Far East Journal of Mathematical Sciences (FJMS)

Published Online: February 2016
http://dx.doi.org/10.17654/MS099060813
Volume 99, Number 6, 2016, Pages 813-830 **ISSN: 0972-0871**

THE HELICITY OF PEKERIS, ACCAD AND SHKOLLER FLOW

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Abstract

Helicity of PAS (Pekeris-Accad-Shkoller) flow is the scalar product of velocity $\boldsymbol{v}$ flow and its curl, in a Euclidean space. Since PAS flow is a Beltrami flow with a property $\boldsymbol{C}\boldsymbol{v} = \operatorname{curl} \boldsymbol{v}$, with $\boldsymbol{C}$ is a constant, the helicity of PAS flow can be determined by integrating the scalar product between vectors $\boldsymbol{v}$ and $\boldsymbol{v}$ and multiply with Λ. This paper will explain the process of calculating the helicity of PAS.

Received: October 4, 2015; Revised: December: 9, 2015; Accepted: December 26, 2015
2010 Mathematics Subject Classification: 00A69.
Keywords and phrases: helicity, flow.
Communicated by Shahrdad G. Sajjadi; Editor: Advances and Applications in Fluid Mechanics: Published by Pushpa Publishing House.

1. Introduction

PAS flow is defined by

$$\boldsymbol{v} = 2\Re\{\boldsymbol{s}_2^2 + \boldsymbol{t}_2^2\}, \tag{1.1}$$

$$s_2^2 = K\Lambda j_2(\Lambda r), \tag{1.2}$$

$$t_2^2 = \Lambda s_2^2(r), \tag{1.3}$$

$$K = \sqrt{6/5},$$

where Λ is the positive root of the second order of Spherical Bessel function $j_2(r)$,

$$j_2(r) = -\frac{3\cos r}{r^2} + \frac{(3 - r^2)\sin r}{r^3} \tag{1.4}$$

and helicity of a flow is defined by [2],

$$H = \int_V \boldsymbol{v} \cdot \nabla \times \boldsymbol{v} dV, \tag{1.5}$$

while the property of Beltrami flow is [3],

$$C\boldsymbol{v} = \nabla \times \boldsymbol{v}, \tag{1.6}$$

So that the integral (1.5) can be substituted by the following form:

$$H = C\int_V \boldsymbol{v} \cdot \boldsymbol{v} dV. \tag{1.7}$$

It is easy to show that PAS flow has

$$C = \Lambda$$

so that the integral (1.7) recast into

$$H = \Lambda\int_V \boldsymbol{v} \cdot \boldsymbol{v} dV. \tag{1.8}$$

2. Main Results

Based on (1.1), it can be stated that

$$\boldsymbol{v} = \boldsymbol{s}_2^2 + \boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2 + \boldsymbol{t}_2^{-2}$$

so that the integrand form in the integral (1.8) can be changed into

$$\begin{aligned}\boldsymbol{v} \cdot \boldsymbol{v} &= (\boldsymbol{s}_2^2 + \boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2 + \boldsymbol{t}_2^{-2}) \cdot (\boldsymbol{s}_2^2 + \boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2 + \boldsymbol{t}_2^{-2}) \\ &= \boldsymbol{s}_2^2\boldsymbol{s}_2^2 + \boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} + \boldsymbol{s}_2^2\boldsymbol{t}_2^2 + \boldsymbol{s}_2^2\boldsymbol{t}_2^{-2} + \boldsymbol{s}_2^{-2}\boldsymbol{s}_2^2 + \boldsymbol{s}_2^{-2}\boldsymbol{s}_2^{-2} + \boldsymbol{s}_2^{-2}\boldsymbol{t}_2^2 \\ &\quad + \boldsymbol{s}_2^{-2}\boldsymbol{t}_2^{-2} + \boldsymbol{t}_2^2\boldsymbol{s}_2^2 + \boldsymbol{t}_2^2\boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2\boldsymbol{t}_2^2 + \boldsymbol{t}_2^2\boldsymbol{t}_2^{-2} + \boldsymbol{t}_2^{-2}\boldsymbol{s}_2^2 + \boldsymbol{t}_2^{-2}\boldsymbol{s}_2^{-2} \\ &\quad + \boldsymbol{t}_2^{-2}\boldsymbol{t}_2^2 + \boldsymbol{t}_2^{-2}\boldsymbol{t}_2^{-2} \end{aligned} \tag{2.1}$$

