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COHERENT FORM OF THE NON-CENTRALITY PARAMETER IN NON-CENTRAL χ^2 DISTRIBUTION

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Abstract

Apart from the theoretical contributions, non-central chi-squared distribution has been applied in mathematical physics, social and behavioral research, communication theory, management science and many other diversified research areas. Non-central chi-squared distribution has been derived mostly by considering the non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \lambda$ and sometimes by taking the non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$. Here we also consider an alternative non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \sqrt{\lambda}$. We calculated the parametric characteristics for each of the three densities using cumulant generating function and recurrence relation. Finally, we made a head-to-head comparison among the characteristics obtained from these three unique densities. It is found that only second non-centrality

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parameter gives us exact correspondence in parametric characteristics calculated using both the methods of calculation.

1. Introduction

Generally non-central chi-squared distribution is termed as generalized Rayleigh distribution (Park [13]) and also commonly known as Rayleigh distribution or Rayleigh-Rice distribution or simply Rice distribution (Miller et al. [12]). This name of the distribution is generally used in mathematical physics and communication theory. Different authors derived the distribution in different ways. Fisher [3] observed the distribution as a limiting case of the distribution of multiple correlation coefficients and at first gave an indirect derivation using a limiting process. He prepared a table for 5% upper point for different values of n and λ. Tang [21] gave the physical proof of the distribution which is considered as first direct derivation of the distribution. This distribution is used by Patnaik [14] to determine the power of chi-squared test procedure. Patnaik [14], Ruben [17] and Guenther [5] gave the geometrical proof of the distribution. Johnson and Leone [7] and Kerridge [8] also derived the distribution. Graybill [4] used moment generation function (*mgf*) to give the proof of the distribution and the contour integration inverted characteristic function has been used by McNolty [11].

2. Motivation

Non-central distributions have been playing a key role not only in theoretical development of statistics but also in practical implications in diversified fields. Non-central chi-squared distribution is the eldest member of non-central family of densities. Theoretically, non-central chi-squared distribution is of great importance because other non-central distributions such as non-central F (Scheffé [19], Patnaik [14] and Chattamvelli [1]), non-central t (Rao [16, p. 139] and Scheffé [19, pp. 135, 415]) and non-central beta (Johnson et al. [6], Posten [15] and Lenth [9]) have been derived using non-central chi-squared density function. Apart from the theoretical importance, this distribution has significant contribution in the social and behavioral researches which deal with model evaluation and power analysis of testing in structural equation modeling (Chun and Shapiro [2]) and also in communication theory (Marcum [10] and Rice [18]).

Conventionally, non-central chi-squared distributions have been derived by using any one of the two non-centrality parameters $\boldsymbol{\mu}'\boldsymbol{\mu} = \lambda$ (Fisher [3] and Patnaik [14]) or $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$ (Searle [20, p. 49]).

Contemplating all the above, the key question in researchers' mind that can we derive the non-central chi-squared distribution using any one of the non-centrality parameters defined above or we must be strict on defining non-centrality parameter. In addition to the earlier non-centrality parameters, we are considering another non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \sqrt{\lambda}$ to facilitate comparison. In the final stage, we wish to suggest which non-centrality parameter we have to consider for deriving a valid non-central chi-squared density function.

3. Theoretical Development

In the subsequent sections of this paper, different densities of non-central chi-squared distribution with different non-centrality parameters have been considered. Calculations have been performed for different parameters of the above mentioned distributions considering each of the densities to represent a unique form of distribution.

3.1. Distribution-I

Let the random vector **x** be $N(\boldsymbol{\mu}, I)$ of order $n \times 1$ with $\boldsymbol{\mu}$ as a vector of distinct means and I as identity variance-covariance matrix of order $n \times n$. Then $\mathbf{x}'\mathbf{x}$ is called *non-central chi-squared variate* with *n* degree of freedom (*df*) and non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \lambda$. Conventionally, the non-central chi-squared variate is denoted by χ'^2 or $\chi'^2_{n,\lambda}$. Adopting the moment generating function (*mgf*) method, the probability density function (*pdf*) of non-central chi-squared variate χ'^2 with *n df* and the above non-centrality parameter is given as (Patnaik [14]):

$$f(\chi'^2) = \sum_{r=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^r e^{-\frac{\chi'^2}{2}} (\chi'^2)^{\frac{n}{2}+r-1}}{r! 2^{\frac{n}{2}+r} \Gamma\left(\frac{n}{2}+r\right)}, \quad 0 \le \chi'^2 < \infty. \tag{1}$$

