \\ \vdots \\ c_{N-1}^{n+1} \\ c_N^{n+1} \end{pmatrix}=\begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ \vdots \\ F_{N-1} \\ F_N \end{pmatrix}.$$

2.2. Initial state

The above system requires two initial time levels at $t = 0,$ and $t = \Delta t = k,$ so we use the exact solution (5) to determine these initial conditions as follows:

Firstly, from equation (5), we can evaluate the initial conditions at $t = 0$ and $t = k$ as follows:

$$f(x) = u(x, 0) = \sqrt[p]{\frac{(p+2)c}{2p} \operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{c}{\mu(c+1)}}(x - x_0)\right)},$$

$$g(x) = u(x, k) = \sqrt[p]{\frac{(p+2)c}{2p} \operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{c}{\mu(c+1)}}(x - (c+1)k - x_0)\right)}. \quad (12)$$

At level time $n = 0\ (t = 0)$

$$U_N(x_j, 0) = \sum_{j=-1}^{N+1} c_j^0 B_j(x_j) = c_{j-1}^0 + 4c_j^0 + c_{j+1}^0 = f(x_j),\ j = 0, 1, ..., N. \quad (13)$$

At level time $n = 1\ (t = k)$

$$U_N(x_j, k) = \sum_{j=-1}^{N+1} c_j^1 B_i(x_j) = c_{j-1}^1 + 4c_j^1 + c_{j+1}^1 = g(x_j),\ j = 0, 1, ..., N. \quad (14)$$

And from boundary condition (3), we have

$$U_N(x_j, 0) = u(x_j, 0) = 0,$$

$$U'_N(x_0, 0) = U'_N(x_N, 0) = 0,$$

$$U'_N(x_0, k) = U'_N(x_N, k) = 0, \quad (15)$$

then, from equations (15), we get

$$-\frac{3}{h}c_{-1}^{0}+\frac{3}{h}c_{1}^{0}=0,$$

$$-\frac{3}{h}c_{N-1}^{0}+\frac{3}{h}c_{N+1}^{0}=0,$$

$$-\frac{3}{h}c_{-1}^{1}+\frac{3}{h}c_{1}^{1}=0,$$

$$-\frac{3}{h}c_{N-1}^{1}+\frac{3}{h}c_{N+1}^{1}=0. \tag{16}$$

So, the equations (12) and (16) lead to a system $(N+3)\times(N+3)$ of the form

$$\begin{pmatrix} -3 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -3 & 0 & -3 \end{pmatrix}\begin{pmatrix} c_{-1}^{0} \\ c_{0}^{0} \\ \vdots \\ c_{N}^{0} \\ c_{N+1}^{0} \end{pmatrix}=\begin{pmatrix} 0 \\ f(x_0) \\ \vdots \\ \vdots \\ f(x_N) \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -3 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -3 & 0 & -3 \end{pmatrix}\begin{pmatrix} c_{-1}^{1} \\ c_{0}^{1} \\ \vdots \\ c_{N}^{1} \\ c_{N+1}^{1} \end{pmatrix}=\begin{pmatrix} 0 \\ g(x_0) \\ \vdots \\ \vdots \\ g(x_N) \\ 0 \end{pmatrix}.$$

Also, we can convert the above systems to tridiagonal system $(N+1)\times(N+1)$ at level time $n=0$ and at level time $n=1$, by using equations (13) and (14) at $j=0$, $j=N$, and equation (16) as in the form

$$\begin{pmatrix} 4 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} c_{0}^{0} \\ c_{1}^{0} \\ \vdots \\ c_{N-1}^{0} \\ c_{N}^{0} \end{pmatrix}=\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \end{pmatrix},$$

in which $c_{-1}^0 = c_1^0$ and $c_{N-1}^0 = c_{N+1}^0$.

$$\begin{pmatrix} 4 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} c_0^1 \\ c_1^1 \\ \vdots \\ c_{N-1}^1 \\ c_N^1 \end{pmatrix} = \begin{pmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ \vdots \\ g(x_{N-1}) \\ g(x_N) \end{pmatrix},$$

in which $c_{-1}^1 = c_1^1$ and $c_{N-1}^1 = c_{N+1}^1$.

Hence, we can easily determine the initial time parameters $(c_0^0, c_1^0, ..., c_{N-1}^0, c_N^0)$ and $(c_0^1, c_1^1, ..., c_{N-1}^1, c_N^1)$ by using the above tridiagonal systems.