with

$$dV = r^2 \sin\theta dr d\theta d\phi,$$

and

$$0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

Integrands (2.1) are composed by 16 parts of scalar product between two vectors. To determine the integral solution of (1.8), the spherical harmonics function is needed to apply in it. This function is defined by [1],

$$Y_n^m(\theta, \phi) = (-)^m \left(\frac{2n+1}{2-\delta_m^0}\right)^{\frac{1}{2}} P_n^m(\cos\theta) e^{im\phi} = (-)^m \overline{Y_n^{-m}}, \tag{2.2}$$

where $\overline{Y_n^{-m}}$ is conjugate of $Y_n^m(\theta, \phi)$,

$$P_n^m(\cos\theta) = \left(\frac{2-\delta_m^0}{(n+m)!}\right)^{\frac{1}{2}} \frac{(1-\cos^2\theta)}{2^n n!} \left[\frac{d}{d\cos\theta}\right]^{m+n} (\cos^2\theta - 1)^n, \tag{2.3}$$

is an associated Legendre polynomial with *n* degree and *m* order, with $0 \le \theta \le \pi$, $e^{im\phi}$ is Euler function with $0 \le \phi \le 2\pi$,

$$\delta_m^0 \text{ is delta Kronecker with } \delta_m^0 = \begin{cases} 1, & m = 0 \\ 0, & m \ne 0. \end{cases}$$

Because in PAS flow $m = 2$ and $n = 2$ then the integrand (2.1) are expanded into spherical harmonic functions of Y_2^2 and Y_2^{-2}, and since the integral of $\int_0^{2\pi} (e^{i2\phi})^2 d\phi$ and $\int_0^{2\pi} (e^{-i2\phi})^2 d\phi$ are zero, the ϕ integral, composed of $Y_2^2 Y_2^2$, and $Y_2^{-2} Y_2^{-2}$ are zero. While the results of the ϕ integral which is composed by $Y_2^2 Y_2^{-2}$, is 2π, because $\int_0^{2\pi} e^{i2\phi} \cdot e^{-i2\phi} d\phi = \int_0^{2\pi} d\phi = 2\pi$. By these properties of the ϕ integral, the 16 parts of integrand (2.1) can be reduced to

$$\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2},\ \boldsymbol{s}_2^2\boldsymbol{t}_2^{-2},\ \boldsymbol{s}_2^{-2}\boldsymbol{s}_2^2,\ \boldsymbol{s}_2^{-2}\boldsymbol{t}_2^2,\ \boldsymbol{t}_2^2\boldsymbol{s}_2^{-2},\ \boldsymbol{t}_2^2\boldsymbol{t}_2^{-2},\ \boldsymbol{t}_2^{-2}\boldsymbol{s}_2^2,\ \boldsymbol{t}_2^{-2}\boldsymbol{t}_2^2.$$

These reduced integrands are then be expanded in the form of spherical harmonic functions as the following.

First, we expand the pair integrand $\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2}$ and $\boldsymbol{s}_2^{-2}\boldsymbol{s}_2^2$,

$$\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} = \left(\frac{6}{r} s_2^2 Y_2^2,\ \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta},\ \frac{1}{e\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}\right)$$

$$\left(\frac{6}{r} s_2^2 Y_2^{-2},\ \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\phi}\right)$$

$$= \frac{36}{r^2}(s_2^2)^2 Y_2^2 Y_2^{-2} + \frac{1}{r^2}\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}$$

$$+ \frac{1}{r^2\sin^2\theta}\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi}, \tag{2.4}$$

$$\boldsymbol{s}_2^{-2}\boldsymbol{s}_2^2 = \left(\frac{6}{r} s_2^2 Y_2^{-2},\ \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\phi}\right)$$

$$\left(\frac{6}{r} s_2^2 Y_2^2,\ \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}\right)$$

$$= \frac{36}{r^2}(s_2^2)^2 Y_2^{-2} Y_2^2 + \frac{1}{r^2}\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^{-2}}{\partial\theta}\frac{\partial Y_2^2}{\partial\theta}$$

$$+ \frac{1}{r^2\sin^2\theta}\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^{-2}}{\partial\phi}\frac{\partial Y_2^2}{\partial\phi}. \tag{2.5}$$

By the properties,

$$\left.\begin{aligned} \frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta} &= \frac{\partial Y_2^{-2}}{\partial\theta}\frac{\partial Y_2^2}{\partial\theta} \\ \frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi} &= \frac{\partial Y_2^{-2}}{\partial\phi}\frac{\partial Y_2^2}{\partial\phi} \end{aligned}\right\} \tag{2.6}$$

then $\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} = \boldsymbol{s}_2^{-2}\boldsymbol{s}_2^2$.

