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FUZZY DIFUNCTIONAL RELATIONS UNDER BALANCED MAPPINGS

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Abstract

The images and preimages of fuzzy difunctional relations are studied under the balanced mappings.

1. Introduction

Ounalli and Jaoua [1] extended difunctional relations in the framework of fuzzy relations with max-min composition for the purpose of gaining a better understanding of their properties and their structure. Seo et al. [3] characterized fuzzy difunctional relations on a set and proved there exists a relationship between fuzzy equivalence relations and fuzzy difunctional relations. Sung [4] characterized balanced mappings and gave sufficient condition for the image of an F-preorder to be an F-preorder under a balanced mapping. As a continuation of these studies, in this paper, we apply the idea of difunctional relation in the setting of balanced mappings and study images and preimages of fuzzy difunctional relations under balanced mappings.

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2. Preliminaries

For details, we refer to [4-6].

Definition 2.1. Let G be a group. A fuzzy subset f of G is call a *fuzzy subgroup* of G if

(i) $f(x, y) \geq f(x) \wedge f(y)$ for all x, y of G,

(ii) $f(x^{-1}) = f(x)$ for all x of G,

(iii) $f(e) = 1$ (e = identity of G).

Definition 2.2. Let R be a fuzzy relation on X. The reflexive(symmetric, transitive) closure R^* of R is defined by:

(i) $R \subseteq R^*$,

(ii) R^* is a fuzzy reflexive(symmetric, transitive) relation,

(iii) for all fuzzy reflexive(symmetric, transitive) relations S that contain R, $R^* \subseteq S$.

Definition 2.3. Let R and S be two fuzzy relations.

We say that

(i) R is *more deterministic* than S if and only if $R^{-1} \circ R \subseteq S^{-1} \circ S$,

(ii) R is *deterministic* if and only if it is more deterministic than the identity I, i.e., $R^{-1} \circ R \subseteq I$.

Theorem 2.4 [1, Proposition 7]**.** *If a fuzzy relation R is symmetric and transitive*, *then R is fuzzy difunctional*.

3. Some Properties

In this section, we deal with some properties of fuzzy difunctional relations in the framework of fuzzy relation.

Theorem 3.1. *Let a t-norm F be idempotent. If a fuzzy relation R is deterministic on X, then* $R \circ R^{-1}$ *is F-transitive.*

Proof. We note that *F*-transitivity is equivalent to transitivity. It suffices to show that $R \circ R^{-1}$ is fuzzy transitive. Indeed,

$$\begin{aligned}(R \circ R^{-1}) \circ (R \circ R^{-1}) &= R \circ (R^{-1} \circ R) \circ R^{-1} \\ &= R \circ I \circ R^{-1}, \text{ as } R \text{ is fuzzy deterministic} \\ &= R \circ R^{-1},\end{aligned}$$

which shows that $R \circ R^{-1}$ is fuzzy transitive.

Theorem 3.2. *Under the above conditions of Theorem* 3.1, *then* $R \circ R^{-1}$ *is fuzzy difunctional.*

Proof. We prove that $R \circ R^{-1}$ is fuzzy transitive, this follows from argument proved earlier in the proof of Theorem 3.1, so we need only to show that $R \circ R^{-1}$ is fuzzy symmetric.

Since $(R \circ R^{-1})^{-1} = R \circ R^{-1}$, $R \circ R^{-1}$ is fuzzy symmetric, this completes the proof.

Theorem 3.3. *Let R fuzzy be a fuzzy symmetric and fuzzy idempotent relation on X. Then the reflexive closure* R^* *of R is fuzzy difunctional.*

Proof. By [6, Theorem 3.1], we note that $R^* = R \cup I$. Then we have

$$\begin{aligned}(R^*)^* &= (R \cup I)^{-1} = R^{-1} \cup I \\ &= R \cup I, \text{ as } R \text{ is fuzzy symmetric} \\ &= R^*.\end{aligned}$$

This means R^* is fuzzy symmetric, also from argument proved earlier in the proof of Theorem 3.1, we see that R^* is fuzzy transitive, hence R^* is fuzzy difunctional.

Theorem 3.4. *Let R be a fuzzy reflexive and symmetric relation on X. Then the transitive closure* R^* *of R is fuzzy difunctional.*

Proof. By [6, Theorem 3.3], we note that R^* is fuzzy idempotent.

