



SHARP INEQUALITIES OF DIFFERENTIAL BOUNDARY VALUES AND SHARP INEQUALITIES OF DIFFERENCE BOUNDARY VALUES

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Abstract

In this paper, we establish sharp inequalities of differential boundary values and sharp inequalities of difference boundary values. Furthermore, an example for the existence of solutions of differential boundary values problems is given. We also hope that these inequalities can play a better role in the study of differential boundary value problems and difference boundary value problems.

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1. Introduction

Let C_T (where $T > 0$) be the set of all real continuous T -periodic functions of the form $x : \mathbb{R} \rightarrow \mathbb{R}$ and let l_ω (where ω is a positive integer) be the set of all real ω -periodic sequences of the form $u = \{u_k\}_{k \in \mathbb{Z}}$.

As is known to all, for any $x(t) \in C^{(1)}(\mathbb{R}, \mathbb{R}) \cap C_T$ and $\xi \in [0, T]$ by fundamental theorem of calculus,

$$\max_{0 \leq t \leq T} |x(t)| \leq |x(\xi)| + \int_0^T |x'(s)| ds. \quad (1)$$

Such inequality has been used, among many things, for finding a priori bounds for T -periodic solutions of differential equations. By means of such a priori bounds, we may then look for T -periodic solutions by means of fixed point theorems such as the continuation theorems (see, e.g., [1]) which are popular (see for examples [1-18]). However, since (1) was applied to find the a priori bounds in these references, and since it is not sharp inequality (as will be seen below), the corresponding existence criteria cannot be a sharp one.

In [19], Li and Wang noted that (1) can be extended to a sharp inequality of the following form:

$$\max_{0 \leq t \leq T} |x(t)| \leq |x(\xi)| + \frac{1}{2} \int_0^T |x'(s)| ds, \quad (2)$$

where the constant factor $1/2$ is the best possible. Then, by using (2), they improved the existence criteria of periodic solution of differential equations, see the most of the literature of this paper.

Similarly, in order to find the priori bounds of periodic solutions of difference equations. Wang and Cheng in the literature [20] gave a sharp inequality for periodic sequences of the following form. Let $u = \{u_k\}_{k \in \mathbb{Z}} \in l_\omega$. Then

$$\max_{0 \leq k \leq \omega-1} |u_k| \leq \min_{0 \leq t \leq \omega-1} |u_k| + \frac{1}{2} \sum_{k=0}^{\omega-1} |\Delta u_k|, \quad (3)$$

where the constant factor $1/2$ is the best possible. Then, by using (3), they improved the existence of periodic solution of difference equations in [21].

In this paper, we will establish sharp inequalities of differential boundary values and sharp inequalities of difference boundary values. It is worth noting that we created inequalities are extended (2) and (3). Furthermore, an example for deriving existence of solutions of differential boundary values problems is given. We also hope that these inequalities can play a better role in the study of differential boundary value problems and difference boundary value problems.

2. Sharp Inequalities

First, we give the Sharpe inequality on differential boundary values in the following.

Theorem 1. *Let $f(t) \in C^{(1)}([a, b], R)$. Then for any $t \in [a, b]$, we have*

$$|f(t)| \leq \frac{1}{2} \left(|f(a) + f(b)| + \int_a^b |f'(s)| ds \right), \quad (4)$$

where the constant factor $1/2$ is the best possible.

Proof. For any $t \in [a, b]$, by Newton-Leibniz integral formula, we have

$$f(t) = f(a) + \int_a^t f'(s) ds \quad (5)$$

and

$$f(t) = f(b) - \int_t^b f'(s) ds. \quad (6)$$

In view of (5) and (6), we see that for any $t \in [a, b]$

$$f(t) = \frac{1}{2} \left(f(a) + f(b) + \int_a^t f'(s) ds - \int_t^b f'(s) ds \right). \quad (7)$$

Thus for any $t \in [a, b]$, we have

$$|f(t)| \leq \frac{1}{2} \left(|f(a) + f(b)| + \int_a^b |f'(s)| ds \right). \quad (8)$$

Now we assert that if α is a constant and $\alpha < 1/2$, then there are $f(t) \in C^{(1)}([a, b], R)$ and $\xi \in [a, b]$ such that

$$|f(\xi)| > \alpha \left(|f(a) + f(b)| + \int_a^b |f'(s)| ds \right). \quad (9)$$

Indeed, let $f(t) = t + |a|$ and $\xi = b$. Then $|f(\xi)| = b + |a|$. Note that $f(a) = a + |a| \geq 0$ and $f(t)$ is strictly increasing on $[a, b]$, so that

$$\begin{aligned} & \alpha \left(|f(a) + f(b)| + \int_a^b |f'(s)| ds \right) \\ &= \alpha \left(f(a) + f(b) + \int_a^b |f'(s)| ds \right) \\ &= \alpha(a + b + 2|a| + b - a) \\ &= 2\alpha(b + |a|) < (b + |a|) = |f(\xi)| \end{aligned} \quad (10)$$

as required. This shows that the constant $1/2$ in (4) is the best possible. The proof is complete.