Second, we expand the pair integrand $\boldsymbol{t}_2^2\boldsymbol{t}_2^{-2}$ and $\boldsymbol{t}_2^{-2}\boldsymbol{t}_2^2$,

$$\boldsymbol{t}_2^2\boldsymbol{t}_2^{-2} = \left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}, -t_2^2\frac{\partial Y_2^2}{\partial\theta}\right)\left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^{-2}}{\partial\phi}, -t_2^2\frac{\partial Y_2^2}{\partial\theta}\right)$$

$$= \frac{(t_2^2)^2}{\sin^2\theta}\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi} + (-t_2^2)^2\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}$$

$$= \frac{(t_2^2)^2}{\sin^2\theta}\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi} + (t_2^2)^2\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}, \tag{2.7}$$

$$\boldsymbol{t}_2^{-2}\boldsymbol{t}_2^2 = \left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^{-2}}{\partial\phi}, -t_2^2\frac{\partial Y_2^{-2}}{\partial\theta}\right)\left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}, -t_2^2\frac{\partial Y_2^2}{\partial\theta}\right)$$

$$= \frac{(t_2^2)^2}{\sin^2\theta}\frac{\partial Y_2^{-2}}{\partial\phi}\frac{\partial Y_2^2}{\partial\phi} + (-t_2^2)^2\frac{\partial Y_2^{-2}}{\partial\theta}\frac{\partial Y_2^2}{\partial\theta}$$

$$= \frac{(t_2^2)^2}{\sin^2\theta}\frac{\partial Y_2^{-2}}{\partial\phi}\frac{\partial Y_2^2}{\partial\phi} + (t_2^2)^2\frac{\partial Y_2^{-2}}{\partial\theta}\frac{\partial Y_2^2}{\partial\theta}. \tag{2.8}$$

By the properties (2.6), equations (2.7) and (2.8) are equal so that $\boldsymbol{t}_2^2\boldsymbol{t}_2^{-2} = \boldsymbol{t}_2^{-2}\boldsymbol{t}_2^2$.

Third, we expand the pair integrands $\boldsymbol{s}_2^2\boldsymbol{t}_2^{-2}$ and $\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2}$,

$$\boldsymbol{s}_2^2\boldsymbol{t}_2^{-2} = \left(\frac{6}{r}s_2^2Y_2^2, \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta}, \frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}\right)$$

$$\cdot\left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^{-2}}{\partial\phi}, -t_2^2\frac{\partial Y_2^{-2}}{\partial\theta}\right)$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\phi} - \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\theta}$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\left(\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\phi} - \frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\theta}\right), \tag{2.9}$$

$$\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2} = \left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}, -t_2^2\frac{\partial Y_2^2}{\partial\theta}\right)$$

$$\cdot\left(\frac{6}{r}s_2^2Y_2^{-2}, \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\theta}, \frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\phi}\right)$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\theta} - \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\phi}$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\left(\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\theta} - \frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\phi}\right). \tag{2.10}$$

By the property,

$$\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\phi} = -\frac{\partial Y_2^{-2}}{\partial\theta}\frac{\partial Y_2^2}{\partial\phi}, \tag{2.11}$$

then equations (2.9) and (2.10) show that $\boldsymbol{s}_2^2\boldsymbol{t}_2^{-2} = -\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2}$.

Fourth, we expand the pairs integrands $\boldsymbol{t}_2^{-2}\boldsymbol{s}_2^2$ and $\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2}$,

$$\boldsymbol{t}_2^{-2}\boldsymbol{s}_2^2 = \left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^{-2}}{\partial\phi}, -t_2^2\frac{\partial Y_2^{-2}}{\partial\theta}\right)$$

$$\cdot\left(\frac{6}{r}s_2^2Y_2^2,\ \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}\right)$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\theta} - \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\theta}\frac{\partial Y_2^{2}}{\partial\phi}$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\left(\frac{\partial Y_2^{-2}}{\partial\phi}\frac{\partial Y_2^{2}}{\partial\theta} - \frac{\partial Y_2^{-2}}{\partial\theta}\frac{\partial Y_2^{2}}{\partial\phi}\right), \tag{2.12}$$