Now, we prove that R^* is fuzzy symmetric.

For $n \geq 1$ and $x, y \in X$, we have

$$\begin{aligned} R^n(x, y) &= \bigvee_{z_1, \ldots, z_{n-1}} (R(x, z_1) \wedge R(z_1, z_2) \wedge \cdots \wedge R(z_{n-1}, y)) \\ &= \bigvee_{z_1, \ldots, z_{n-1}} (R(y, z_{n-1}) \wedge R(z_{n-1}, z_{n-2}) \wedge \cdots \wedge R(z_1, x)) \\ &= R^n(y, x) \text{ for all } x, y \in X, \end{aligned}$$

which yields R^n is fuzzy symmetric for $n \geq 1$.

Therefore, R^* is fuzzy symmetric. But, since R^* is fuzzy symmetric and fuzzy idempotent, $R^* \circ (R^*)^{-1} \circ (R^*) = R^* \circ R^* \circ R^* = R^* \circ R^* = R^*$, which shows that R^* is fuzzy difunctional.

Theorem 3.5. *Let f be a fuzzy subgroup of a group G. Then the fuzzy relation R on G defined by* $R_f(a, b) = f(ab^{-1})$ *for all* (a, b) *of* $G \times G$, *is fuzzy difunctional.*

Proof. We prove that R^f is fuzzy symmetric and transitive. Let $a, b \in G$. Then

$$\begin{aligned} R_f(a, b) &= f(ab^{-1}) \\ &= f((ab^{-1})^{-1}), \text{ as } f \text{ is a fuzzy subgroup on } G \\ &= f(ba^{-1}) \\ &= R_f(b, a), \end{aligned}$$

which yields R_f is fuzzy symmetric.

For $a, b \in G$, we have

$$R_f \circ R_f(a, b) = \bigvee_t (R_t(a, t) \wedge R_f(t, b))$$

$$= \bigvee_t (f(at^{-1}) \wedge f(tb^{-1}))$$

$$\leq f(ab^{-1}), \text{ as } f \text{ is a fuzzy subgroup on } G$$

$$= R_f(a, b),$$

which yields R_f is fuzzy transitive. Therefore, R_f is fuzzy difunctional.

Theorem 3.6. *Let R be a fuzzy reflexive relation on X. If R is fuzzy difunctional, then for a given* $k \in [0, 1]$, *a fuzzy relation p given by* $p(x, y) = R(x, y) \vee k$ *for all* $x, y \in X$, *is fuzzy difunctional*.

Proof. For $x, y \in X$, we have

$$P(x, y) = \bigvee_t (p(x, t) \wedge p(t, y))$$

$$= \bigvee_t (R(x, t) \wedge R(t, y)) \vee k$$

$$= R \circ R(x, y) \vee k$$

$$\leq R(x, x) \vee k, \text{ as } R \text{ is fuzzy difunctional}$$

$$= P(x, y) \text{ for all } x, y \in X,$$

which shows that *p* is fuzzy transitive. Therefore, *P* is fuzzy difunctional.

Theorem 3.7 [5, Theorem 3.14]**.** *Let R be a fuzzy relation on X. Then the following conditions are equivalent*:

(i) *R is fuzzy difunctional*,

(ii) ${}_xR_\alpha^{-1} \cap {}_yR_\alpha^{-1} = \varnothing$, $x, y \in X$ *and* $\alpha > 0$ *implies* ${}_xR_\alpha^{-1} = {}_yR_\alpha^{-1}$.

Theorem 3.8. *Let* $f : X \to Y$ *be a function. If Q is fuzzy difunctional on Y, then function* $R(x, y) = Q(f(x), f(y))$ *for all* $x, y \in X$, *is fuzzy difunctional on X.*