2.3. Stability analysis

The Fourier method has been applying to investigate the stability of the cubic scheme, assuming U in the nonlinear term as a constant. Firstly, rewrite the equation (9) in linearized form as

$$\left(1 - \frac{6\mu}{h^2}\right)c_{j-1}^{n+1} + \left(4 + \frac{12\mu}{h^2}\right)c_j^{n+1} + \left(1 - \frac{6\mu}{h^2}\right)c_{j+1}^{n+1}$$

$$-6\left(\frac{k}{h} + \frac{\varepsilon u^p k}{h}\right)c_{j-1}^n + 6\left(\frac{k}{h} + \frac{\varepsilon U^p k}{h}\right)c_{j+1}^n - \left(1 - \frac{6\mu}{h^2}\right)c_{j-1}^{n-1}$$

$$-\left(4 + \frac{12\mu}{h^2}\right)c_j^{n-1} - \left(1 - \frac{6\mu}{h^2}\right)c_{j+1}^{n-1} = 0, \; j = 0, 1, ..., N. \tag{17}$$

Now using the Fourier method, we have

$$c_j^n = \xi^n e^{ikjh}, \quad i = \sqrt{-1}, \tag{18}$$

where k is mode number.

Substituting equation (18) into equation (17) yields

$$\alpha\xi^{n+1}e^{ik(j-1)h} + \beta\xi^{n+1}e^{ikjh} + \alpha\xi^{n+1}e^{ik(j+1)h}$$
$$- \gamma\xi^{n}e^{ik(j-1)h} + \gamma\xi^{n}e^{ik(j+1)h} - \alpha\xi^{n-1}e^{ik(j-1)h}$$
$$- \beta\xi^{n-1}e^{ikjh} - \alpha\xi^{n-1}e^{ik(j+1)h} = 0,\ i = 0, 1, \ldots, N, \tag{19}$$

where $\alpha = \left(1 - \frac{6\mu}{h^2}\right)$, $\beta = \left(4 + \frac{12\mu}{h^2}\right)$, $\gamma = 6\left(\frac{k}{h} + \frac{\varepsilon U^p k}{h}\right)$.

In case of applying the Von-Neumann stability theory, the growth of Fourier mode is given by

$$\xi^{n+1} = g\xi^{n}, \quad g^2 = \frac{\zeta^{n+1}}{\zeta^{n-1}}, \tag{20}$$

where g is the growth factor.

Now using equation (20) into (19) yields

$$(2\alpha\cos\varphi + \beta)g^2 + (2\gamma i\sin\varphi)g - (2\alpha\cos\varphi + \beta) = 0, \tag{21}$$

where $\varphi + kh$,

or

$$g^2 + 2i\left(\frac{\gamma\sin\varphi}{2\alpha\cos\varphi + \beta}\right)g - 1 = 0. \tag{22}$$

Lemma. *For* h, Δt, φ *and* U *are defined as above* [11]

$$\left|\frac{\gamma\sin\varphi}{2\alpha\cos\varphi + \beta}\right| \le 1, \quad \forall\varphi, \tag{23}$$

where $\alpha = \left(1 - \frac{6\mu}{h^2}\right)$, $\beta = \left(4 + \frac{12\mu}{h^2}\right)$, $\gamma = 6\left(\frac{\Delta t}{h} + \frac{\varepsilon U^p \Delta t}{h}\right)$.

Proof. To prove this lemma you can see [10].

Next, from the above lemma, we can put

$$\frac{\gamma \sin \varphi}{2\alpha \cos \varphi + \beta} = \sin \theta. \tag{24}$$

For any angle θ.

So, equation (22) becomes

$$g^2 + 2ig \sin \theta - 1 = 0, \tag{25}$$

Thus, $g_1 = e^{i\theta} = \cos\theta + i\sin\theta$ and $g_2 = -e^{-i\theta} = -\cos\theta + i\sin\theta$ because $g_1 g_2 = -e^{i\theta}e^{-i\theta} = -1$ and $g_1 + g_2 = 2i \sin \theta$.

Therefore, we have

$$|g_1| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1,$$

$$|g_2| = |-\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1. \tag{26}$$

Showing that our scheme is unconditionally stable.