3.1.1. Cumulant generating function and cumulants from Distribution-I

The *mgf* of the above distribution is obtained as:

$$M_{\chi'^2}(t) = E(e^{t\chi'^2}) = (1-2t)^{\frac{-n}{2}} e^{\frac{\lambda t}{1-2t}}, \quad |t| < \frac{1}{2}.$$

By definition, the cumulant generating function (*cgf*) is obtained as:

$$K_{\chi'^2}(t) = \log(M_{\chi'^2}(t))$$

$$= -\frac{n}{2}\log(1-2t) + \lambda t(1-2t)^{-1}$$

$$= \frac{n}{2}\left[2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \cdots\right] + \lambda t[1 + (2t)^2 + (2t)^3 + \cdots]$$

$$\left(\text{since } \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right.$$

$$\left.\text{and } (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\right)$$

$$= \left(n\frac{t}{1!} + 2n\frac{t^2}{2!} + 8n\frac{t^3}{3!} + 48n\frac{t^4}{4!} + \cdots\right)$$

$$+ \left(\lambda\frac{t}{1!} + 4n\frac{t^2}{2!} + 24\lambda\frac{t^3}{3!} + 192\lambda\frac{t^4}{4!} + \cdots\right)$$

$$\Rightarrow K_{\chi'^2}(t) = (n+\lambda)\frac{t}{1!} + 2(n+2\lambda)\frac{t^2}{2!} + 8(n+3\lambda)\frac{t^3}{3!}$$

$$+\, 48(n + 4\lambda)\frac{t^4}{4!} + \cdots + 2^{r-1}(r-1)!(n + r\lambda)\frac{t^r}{r!} + \cdots$$

which is the required *cgf* of the above mentioned distribution.

We know

$$K_r = r\text{th cumulant} = \text{coefficient of } \frac{t^r}{r!} \text{ in } K_{\chi'^2}(t)$$

$$\Rightarrow K_r = 2^{r-1}(r-1)!(n + r\lambda).$$

Hence,

$$k_1 = \text{First cumulant} = (n + \lambda) = \mu_1' = \text{Mean},$$

$$k_2 = \text{Second cumulant} = 2(n + 2\lambda) = \mu_2 = \text{Variance},$$

$$k_3 = \text{Third cumulant} = 8(n + 3\lambda) = \mu_3,$$

$$k_4 = \text{Fourth cumulant} = 48(n + 4\lambda).$$

We also know $\mu_4 = k_4 + 3k_2^2 = 48(n + 4\lambda) + 12(n + 2\lambda)^2.$

Therefore, the coefficient of skewness (β_1) and coefficient of kurtosis (β_2) are obtained as:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[8(n + 3\lambda)]^2}{[2(n + 2\lambda)]^3} = \frac{8(n + 3\lambda)^2}{(n + 2\lambda)^3}$$

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{48(n + 4\lambda) + 12(n + 2\lambda)^2}{[2(n + 2\lambda)]^2} = 3 + \frac{12(n + 4\lambda)}{(n + 2\lambda)^2}.$$

3.1.2. Recurrence relation for the cumulants of Distribution-I

We already obtained the *r*th cumulant of the defined distribution which is:

$$K_r = 2^{r-1}(r-1)!(n + r\lambda). \tag{2}$$

Now putting $r = r - 1$, we get

$$K_{r-1} = 2^{r-2}(r-2)![n + (r-1)\lambda]. \tag{3}$$

Differentiating (3) w.r.t. λ, we get

$$\frac{\partial}{\partial\lambda}(K_{r-1}) = \frac{\partial}{\partial\lambda}[2^{r-2}(r-2)!(n + (r-1)\lambda)] = 2^{r-2}(r-1)!$$

$$\Rightarrow \frac{\partial}{\partial\lambda}(K_{r-1}) = \frac{K_r}{2(n + r\lambda)} \text{ (using (2))}$$

$$\Rightarrow K_r = 2(n + r\lambda)\frac{\partial}{\partial\lambda}(K_{r-1}) \tag{4}$$

which is the recurrence relation for the cumulants of the distribution.