$$\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2} = \left(0,\ \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi},\ -t_2^2\frac{\partial Y_2^2}{\partial\theta}\right)$$

$$\cdot\left(\frac{6}{r}s_2^2Y_2^{-2},\ \frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{-2}}{\partial\phi}\right)$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{2}}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\theta} - \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^{2}}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\phi}$$

$$= \frac{t_2^2}{r\sin\theta}\frac{d}{dr}(rs_2^2)\left(\frac{\partial Y_2^{2}}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\theta} - \frac{\partial Y_2^{2}}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\phi}\right). \tag{2.13}$$

According to the property (2.11), the results (2.12) and (2.13) show that $\boldsymbol{t}_2^{-2}\boldsymbol{s}_2^2 = -\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2}$.

From the expansions of integrands (2.1), therefore we find two pairs of integrands with same sign,

$$\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} = \boldsymbol{s}_2^{-2}\boldsymbol{s}_2^2,$$

and

$$\boldsymbol{t}_2^2\boldsymbol{t}_2^{-2} = \boldsymbol{t}_2^{-2}\boldsymbol{t}_2^2,$$

and two integrands with opposite sign,

$$\boldsymbol{s}_2^2\boldsymbol{t}_2^{-2} = -\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2}$$

and

$$\boldsymbol{t}_2^{-2}\boldsymbol{s}_2^2 = -\boldsymbol{t}_2^2\boldsymbol{s}_2^{-2}.$$

Since the sum of two integrands in (2.1) with opposite sign is zero, the scalar product of vector $\boldsymbol{v}$ and $\boldsymbol{v}$ in (2.1) can be reduced as the following:

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{v} &= \boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} + \boldsymbol{s}_2^{-2}\boldsymbol{s}_2^2 + \boldsymbol{t}_2^2\boldsymbol{t}_2^{-2} + \boldsymbol{t}_2^{-2}\boldsymbol{t}_2^2 \\ &= 2(\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2\boldsymbol{t}_2^{-2}). \qquad (2.14)\end{aligned}$$

Substitute (2.14) into (1.8), the integral form becomes

$$\begin{aligned}H &= \Lambda\int_V 2(\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2\boldsymbol{t}_2^{-2})dV \\ &= \Lambda\int_0^{2\pi}\int_0^{\pi}\int_0^1 2(\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2\boldsymbol{t}_2^{-2})r^2\sin\theta dr d\theta d\phi. \qquad (2.15)\end{aligned}$$

Substitute (2.4) and (2.7) into the integrand (2.15), will obtain

$$\begin{aligned}&2(\boldsymbol{s}_2^2\boldsymbol{s}_2^{-2} + \boldsymbol{t}_2^2\boldsymbol{t}_2^{-2})r^2\sin\theta \\ &= 2\Bigg(\Bigg\{\frac{36}{r^2}(s_2^2)^2Y_2^2Y_2^{-2} \\ &\quad + \frac{1}{r^2}\left[\frac{d}{dr}(rs_2^2)\right]^2\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\left[\frac{d}{dr}(rs_2^2)\right]^2\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi}\Bigg\} \\ &\quad + \left\{\frac{(t_2^2)^2}{\sin^2\theta}\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi} + (t_2^2)^2\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}\right\}\Bigg)r^2\sin\theta \\ &= 2\Bigg\{\frac{36}{r^2}(s_2^2)^2Y_2^2Y_2^{-2} \\ &\quad + \frac{1}{r^2}\left[\frac{d}{dr}(rs_2^2)\right]^2\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\left[\frac{d}{dr}(rs_2^2)\right]^2\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi}\Bigg\}r^2\sin\theta \\ &\quad + 2\left\{\frac{(t_2^2)^2}{\sin^2\theta}\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi} + (t_2^2)^2\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}\right\}r^2\sin\theta\end{aligned}$$

$$= \underbrace{72(s_2^2)^2 Y_2^2 Y_2^{-2} \sin\theta}_{1} + \underbrace{2\sin\theta\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}}_{2}$$

$$+ \underbrace{\frac{2}{\sin\theta}\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi}}_{3}$$

$$+ \underbrace{\frac{2r^2(t_2^2)^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi}}_{4} + \underbrace{2r^2(t_2^2)^2 \sin\theta \frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}}_{5}. \tag{2.16}$$

So the integral (2.14) is the sum of five integrals with the integrand 1 until 5 in (2.16). To solve the integral (2.15), we expand these integrands to the following spherical harmonic functions form.