2.4. Numerical tests and results of GRLW equation

In this section, we present some numerical examples to test validity of our scheme for solving GRLW equation (2). For this purpose, we aim to simulate motion of single solitary wave, interaction of two and three solitary waves, wave undulation and propagation of wave with the Maxwellian initial condition. Accuracy of the scheme is measured by using the following error norms:

$$L_2 = \| u^E - u^N \| = \sqrt{h\sum_{i=0}^{N}(u_j^E - u_j^N)^2},$$

$$L_\infty = \max_j | u_j^E - u_j^N |, \ j = 0, 1, ..., N, \tag{27}$$

where u^E is the exact solution u and u^N is the approximation solution U_N.

2.4.1. The motion of single solitary waves

In previous section, we have provided the cubic B-spline scheme for the GRLW equation, and we can take the following as an initial condition:

$$u(x, 0) = \sqrt[p]{\frac{(p+2)c}{2p}\operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{c}{\mu(c+1)}}(x - x_0)\right)}. \tag{28}$$

The norms L_2 and L_∞ are used to compare the numerical solution with the analytical solution and the quantities I_1, I_2 and I_3 are shown to measure conservation for the schemes.

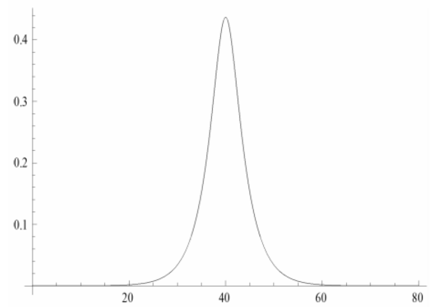
Now, we consider two test problems.

First test problem

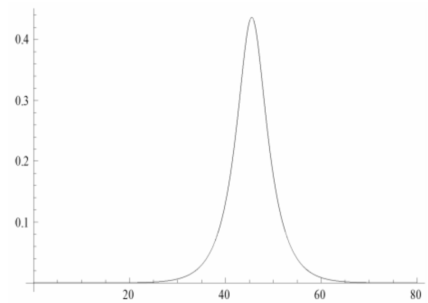
Now, we consider a test problem where $p = 3$, $c = 0.1$, $\mu = 1$, $h = 0.1$, $x_0 = 40$, $\Delta t = k = 0.1$ with range $[0, 80]$. The simulations are done up to $t = 5$. The invariants I_1, I_2 and I_3 are approach to zero in the computer program for the scheme. Errors, also, at time 5 are satisfactorily small L_2-error $= 5.29594 \times 10^{-4}$, and L_∞-error $= 2.89598 \times 10^{-4}$, for the scheme. Our results are recorded in Table 2 and the motion of solitary wave is plotted at different time levels in Figure 1.

Table 2. Invariants and errors for single solitary wave $p = 3$, $c = 0.1$, $h = 0.1$, $k = 0.1$ and $x_0 = 40$, $0 \le x \le 80$

t	I_1	I_2	I_3	L_2-norm	L_∞-norm
0	4.06257	1.09135	0.13542	0.0	0.0
1	4.06257	1.09135	0.13523	1.08411E-5	6.21631E-5
2	4.06256	1.09135	0.13524	2.15815E-4	9.96642E-5
3	4.06255	1.09135	0.13517	3.27326E-4	1.44133E-4
4	4.06254	1.09134	0.13533	4.32599E-4	2.44683E-4
5	4.06252	1.09134	0.13534	5.29594E-4	2.89598E-4



(a)



(b)

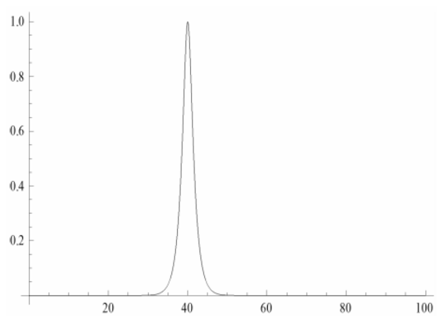
Figure 1. Single solitary wave with $c = 0.1$, $h = 0.1$, $k = 0.1$ and $x_0 = 40$, $0 \le x \le 80$, $t = 0, 5$.