Plugging in $r = 1, 2, 3$ and 4 in expression (4), we get

$$k_1 = 2(n + \lambda)\frac{\partial}{\partial\lambda}(K_0) = 2(n + \lambda)\left(\text{since } \frac{\partial}{\partial\lambda}(K_0) = 1\right),$$

$$k_2 = 2(n + 2\lambda)\frac{\partial}{\partial\lambda}(K_1) = 2(n + 2\lambda)\frac{\partial}{\partial\lambda}[2(n + \lambda)] = 4(n + 2\lambda),$$

$$k_3 = 2(n + 3\lambda)\frac{\partial}{\partial\lambda}(K_2) = 2(n + 3\lambda)\frac{\partial}{\partial\lambda}[4(n + 2\lambda)] = 16(n + 3\lambda),$$

$$k_4 = 2(n + 4\lambda)\frac{\partial}{\partial\lambda}(K_3) = 2(n + 4\lambda)\frac{\partial}{\partial\lambda}[16(n + 3\lambda)] = 96(n + 4\lambda).$$

For non-central chi-squared distribution with *n df* and non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \lambda$, the cumulants obtained from *cgf* reflect the dissimilarity with the cumulants obtained from the recurrence relation for cumulants of the same distribution. Hence, mean, variance, coefficient of skewness and coefficient of kurtosis reflect the anomalous results obtained by the two methods.

3.2. Distribution-II

Let the random vector **x** be $N(\boldsymbol{\mu}, I)$ of order $n \times 1$ with $\boldsymbol{\mu}$ as a vector of distinct means and I as identity variance-covariance matrix of order $n \times n$.

Then $\mathbf{x}'\mathbf{x}$ is called *non-central chi-squared variate* with *n df* and non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$. Conventionally, the non-central chi-squared variate is denoted by χ'^2 or $\chi'^2_{n,\lambda}$. Adopting the *mgf* method, the *pdf* of non-central chi-squared variate χ'^2 with *n df* and the above non-centrality parameter is given as (Searle [20]):

$$f(\chi'^2) = \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r e^{-\frac{\chi'^2}{2}} (\chi'^2)^{\frac{n}{2}+r-1}}{r!\,2^{\frac{n}{2}+r}\,\Gamma\left(\frac{n}{2}+r\right)}, \quad 0 \le \chi'^2 \le \infty. \tag{5}$$

3.2.1. Cumulant generating function and cumulants from Distribution-II

The *mgf* of the above distribution is obtained as: (On simplification)

$$M_{\chi'^2}(t) = E(e^{t\chi'^2}) = (1-2t)^{\frac{-n}{2}} e^{\frac{2\lambda t}{1-2t}}, \quad |\,t\,| < \frac{1}{2}.$$

By definition, the *cgf* is obtained as:

$$K_{\chi'^2}(t) = \log(M_{\chi'^2}(t)) = -\frac{n}{2}\log(1-2t) + 2\lambda t(1-2t)^{-1}$$

$$= \frac{n}{2}\left[2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \cdots\right]$$

$$+ 2\lambda t[1 + (2t) + (2t)^2 + (2t)^3 + \cdots]$$

$$= \left(n\frac{t}{1!} + 2n\frac{t^2}{2!} + 8n\frac{t^3}{3!} + 48n\frac{t^4}{4!} + \cdots\right)$$

$$+ \left(2\lambda\frac{t}{1!} + 8\lambda\frac{t^2}{2!} + 48\lambda\frac{t^3}{3!} + 384\lambda\frac{t^4}{4!} + \cdots\right)$$

$$\Rightarrow K_{\chi'^2}(t) = (n+2\lambda)\frac{t}{1!} + 2(n+4\lambda)\frac{t^2}{2!} + 8(n+6\lambda)\frac{t^3}{3!}$$

$$+ 48(n + 8\lambda)\frac{t^4}{4!} + \cdots + 2^{r-1}(r-1)!(n + 2r\lambda)\frac{t^r}{r!} + \cdots$$

which is the *cgf* of the above density.