Integrand 1

$$72(s_2^2)^2 Y_2^2 Y_2^{-2} \sin\theta = 135K^2\Lambda^2(j_2(\Lambda r))^2 \sin^4\theta \sin\theta. \tag{2.17}$$

Integrand 2

$$2\sin\theta\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta}$$

$$= 2(K^2\Lambda^4 r^2(j_1(\Lambda r))^2 - 4K^2\Lambda^3 r j_1(\Lambda r)\, j_2(\Lambda r)$$

$$+ 4K^2\Lambda^2(j_2(\Lambda r))^2)\left(\frac{15}{2}\sin^2\theta\cos^2\theta\sin\theta\right) \text{ [4].} \tag{2.18}$$

Integrand 3

$$\frac{2}{\sin\theta}\left[\frac{d}{dr}(rs_2^2)\right]^2 \frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi}$$

$$= 2(K^2\Lambda^4 r^2(j_1(\Lambda r))^2 - 4K^2\Lambda^3 r j_1(\Lambda r)\, j_2(\Lambda r)$$

$$+ 4K^2\Lambda^2(j_2(\Lambda r))^2)(30\cos^2\theta\sin\theta). \tag{2.19}$$

Integrand 4

$$\frac{2r^2(t_2^2)^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}\frac{\partial Y_2^{-2}}{\partial\phi} = 2r^2K^2\Lambda^4(j_2(\Lambda r))^2(30\cos^2\theta\sin\theta). \quad (2.20)$$

Integrand 5

$$2r^2(t_2^2)^2\sin\theta\frac{\partial Y_2^2}{\partial\theta}\frac{\partial Y_2^{-2}}{\partial\theta} = 2r^2K^2\Lambda^4(j_2(\Lambda r))^2\sin\theta\left(\frac{15}{2}\right)\sin^2\theta\cos^2\theta. \quad (2.21)$$

The next step is, to integrate each form of integrands from (2.17) until (2.21).

Since the three variables of integral, ϕ, θ, r are mutually independent in the integral (2.15), we solve the integral as the multiplication of $(I_\phi)(I_\theta)(I_r)$. These integrals are associated to every part of the five integrands (2.16). We solve the integrals as the following order.

Integral 1, I_1 with integrand (2.17).

Integral ϕ,

$$I_{1\phi} = \int_0^{2\pi} d\phi = 2\pi.$$

Integral θ,

$$I_{1\theta} = \int_0^{\pi}\sin^4\theta\sin\theta d\theta = \frac{16}{15}.$$

Integral r,

$$I_{1r} = \int_0^1 135K^2\Lambda^2(j_2(\Lambda r))^2 dr.$$

So the result is integral 1 is

$$I_1 = (I_{1\phi})(I_{1\theta})(I_{1r}) = 288\pi\int_0^1 K^2\Lambda^2(j_2(\Lambda r))^2 dr.$$

Integral 2, I_2 with integrand (2.18).

Integral ϕ,

$$I_{2\phi} = \int_0^{2\pi} d\phi = 2\pi.$$

Integral θ,

$$I_{2\theta} = \int_0^{\pi} \frac{15}{2} \sin^2 \theta \cos^2 \theta \sin \theta d\theta = 2.$$

Integral r,

$$I_{2r} = \int_0^1 2(K^2 \Lambda^4 r^2 (j_1(\Lambda r))^2 - 4K^2 \Lambda^3 r j_1(\Lambda r) j_2(\Lambda r)$$

$$+ 4K^2 \Lambda^2 (j_2(\Lambda r))^2) dr.$$

Result integral 2 is

$$I_2 = (I_{2\phi})(I_{2\theta})(I_{2r}) = 8\pi \int_0^1 (K^2 \Lambda^4 r^2 (j_1(\Lambda r))^2$$

$$- 4K^2 \Lambda^3 r j_1(\Lambda r) j_2(\Lambda r) + 4K^2 \Lambda^2 (j_2(\Lambda r))^2) dr.$$

Integral 3, I_3 with integrand (2.19).