Remark. By Theorem 1, it is easy to obtain the formula (2). Indeed, for $x(t) \in C^{(1)}(R, R) \cap C_T$, then there is a $\xi \in R$, such that $|x(\xi)| = \min_{0 \leq t \leq T} |x(t)|$. Thus, if we let $f(t) = x(t)$, $a = \xi$ and $b = \xi + T$. Then by (4), we know that (2) is established.

Theorem 2. Let $u = (u_1, u_2, \dots, u_{n+1})^T \in R^{n+1}$. Then for any $k \in \{1, 2, \dots, n\}$, we have

$$|u_k| \leq \frac{1}{2} \left\{ |u_1 + u_{n+1}| + \sum_{k=1}^n |\Delta u_k| \right\}, \quad (11)$$

where the constant factor $1/2$ is the best possible.

Proof. For any $\xi \in \{1, 2, \dots, n\}$, we have

$$u_\xi = u_{n+1} - \sum_{k=\xi}^n \Delta u_k \quad (12)$$

and

$$u_\xi = u_1 + \sum_{k=1}^{\xi-1} \Delta u_k, \quad (13)$$

where we define $\sum_{k=1}^{\xi-1} \Delta u_k = 0$, if $\xi = 1$. In view of (12) and (13), we see that for any $\xi \in \{1, 2, \dots, n\}$,

$$u_\xi = \frac{1}{2} \left\{ u_1 + u_{n+1} + \sum_{k=1}^{\xi-1} \Delta u_k - \sum_{k=\xi}^n \Delta u_k \right\}. \quad (14)$$

Thus for any $\xi \in \{1, 2, \dots, n\}$, we have

$$|u_\xi| \leq \frac{1}{2} \left\{ |u_1 + u_{n+1}| + \sum_{k=1}^n |\Delta u_k| \right\}. \quad (15)$$

Now we assert that if β is a constant and $\beta < 1/2$, then there are $u = (u_1, u_2, \dots, u_{n+1})^T \in R^{n+1}$ and $\xi \in \{1, 2, \dots, n\}$ such that

$$|u_\xi| > \beta \left\{ |u_1 + u_{n+1}| + \sum_{k=1}^n |\Delta u_k| \right\}. \quad (16)$$

Indeed, let $u = (u_1, u_2, \dots, u_{n+1})^T \in R^{n+1}$ such that

$$u_k = \begin{cases} k, & \text{for } k = 1, 2, \dots, n, \\ 0, & \text{for } n+1, \end{cases}$$

and let $\xi = n$. Then

$$|u_\xi| = n > 2\beta n = \beta \left\{ |u_1 + u_{n+1}| + \sum_{k=1}^n |\Delta u_k| \right\} \quad (17)$$

as required. This shows that the constant $1/2$ in (11) is the best possible. The proof is complete.

Remark. By formula (11), it is easy to obtain the formula (3). Indeed, for $u = \{u_k\}_{k \in \mathbb{Z}} \in l_\omega$, then there is a $\xi \in \{0, 1, 2, \dots, \omega-1\}$, such that $|u_\xi| = \min_{0 \leq k \leq \omega-1} |u_k|$. Since u is periodic, by (11) we know that for any $k \in \{\xi, \xi+1, \dots, \xi+\omega-1\}$,

$$\begin{aligned} |u_k| &\leq \frac{1}{2} \left\{ |u_\xi + u_{\xi+\omega}| + \sum_{k=\xi}^{\xi+\omega-1} |\Delta u_k| \right\} \\ &= |u_\xi| + \frac{1}{2} \sum_{k=0}^{\omega-1} |\Delta u_k|. \end{aligned}$$

Thus, (3) is obtained.

3. Applications

In this section, we provide an example to illustrate the application of inequality (4). We also hope that the inequality (11) can play a good role in the study of difference boundary value problems.

As far as we know, in [22-26] the authors study the existence of solutions of two point boundary value problem of the form

$$x''(t) = f(t, x(t), x'(t)), \quad t \in [a, b], \quad (18)$$

$$x(a) = x(b) = 0, \quad (19)$$

where f be a real continuous function defined on R^3 .

Now we use the sharp inequalities we establishes in this paper to estimated the prior bounds of the solution of this problem, and then combined with the continuation theorem, to get a new sharp condition for the existence of solutions of this problem.

For the sake of convenience, for any $x \in C([a, b], R)$, we define

$$\|x\|_{\infty} = \max_{a \leq t \leq b} |x(t)| \text{ and } \|x\|_1 = \int_a^b |x(s)| ds. \quad (20)$$

Theorem 3. *Suppose there exist nonnegative real continuous functions p , q and r which are all defined on $[a, b]$, and there is a constant $D > 0$, such that*

$$(i) |f(t, u, v)| \leq p(t)|u| + q(t)|v| + r(t), \text{ for } t \in [a, b] \text{ and } (u, v) \in R^2,$$

$$(ii) uf(t, u, v) > 0, \text{ for } |u| \geq D, t, v \in R.$$

Then for $(b-a)\|p\|_1 + 2\|q\|_1 < 2$, the boundary value problem (18)-(19) has at least one solution.