We have

$$\Rightarrow K_r = 2^{r-1}(r-1)!(n + 2r\lambda).$$

Hence,

$$k_1 = (n + 2\lambda), \quad k_2 = 2(n + 4\lambda), \quad k_3 = 8(n + 6\lambda), \quad k_4 = 48(n + 8\lambda).$$

Therefore, $\mu_4 = 48(n + 8\lambda) + 12(n + 4\lambda)^2$.

Hence,

$$\beta_1 = \frac{[8(n + 6\lambda)]^2}{[2(n + 4\lambda)]^3} = \frac{8(n + 6\lambda)^2}{(n + 4\lambda)^3}$$

and

$$\beta_2 = \frac{48(n + 8\lambda) + 12(n + 4\lambda)^2}{[2(n + 4\lambda)]^2} = 3 + \frac{12(n + 8\lambda)}{(n + 4\lambda)^2}.$$

3.2.2. Recurrence relation for the cumulants of Distribution-II

The *r*th cumulant of this distribution is:

$$K_r = 2^{r-1}(r-1)!(n + 2r\lambda). \tag{6}$$

Now putting $r = r - 1$, we get

$$K_{r-1} = 2^{r-2}(r-2)![n + 2(r-1)\lambda]. \tag{7}$$

Differentiating (7) w.r.t. λ, we get

$$\frac{\partial}{\partial\lambda}(K_{r-1}) = \frac{\partial}{\partial\lambda}[2^{r-2}(r-2)!(n + 2(r-1)\lambda)] = 2^{r-1}(r-1)!$$

$$\Rightarrow \frac{\partial}{\partial\lambda}(K_{r-1}) = \frac{K_r}{(n + 2r\lambda)} \text{ (using (6))}$$

$$\Rightarrow K_r = (n + 2r\lambda)\frac{\partial}{\partial\lambda}(K_{r-1}) \tag{8}$$

which is the recurrence relation for the cumulants of the above mentioned distribution.

Plugging in $r = 1, 2, 3$ and 4 in equation (8), we get

$$k_1 = (n + 2\lambda)\frac{\partial}{\partial\lambda}(K_0) = (n + 2\lambda),$$

$$k_2 = (n + 4\lambda)\frac{\partial}{\partial\lambda}(K_1) = (n + 4\lambda)\frac{\partial}{\partial\lambda}[(n + 2\lambda)] = 2(n + 4\lambda),$$

$$k_3 = (n + 6\lambda)\frac{\partial}{\partial\lambda}(K_2) = (n + 6\lambda)\frac{\partial}{\partial\lambda}[2(n + 4\lambda)] = 8(n + 6\lambda),$$

$$k_4 = (n + 8\lambda)\frac{\partial}{\partial\lambda}(K_3) = (n + 8\lambda)\frac{\partial}{\partial\lambda}[8(n + 6\lambda)] = 48(n + 8\lambda).$$

Here, it is obvious that, for non-central chi-squared distribution with *n df* and non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda,$ the cumulants obtained from *cgf* are exactly similar with those obtained form the recurrence relation for cumulants for Distribution-II. In addition, since cumulants are similar in both the methods, the coefficient of skewness and coefficient of kurtosis will also be similar for this distribution.

3.3. Distribution-III

Let the random vector **x** be $N(\boldsymbol{\mu}, I)$ of order $n \times 1$ with $\boldsymbol{\mu}$ as a vector of distinct means and I as identity variance-covariance matrix of order $n \times n.$ Then $\mathbf{x}'\mathbf{x}$ is called *non-central chi-squared variate* with *n df* and non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \sqrt{\lambda}.$ Conventionally, the non-central chi-squared variate is denoted by χ'^2 or $\chi'^2_{n,\lambda}.$ Adopting the *mgf* method, the *pdf* of non-central chi-squared variate χ'^2 with *n df* and the above non-centrality parameter is given as:

$$f(\chi'^2) = \sum_{r=0}^{\infty} \frac{e^{-\frac{\sqrt{\lambda}}{2}} \left(\frac{\sqrt{\lambda}}{2}\right)^r e^{-\frac{\chi'^2}{2}} (\chi'^2)^{\frac{n}{2}+r-1}}{r! 2^{\frac{n}{2}+r} \Gamma\left(\frac{n}{2}+r\right)}, \quad 0 \le \chi'^2 \le \infty. \tag{9}$$