Integral ϕ,

$$I_{3\phi} = \int_0^{2\pi} d\phi = 2\pi.$$

Integral θ,

$$I_{3\theta} = \int_0^{\pi} \sin^3 \theta d\theta = \frac{4}{3}.$$

Integral r,

$$I_{3r} = \int_0^1 15(K^2 \Lambda^4 r^2 (j_1(\Lambda r))^2 - 4K^2 \Lambda^3 r j_1(\Lambda r) j_2(\Lambda r)$$

$$+ 4K^2 \Lambda^2 (j_2(\Lambda r))^2) dr.$$

Result integral 3 is

$$I_3 = (I_{3\phi})(I_{3\theta})(I_{3r}) = 40\pi \int_0^1 (K^2\Lambda^4 r^2 (j_1(\Lambda r))^2$$

$$- 4K^2\Lambda^3 r j_1(\Lambda r) j_2(\Lambda r) + 4K^2\Lambda^2 (j_2(\Lambda r))^2) dr.$$

Integral 4, I_4 with integrand (2.20).

Integral ϕ,

$$I_{4\phi} = \int_0^{2\pi} d\phi = 2\pi.$$

Integral θ,

$$I_{4\theta} = \int_0^{\pi} \sin^3 \theta d\theta = \frac{4}{3}.$$

Integral r,

$$I_{4r} = \int_0^1 15K^2\Lambda^4 r^2 (j_2(\Lambda r))^2 dr.$$

Result integral 4 is

$$I_4 = (I_{4\phi})(I_{4\theta})(I_{4r}) = 40\pi \int_0^1 K^2\Lambda^4 r^2 (j_2(\Lambda r))^2 dr.$$

Integral 5, I_5 with integrand (2.21).

Integral ϕ,

$$I_{5\phi} = \int_0^{2\pi} d\phi = 2\pi.$$

Integral θ,

$$I_{5\theta} = \int_0^{\pi} \frac{15}{2} \sin^2 \theta \cos^2 \theta \sin \theta d\theta = 2.$$

Integral r,

$$I_{5r} = \int_0^1 2K^2\Lambda^4 r^2 (j_2(\Lambda r))^2 \, dr.$$

Result integral 5 is

$$I_5 = (I_{5\phi})(I_{5\theta})(I_{5r}) = 8\pi \int_0^1 K^2\Lambda^4 r^2 (j_2(\Lambda r))^2 \, dr.$$

With these five results we can formulate the helicity of PAS flow as

$$H = \Lambda(I_1 + I_2 + I_3 + I_4 + I_5) = \Lambda I_1 + \Lambda I_2 + \Lambda I_3 + \Lambda I_4 + \Lambda I_5.$$

Hence,

$$\begin{aligned}
H &= \Lambda\left(288\pi \int_0^1 K^2\Lambda^2 (j_2(\Lambda r))^2 \, dr\right) \\
&\quad + \Lambda\Big(8\pi \int_0^1 (K^2\Lambda^4 r^2 (j_1(\Lambda r))^2 - 4K^2\Lambda^3 r j_1(\Lambda r)\, j_2(\Lambda r) \\
&\qquad + 4K^2\Lambda^2 (j_2(\Lambda r))^2)\, dr\Big) \\
&\quad + \Lambda\Big(40\pi \int_0^1 (K^2\Lambda^4 r^2 (j_1(\Lambda r))^2 - 4K^2\Lambda^3 r j_1(\Lambda r)\, j_2(\Lambda r) \\
&\qquad + 4K^2\Lambda^2 (j_2(\Lambda r))^2)\, dr\Big) \\
&\quad + \Lambda\left(40\pi \int_0^1 K^2\Lambda^4 r^2 (j_2(\Lambda r))^2 \, dr\right) + \Lambda\left(8\pi \int_0^1 K^2\Lambda^4 r^2 (j_2(\Lambda r))^2 \, dr\right) \\
&= \Lambda\left(288\pi \int_0^1 K^2\Lambda^4 (j_2(\Lambda r))^2 \, dr\right) \\
&\quad + \Lambda\Big(48\pi \int_0^1 (K^2\Lambda^4 r^2 (j_1(\Lambda r))^2 - 4K^2\Lambda^3 r j_1(\Lambda r)\, j_2(\Lambda r) \\
&\qquad + 4K^2\Lambda^2 (j_2(\Lambda r))^2)\, dr\Big) + \Lambda\left(48\pi \int_0^1 K^2\Lambda^4 r^2 (j_2(\Lambda r))^2 \, dr\right)
\end{aligned}$$