Second test problem

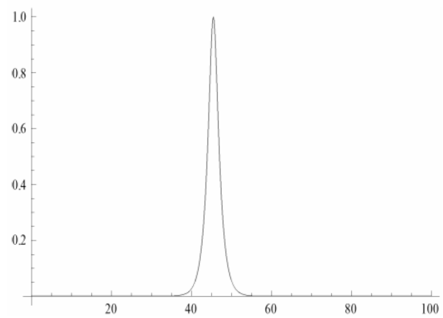
Now, we consider a test problem where $p = 3$, $c = 1.2$, $\mu = 1$, $h = 0.1$, $x_0 = 40$, $\Delta t = k = 0.025$ with range $[0, 100]$. The simulations are done up to $t = 2.5$. The invariants I_1 approach to zero, I_2 and I_3 are changed by less than 2×10^{-4} and 4×10^{-5}, respectively in the computer program for the scheme. Errors, also, at time 2.5 are satisfactorily small L_2-error = 3.72171×10^{-4} and L_∞-error = 2.47808×10^{-4} for the scheme. Our results are recorded in Table 3 and the motion of solitary wave is plotted at different time levels in Figure 2.

Table 3. Invariants and errors for single solitary wave $p = 3$, $c = 1.2$, $h = 0.1$, $k = 0.025$ and $x_0 = 40$, $0 \le x \le 100$

t	I_1	I_2	I_3	L_2-norm	L_∞-norm
0	3.79713	2.33531	1.51885	0.0	0.0
0.5	3.79713	2.33583	1.51888	2.33182E-4	2.15669E-4
1.0	3.79713	2.33546	1.51884	3.52069E-4	2.15952E-4
1.5	3.79713	2.33551	1.51883	3.76971E-4	2.41288E-4
2	3.79713	2.33551	1.51881	3.76226E-4	2.41641E-4
2.5	3.79713	2.33551	1.51881	3.72171E-4	2.47808E-4



(a)



(b)

Figure 2. Single solitary wave with $c = 1.2,$ $h = 0.1,$ $k = 0.025$ and $x_0 = 40,\ 0 \leq x \leq 100,\ t = 0,\ 2.$

2.4.2. Interaction of two solitary waves

The interaction of two GRLW solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider GRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

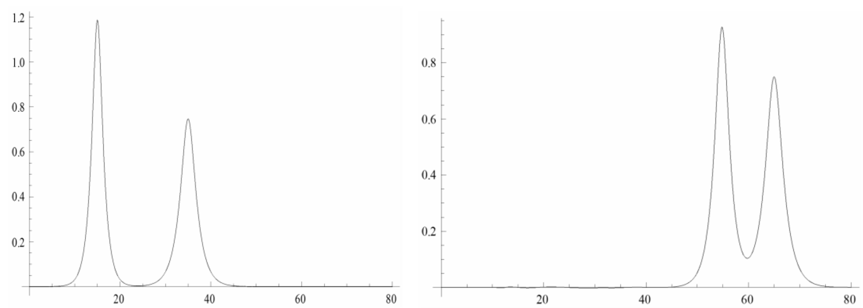
$$u(x,\ 0) = \sum_{j=1}^{2} \sqrt[p]{\frac{(p+2)c_j}{2p} \operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{c_j}{\mu(c_j+1)}}(x - x_j)\right)}, \tag{29}$$

where $j = 1, 2,$ x_j and c_j are arbitrary constants. In our computational work, now, we discuss the Interaction of two solitary waves for GRLW equation.

Now, we choose $c_1 = 1,\ c_2 = 0.5,\ x_1 = 15,\ x_2 = 35,\ \mu = 1,\ h = 0.1,\ k = 0.1$ with interval $[0,\ 80].$ In Figure 3, the interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the three invariants for this case are shown in Table 4. The invariants $I_1,\ I_2$ and I_3 are changed by less than $2.8 \times 10^{-4},\ 1.05 \times 10^{-3}$ and $5.27 \times 10^{-3},$ respectively for the scheme.

Table 4. Invariants of interaction two solitary waves of GRLW equation $c_1 = 1,\ c_2 = 0.5,\ x_1 = 15,\ x_2 = 35,\ 0 \le x \le 80$

t	I_1	I_2	I_3
0	7.35999	3.82676	1.84872
2	7.35999	3.82348	1.89851
4	7.35977	3.82348	1.85161
6	7.35958	3.81707	1.84374
8	7.35933	3.81606	1.84451
10	7.35893	3.81781	1.84351



(a) (b)

Figure 3. Interaction two solitary waves with $c_1 = 2,\ c_2 = 0.5,\ x_1 = 15,\ x_2 = 35,\ 0 \le x \le 80$ at times $t = 0,\ 20.$