3.3.1. Cumulant generating function and cumulants from Distribution-III

The *mgf* of the above distribution is obtained as:

$$M_{\chi'^2}(t) = E(e^{t\chi'^2}) = (1-2t)^{\frac{-n}{2}} e^{\frac{\sqrt{\lambda}t}{1-2t}}, \quad |t| < \frac{1}{2}.$$

By definition, the *cgf* is obtained as:

$$\begin{aligned}
K_{\chi'^2}(t) = \log(M_{\chi'^2}(t)) &= -\frac{n}{2}\log(1-2t) + \sqrt{\lambda}t(1-2t)^{-1} \\
&= \frac{n}{2}\left[2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \cdots\right] \\
&\quad + \sqrt{\lambda}t[1 + (2t) + (2t)^2 + (2t)^3 + \cdots] \\
&= \left(n\frac{t}{1!} + 2n\frac{t^2}{2!} + 8n\frac{t^3}{3!} + 48n\frac{t^4}{4!} + \cdots\right) \\
&\quad + \left(\sqrt{\lambda}\frac{t}{1!} + 4\sqrt{\lambda}\frac{t^2}{2!} + 24\sqrt{\lambda}\frac{t^3}{3!} + 192\sqrt{\lambda}\frac{t^4}{4!} + \cdots\right)
\end{aligned}$$

$$\Rightarrow K_{\chi'^2}(t) = (n+\sqrt{\lambda})\frac{t}{1!} + 2(n+2\sqrt{\lambda})\frac{t^2}{2!} + 8(n+3\sqrt{\lambda})\frac{t^3}{3!}$$

$$+ 48(n+4\sqrt{\lambda})\frac{t^4}{4!} + \cdots + 2^{r-1}(r-1)!(n+r\sqrt{\lambda})\frac{t^r}{r!} + \cdots$$

which is the *cgf* of the above distribution.

Thus,

$$\Rightarrow K_r = 2^{r-1}(r-1)!(n+r\sqrt{\lambda}).$$

Hence,

$$k_1 = (n+\sqrt{\lambda}), \quad k_2 = 2(n+2\sqrt{\lambda}), \quad k_3 = 8(n+3\sqrt{\lambda}), \quad k_4 = 48(n+4\sqrt{\lambda}).$$

Thus, $\mu_4 = 48(n+4\sqrt{\lambda}) + 12(n+2\sqrt{\lambda})^2.$

Therefore,

$$\beta_1 = \frac{[8(n+3\sqrt{\lambda})]^2}{[2(n+2\sqrt{\lambda})]^3} = \frac{8(n+3\sqrt{\lambda})^2}{(n+2\sqrt{\lambda})^3}$$

and

$$\beta_2 = \frac{48(n+4\sqrt{\lambda}) + 12(n+2\sqrt{\lambda})^2}{[2(n+2\sqrt{\lambda})]^2} = 3 + \frac{12(n+4\sqrt{\lambda})}{(n+2\sqrt{\lambda})^2}.$$

3.3.2. Recurrence relation for the cumulants of Distribution-III

We already obtained the *r*th cumulant of this density function as:

$$K_r = 2^{r-1}(r-1)!(n+r\sqrt{\lambda}). \tag{10}$$

Now putting $r = r-1$, we get

$$K_{r-1} = 2^{r-2}(r-2)![n+(r-1)\sqrt{\lambda}]. \tag{11}$$

Differentiating (11) w.r.t. λ, we get

$$\frac{\partial}{\partial\lambda}(K_{r-1}) = \frac{\partial}{\partial\lambda}[2^{r-2}(r-2)!(n+(r-1)\sqrt{\lambda})] = \frac{2^{r-3}(r-1)!}{\sqrt{\lambda}}$$

$$\Rightarrow \frac{\partial}{\partial\lambda}(K_{r-1}) = \frac{K_r}{4\sqrt{\lambda}(n+r\sqrt{\lambda})} \text{ (using (10))}$$

$$\Rightarrow K_r = 4(n\sqrt{\lambda}+r\lambda)\frac{\partial}{\partial\lambda}(K_{r-1}) \tag{12}$$

which is the recurrence relation for the cumulants of the above distribution.