$$= 288\pi\int_0^1 K^2\Lambda^3(j_2(\Lambda r))^2 dr + 48\pi\int_0^1 K^2\Lambda^5 r^2(j_1(\Lambda r))^2 dr$$

$$-192\pi\int_0^1 K^2\Lambda^4 rj_1(\Lambda r)\, j_2(\Lambda r)dr$$

$$+192\pi\int_0^1 K^2\Lambda^3(j_2(\Lambda r))^2 dr + 48\pi\int_0^1 K^2\Lambda^5 r^2(j_2(\Lambda r))^2 dr$$

$$= 480\pi\int_0^1 K^2\Lambda^3(j_2(\Lambda r))^2 dr + 48\pi\int_0^1 K^2\Lambda^5 r^2(j_1(\Lambda r))^2 dr$$

$$+ 48\pi\int_0^1 K^2\Lambda^5 r^2(j_2(\Lambda r))^2 dr - 192\pi\int_0^1 K^2\Lambda^4 rj_1(\Lambda r)\, j_2(\Lambda r)dr. \quad (2.22)$$

Write the integrals of (2.22) as

$$H = \int_0^1 (480\pi K^2\Lambda^3(j_2(\Lambda r))^2 + 48\pi K^2\Lambda^5 r^2(j_1(\Lambda r))^2$$

$$+ 48\pi K^2\Lambda^5 r^2(j_2(\Lambda r))^2 - 192\pi K^2\Lambda^4 rj_1(\Lambda r)\, j_2(\Lambda r))dr. \quad (2.23)$$

The integral of (2.23) is a globally *helicity*, which is calculated in a volume of a spherical space. The *helicity* could also calculated as a function of radius of a ball with r between 0 and 1.

This function is the integrand of integral (2.23), which is formulated as

$$H(r) = 480\pi K^2\Lambda^3(j_2(\Lambda r))^2 + 48\pi K^2\Lambda^5 r^2(j_1(\Lambda r))^2$$

$$+ 48\pi K^2\Lambda^5 r^2(j_2(\Lambda r))^2 - 192\pi K^2\Lambda^4 rj_1(\Lambda r)\, j_2(\Lambda r), \quad 0 < r \le 1. \quad (2.24)$$

We solve numerically the integral by software of Wolfram Mathematica 7, version 7.0.1. Since $K = \sqrt{6/5}$, $K^2 = 1.2$, so the numerical calculation of (2.23) is written in code

NIntegrate[480 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^3

∗ SphericalBesselJ[2, Λ ∗ r]^2+48 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^5 ∗ r^2

∗ SphericalBesselJ[1,Λ ∗ r]^2+48 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^5 ∗ r^2

∗ SphericalBesselJ [2,Λ ∗ r]^2–192 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^4 ∗ r

∗ SphericalBesselJ[1,Λ ∗ r]SphericalBesselJ[2, Λ ∗ r], {r, 0, 1}]

and the graph of (2.24) is written with code

Plot[480 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^3

∗ SphericalBesselJ[2,Λ ∗ *r*]^2+48 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^5 ∗ *r*^2

∗ SphericalBesselJ[1,Λ ∗ *r*]^2+48 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^5 ∗ *r*^2

∗ SphericalBesselJ[2,Λ ∗ *r*]^2–192 ∗ *N*[Pi] ∗ 1.2 ∗ Λ^4 ∗ *r*

∗ SphericalBesselJ[1,Λ ∗ *r*]SphericalBesselJ[2,Λ ∗ *r*],

{*r*, 0, 1}, Frame → True, GridLines →∗ Automatic]

Since the model of PAS flow, use $\Lambda = 12.3229$ [5], the outcomes of integral (2.23) would be generated by the following codes:

NIntegrate[480 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^3

∗ SphericalBesselJ[2,12.3229 ∗ *r*]^2+48 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^5 ∗ *r*^2

∗ SphericalBesselJ[1,12.3229 ∗ *r*]^ 2 + 48 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^5 ∗ *r*^2

∗ SphericalBesselJ[2,12.3229 ∗ *r*]^2–192 ∗ *N*[Pi] ∗ 1.2 ∗12.3229^4 ∗ *r*

∗ SphericalBesselJ[1,12.3229 ∗ *r*]SphericalBesselJ[2,12.3229 ∗ *r*],

{*r*, 0, 1}]

and the outcomes is $H = 331932$.