Proof. We let

$$x''(t) = \lambda f(t, x(t), x'(t)), t \in [a, b], \quad (21)$$

$$x(a) = x(b) = 0, \quad (22)$$

where $\lambda \in (0, 1)$. By continuation theorems and from the literature [1] fortieth pages of remark, we know that, it suffices to prove that for any solution $x(t)$ of (21)-(22), there exist constants M_0 and M_1 , which are independent from $x(t)$ and λ , such that

$$\|x\|_{\infty} \leq M_0 \text{ and } \|x'\|_{\infty} \leq M_1. \quad (23)$$

First of all, note that $x(a) = x(b)$, we may show that there is a $t^* \in (a, b)$ such that

$$|x'(t^*)| = 0. \quad (24)$$

By (4), (21) and (24), we know that for any $t \in [a, b]$,

$$\begin{aligned}
 |x'(t)| &= \left| \int_{t^*}^t x''(s) ds \right| = \lambda \left| \int_{t^*}^t f(s, x(s), x'(s)) ds \right| \\
 &\leq \int_a^b |f(s, x(s), x'(s))| ds \\
 &\leq \int_a^b (p(s)|x(s)| + q(s)|x'(s)| + r(s)) ds \\
 &\leq \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 \\
 &\leq \|p\|_1 \left(\frac{|x(a) + x(b)|}{2} + \frac{1}{2} \int_a^b |x'(s)| ds \right) + \|q\|_1 \|x'\|_\infty + \|r\|_1 \\
 &\leq \frac{1}{2} \|p\|_1 \int_a^b |x'(s)| ds + \|q\|_1 \|x'\|_\infty + \|r\|_1 \\
 &\leq \left(\frac{b-a}{2} \|p\|_1 + \|q\|_1 \right) \|x'\|_\infty + \|r\|_1.
 \end{aligned} \tag{25}$$

It follows that

$$\|x'\|_\infty \leq \left(\frac{b-a}{2} \|p\|_1 + \|q\|_1 \right) \|x'\|_\infty + \|r\|_1. \tag{26}$$

By (26) and note that $(b-a)\|p\|_1 + 2\|q\|_1 < 2$, we know that, there is a constant M_1 such that

$$\|x'\|_\infty \leq M_1. \tag{27}$$

In view of (4), (22) and (27), we have

$$\begin{aligned}
 \|x\|_\infty &\leq \frac{|x(a) + x(b)|}{2} + \frac{1}{2} \int_a^b |x'(s)| ds \\
 &= \frac{1}{2} \int_a^b |x'(s)| ds \leq M_0,
 \end{aligned} \tag{28}$$

where $M_0 = (b-a)M_1/2$. The proof is complete.

Corollary 1. *Suppose there exist constants $p \geq 0$, $q \geq 0$, $D > 0$, and there is a nonnegative real continuous function r defined on $[a, b]$, such that*

$$(i) \quad |f(t, u, v)| \leq p|u| + q|v| + r(t), \text{ for } t \in [a, b] \text{ and } (u, v) \in \mathbb{R}^2,$$

$$(ii) \quad uf(t, u, v) > 0, \text{ for } |u| \geq D \text{ and } t, v \in \mathbb{R}.$$

Then for $(b-a)((b-a)p + 2q) < 2$, the boundary value problem (18)-(19) has at least one solution.

Example 1. Consider the two point boundary value problem of the form

$$x''(t) = px(t) + r(t), \quad t \in [a, b], \quad (29)$$

$$x(a) = x(b) = 0, \quad (30)$$

where $p > 0$ is a constant and r is a nonnegative real continuous function. We may show that the boundary value problem (29)-(30) has only one solution provided $(b-a)(b-a)p < 2$.

Indeed, if we let $q = 0$ and $D > 0$, then all conditions of Corollary 1 are satisfied. Thus, the boundary value problem (29)-(30) has at least one solution. Next, we go to prove the uniqueness of the solution. Suppose that $x(t)$ and $y(t)$ are solutions of (29)-(30). Let $z(t) = x(t) - y(t)$. It is easy to see that $z(t)$ is a solution of the boundary value problem of the form

$$z''(t) = pz(t), \quad t \in [a, b], \quad (31)$$

$$z(a) = z(b) = 0. \quad (32)$$

By (31) we know that

$$z(t) = c_1 \exp(\sqrt{p}t) + c_2 \exp(-\sqrt{p}t), \quad (33)$$

where c_1 and c_2 are constants. It follows from (32) and (33) that $c_1 = c_2 = 0$, that is $z(t) = 0$, $t \in [a, b]$. Thus $x(t) = y(t)$, $t \in [a, b]$. Therefore, our assertion is true. This example shows that Corollary 1 is sharp.

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