Plugging in $r = 1, 2, 3$ and 4 in expression (12), we get

$$k_1 = 4(n\sqrt{\lambda} + \lambda)\frac{\partial}{\partial\lambda}(K_0) = 4(n\sqrt{\lambda} + \lambda),$$

$$k_2 = 4(n\sqrt{\lambda} + 2\lambda)\frac{\partial}{\partial\lambda}(K_1) = 4(n\sqrt{\lambda} + 2\lambda)\frac{\partial}{\partial\lambda}[4(n\sqrt{\lambda} + \lambda)] = 8(n + 2\sqrt{\lambda})^2,$$

$$k_3 = 4(n\sqrt{\lambda} + 3\lambda)\frac{\partial}{\partial\lambda}(K_2) = 4(n\sqrt{\lambda} + 3\lambda)\frac{\partial}{\partial\lambda}[8(n + 2\sqrt{\lambda})^2] = 64(n + 3\sqrt{\lambda}),$$

$$k_4 = 4(n\sqrt{\lambda} + 4\lambda)\frac{\partial}{\partial\lambda}(K_3) = (n + 4\lambda)\frac{\partial}{\partial\lambda}[64(n + 3\sqrt{\lambda})] = 384(n + 4\sqrt{\lambda}).$$

From the above results of cumulants, it is clear that there is also a dissimilarity among the cumulants obtained from *cgf* and those from recurrence relation for Distribution-III with non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \sqrt{\lambda}$. Hence, the measurements of shape characteristics will also be dissimilar accordingly.

3.4. Discussion

Contemplating the above results, it is clear that for non-central chi-squared distribution with n *df* and non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \lambda$, the cumulants obtained from the cumulant generating function showing dissimilarity with those obtained from cumulant generating recurrence relation. Nonetheless, the coefficient of skewness and the coefficient of kurtosis are also different in both the methods of calculation. In addition, similar scenarios have been observed for the distribution with non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = \sqrt{\lambda}$. Surprisingly, disregarding the methods of calculation of the parametric characteristics, the coherence in the results have been observed only for the distribution with non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$.

Moreover, it is the established fact that if a certain distribution is correctly derived, then all of its parametric characteristics obtained by different generating method should be identical. If there is any discrepancy in the parametric forms obtained using different methods, then it is presumed that the distribution is not derived correctly. Here in all three forms of non-central chi-squared distribution, the degree of freedoms is same but only the non-centrality parameters are different. Thus, the variation in results by two methods for the same distribution might be due to non-centrality parameters only.

Till today, most of the authors derived this distribution in their works considering $\boldsymbol{\mu}'\boldsymbol{\mu} = \lambda$ as non-centrality parameter which misleads the derived distribution. However, we appreciate and accept the inventory of researchers who derived the distribution using $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$ but we may void the derivation of the non-central chi-squared distribution using the non-centrality parameters $\boldsymbol{\mu}'\boldsymbol{\mu} = \lambda$ and $\boldsymbol{\mu}'\boldsymbol{\mu} = \sqrt{\lambda}$ as well.

Non-central chi-squared distribution is commonly known as the mother distribution among the non-central family of densities because non-central *t*, non-central *F* and non-central *beta* distributions have been derived from this distribution. If non-central chi-squared distribution has not been derived with correct non-centrality parameter, then it will mislead the other parametric characteristics of the distribution. Accordingly, disregarding the method of derivation and calculation of all other non-central distributions will be misled and hence their parametric characteristics will not be correct too. More concisely, if non-central chi-squared distribution has not been derived with the non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$, then one cannot derive other non-central distributions correctly.

Finally, the use of quantile tables for all non-central distributions will be of use for the unique non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$. The existing quantile tables have used variants of non-centrality parameter and hence a caution is needed while reading the tables. This unique representation would eliminate the difficulties in reading the tables.

Lastly, we are suggesting that the non-central chi-squared distribution must be derived with the one and only one non-centrality parameter $\boldsymbol{\mu}'\boldsymbol{\mu} = 2\lambda$ for best sake of theoretical and applied statistical research.

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