The graph (2.24) is generated by the following code

Plot[480 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^3

∗ SphericalBesselJ[2,12.3229 ∗ *r*]^2 + 48 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^5 ∗ *r*^2

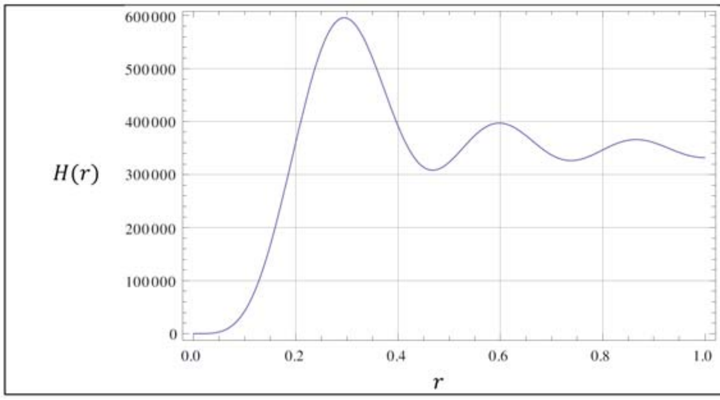
∗ SphericalBesselJ[1,12.3229 ∗ *r*]^2 + 48 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^5 ∗ *r*^2

∗ SphericalBesselJ[2,12.3229 ∗ *r*]^2–192 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^4 ∗ *r*

∗ SphericalBesselJ[1,12.3229 ∗ *r*]SphericalBesselJ[2,12.3229 ∗ *r*],

{*r*, 0, 1}, Frame → True, GridLines → Automatic]

and the result is the following graph:



r	$H(r)$
0.10	42443
0.20	360531
0.30	594923
0.40	391466
0.60	397039
0.80	346452
1.00	331932

Graph of Helicity of PAS Flow with $\Lambda = 12.3229$.

The table in the right side of the graph shows the value of $H(r)$ in a certain value of r, which is calculated as

N[480 ∗ *N*[Pi] ∗ 1.2 ∗ l2.3229^3

∗ SphericalBesselJ[2,12.3229 ∗ *r*]^2+48 ∗ *N*[Pi] ∗ 1.2 * 12.3229^5 ∗ *r*^2

∗ SphericalBesselJ[1,12.3229 ∗ *r*]^2+48 ∗ *N*[Pi] ∗1.2 ∗ 12.3229^5 ∗ *r*^2

∗ SphericalBesselJ[2,12.3229 ∗ *r*]^2 – 192 ∗ *N*[Pi] ∗ 1.2 ∗ 12.3229^4 * r

∗ SphericalBesselJ[1,12.3229 ∗ *r*]SphericalBesselJ[2,12.3229 ∗ *r*]]

As examples, we substitute r with the values of 0.10, 0.20, 0.40, 0.60, 0.80, and 1.00. Then the values of $H(r)$ are $H(0.10) = 42443$, $H(0.20) = 360531$, $H(0.40) = 391446$, $H(0, 60) = 397039$, $H(0.80) = 346452$, and $H(1.00) = 331932$.

3. Conclusion

The *helicity* of any flow is determined by integral that defined by Moffatt,

$$H = \int_V \boldsymbol{v} \cdot \nabla \times \boldsymbol{v} dV.$$

Since PAS flow is Beltrami flow, which $\boldsymbol{v}$ vector is parallel to its vorticity, $C\boldsymbol{v} = \nabla \times \boldsymbol{v},$ we can substitute the above integral to the following integral:

$$H = \int_V \boldsymbol{v} \cdot C\boldsymbol{v} dV$$

where C is a constant, so that the integral becomes

$$H = C\int_V \boldsymbol{v} \cdot \boldsymbol{v} dV.$$

If we determine the PAS flow directly by the definition of integral in (1.5), we have to expand vector $\boldsymbol{v}$ and $\nabla \times \boldsymbol{v},$ and then expand the scalar product of them. However, if we use the Beltrami property to determine the *helicity* of PAS flow, we need only to expand the product scalar between vector $\boldsymbol{v}$ and the same vector. Therefore, Beltrami property make simpler in determining the *helicity* of PAS flow. With Beltrami property, we can derive that the value of C which is equal to Λ, the positive roots of second order of Bessel function.

The result of global *helicity* calculated in a space of spherical volume and calculated as a function of radius of a ball are equal when $r = 1,$ that is $H = H(1) = 331932.$

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions which let to the improvement of the manuscript.

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