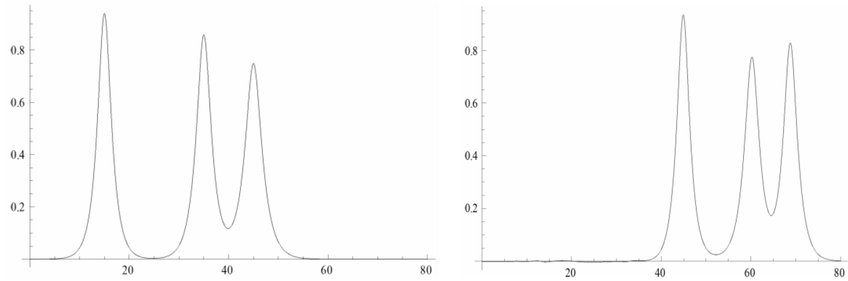
2.4.3. Interaction of three solitary waves

The interaction of three GRLW solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the GRLW equation with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes:

$$u(x,\ 0) = \sum_{j=1}^{3} \sqrt[p]{\frac{(p+2)c_j}{2p}}\, \operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{c_j}{\mu(c_j+1)}}(x - x_j)\right), \tag{30}$$

where $j = 1, 2, 3,$ x_j and c_j are arbitrary constants. In our computational work.

Now we discuss the interaction of three solitary waves for GRLW equation.



(a) (b)

Figure 4. Interaction three solitary waves with $c_1 = 1, c_2 = 0.75, c_3 = 0.5,$ $x_1 = 15, x_2 = 35, x_3 = 45, 0 \le x \le 80$ at time $t = 0, 15.$

Now, we choose $c_1 = 1, c_2 = 0.75, c_3 = 0.5, x_1 = 15, x_2 = 35, x_3 = 45$ with interval $[0, 80].$ In Figure 4, the interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the three invariants for this case are shown in Table 5. The invariants I_1, I_2 and I_3 are changed by less than $3 \times 10^{-4}, 1.604 \times 10^{-2}$ and $7.9 \times 10^{-3},$ respectively for the scheme.

Table 5. Invariants of interaction three solitary waves of GRLW equation. $c_1 = 1, c_2 = 0.75, c_3 = 0.5, x_1 = 15, x_2 = 35, x_3 = 45, 0 \le x \le 80$

t	I_1	I_2	I_3
0	11.0225	5.82819	2.79641
2	11.0226	5.83156	2.79428
4	11.0227	5.83475	2.79081
6	11.0227	5.84826	2.78668
8	11.0228	5.85315	2.78531
10	11.0228	5.85423	2.78551

2.4.4. The Maxwellian initial condition

In final series of numerical experiments, the development of the Maxwellian initial condition

$$u(x, 0) = \exp(-(x - 40)^2). \tag{31}$$

Into a train of solitary waves is examined. We apply it to the problem for different cases: (I) $\mu = 0.1$, (II) $\mu = 0.05$, (III) $\mu = 0.04$, (IV) $\mu = 0.015$ and (V) $\mu = 0.01$. Now we discusses it for GRLW equation.

When μ is large such as Case (I), only single soliton is generated as shown in Figure 5(a). However, when μ reduced, more and more solitary waves is are formed, since for Case (II), two solitary waves are generated as shown in Figure 5(b), and for Case (III) the Maxwellian pulse breaks up into a train of at least two solitary waves as shown in Figure 6(a). Finally, for (IV) and (V) cases, the Maxwellian initial condition has decayed into three stable solitary waves as shown in Figure 6(b) and Figure 7. The peaks of the well-developed wave lie on a straight line so that their velocities are linearly dependent on their amplitudes and we observe a small oscillating tail appearing behind the last wave as shown in Figures 5, 6 and 7, and all states at $t = 5$. Moreover, the total number of solitary waves which are generated from the Maxwellian initial condition according to the results obtained from the numerical scheme in test problem as shown in Table 7, can be shown to follow approximately the relation:

$$N \cong [1/\sqrt[4]{\mu}]. \tag{32}$$

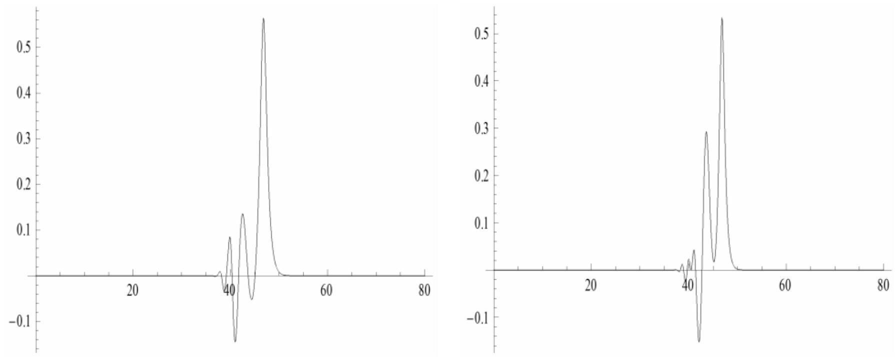
The values of the quantities I_1, I_2 and I_3 for the cases: $\mu = 0.1$, $\mu = 0.05$, $\mu = 0.04$, $\mu = 0.015$ and $\mu = 0.01$ are given in Table 6.

Table 6

μ	t	I_1	I_2	I_3
0.1	3	1.77245	1.25331	0.886227
	4	1.77232	1.22331	0.886244
	5	1.77211	1.24331	0.886267
0.05	3	1.67245	1.25356	0.886227
	4	1.67211	1.25324	0.866629
	5	1.66233	1.25367	0.856824
0.04	3	1.77245	1.15491	0.786234
	4	1.77342	1.15345	0.766298
	5	1.77123	1.14324	0.756278
0.015	3	1.87245	1.15333	0.776227
	4	1.87233	1.15356	0.766277
	5	1.87212	1.15567	0.756267
0.01	3	1.57245	1.05323	0.784527
	4	1.57124	1.05334	0.782326
	5	1.57298	1.05334	0.783226

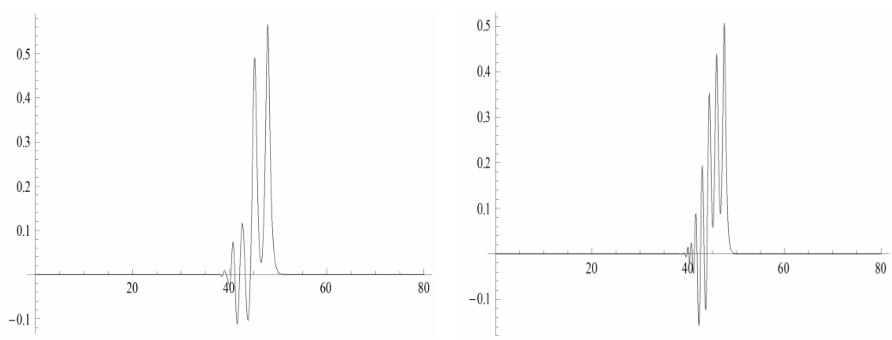
Table 7. Solitary waves generated from a Maxwellian initial condition

μ	Number of solitary waves
0.1	1
0.05	2
0.04	2
0.015	3
0.01	4



(a) (b)

Figure 5. The Maxwellian initial condition at (a) $\mu = 0.1$, (b) $\mu = 0.05$ and $t = 5$.



(a) (b)

Figure 6. The Maxwellian initial condition at (a) $\mu = 0.04$, (b) $\mu = 0.015$ and $t = 5$.

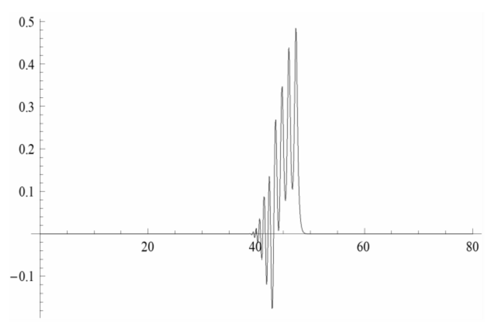


Figure 7. The Maxwellian initial condition at $\mu = 0.01$ and $t = 5$.

3. Conclusion

In this paper, we applied a collocation method using cubic B-splines to study solitary waves of GRLW equation, and shown that our scheme are unconditionally stable. We tested our scheme through single solitary wave in which the analytic solution is known, and then extend them to study the interaction of solitons where no analytic solution is known during the interaction. The Maxwellian initial condition for GRLW equation is used. The obtained approximate numerical solutions maintain good accuracy compared with the exact solutions.

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