Proof. Let $\alpha > 0$ be fixed and ${}_xR_\alpha^{-1} \cap {}_yR_\alpha \neq \varnothing$, $x, y \in X$. Then we prove that ${}_xR_\alpha^{-1} = {}_yR_\alpha^{-1}$. Now, let $z \in {}_xR_\alpha^{-1}$. Then $R(z, x) \geq \alpha$, by ${}_xR_\alpha^{-1} \cap {}_yR_\alpha \neq \varnothing$, this means that $R(w, x) \geq \alpha$ and $R(w, y) \geq \alpha$. On the other hand, we have

$$\begin{aligned} Q(f(z), f(y)) &\geq Q \circ Q^{-1} \circ Q(f(z), f(y)), \text{ as } Q \text{ is fuzzy difunctional} \\ &= \bigvee_t (Q(f(z), t)) \wedge (Q^{-1} \circ Q)(t, f(y)) \\ &\geq Q(f(z), f(x)) \wedge (\bigvee_s (Q(s, f(x)) \wedge Q(s, f(y)))) \\ &\geq Q(f(z), f(x)) \wedge (Q(f(w, f(x)) \wedge Q(f(w), f(y)))) \\ &\quad \text{for } s = f(w) \\ &\geq \alpha, \end{aligned}$$

which yields $R(z, y) \geq \alpha$, thus we get $z \in {}_yR_\alpha$. Hence, ${}_xR_\alpha^{-1} \subseteq {}_yR_\alpha^{-1}$.

We can prove in the same method that ${}_yR_\alpha^{-1} \subseteq {}_xR_\alpha^{-1}$. Therefore, ${}_xR_\alpha^{-1} = {}_yR_\alpha^{-1}$.

Let *F* be a *t*-norm and *R* be an *F*-preorder. We denote $F(x, y) = x * y$ for $x, y \in R$. Now, we define $x * y$ if and only if $R(x, y) = 1$ and $R(y, x) = 1$. clearly, ~ is an equivalence relation on *X*. Let $({}^R/_\sim)(u, v) = R(x, y)$ for $x \in u \in {}^X/_\sim$, $y \in v \in {}^X/_\sim$.

Then we get the following result.

Theorem 3.9. *Let F be a t-norm and R be an F-preorder. If R is fuzzy difunctional on X, then ${}^{R}/_{\sim}$ is fuzzy difunctional on ${}^{X}/_{\sim}$.*

Proof. Let $\alpha > 0$ be fixed and $u({}^{R}/_{\sim})_{\alpha}^{-1} \cap v({}^{R}/_{\sim})_{\alpha}^{-1} \neq \emptyset$ for $u, v \in {}^{X}/_{\sim}$. Then we prove that $u({}^{R}/_{\sim})_{\alpha}^{-1} = v({}^{R}/_{\sim})_{\alpha}^{-1}$. Let $w \in u({}^{X}/_{\sim})$, since $u, v, w \in {}^{X}/_{\sim}$, there exist $x, y, z \in X$ such that $x \in u$, $y \in v$, $z \in w$, since $w \in u({}^{R}/_{\sim})$, this means that $({}^{R}/_{\sim})(w, u) \geq \alpha$ and so $R(z, x) \geq \alpha$. By $u({}^{R}/_{\sim})_{\alpha}^{-1} \cap v({}^{R}/_{\sim})_{\alpha}^{-1} \neq \emptyset$, there exists $w' \in {}^{X}/_{\sim}$ such that $w' \in u({}^{R}/_{\sim})_{\alpha}^{-1} \cap v({}^{R}/_{\sim})_{\alpha}^{-1}$, this means that $({}^{R}/_{\sim})(w', u) \geq \alpha$ and $({}^{R}/_{\sim})(w', v) \geq \alpha$, this leads to $R(z', x) \geq \alpha$ and $R(z', y) \geq \alpha$. On the other hand, we have

$$
\begin{aligned}
R(z, y) &\geq R \circ R^{-1} \circ R(z, y), \text{ as } R \text{ is fuzzy difunctional} \\
&= \bigvee_{t}(R(z, t) \wedge (R^{-1} \circ R)(t, y)) \\
&\geq R(x, x) \wedge R^{-1} \circ R(x, y) \text{ for } t = x \\
&\geq R(z, x) \wedge (\bigvee_{s}(R(s, x) \wedge R(s, y))) \text{ for } s \in z' \\
&\geq R(z, x) \wedge R(z', x) \wedge R(z', y) \text{ for } s = z' \\
&\geq \alpha,
\end{aligned}
$$

which yields $({}^{R}/_{\sim})(w, v) \geq \alpha$, hence we have $u({}^{R}/_{\sim})_{\alpha}^{-1} \subseteq v({}^{R}/_{\sim})_{\alpha}^{-1}$, we are done $v({}^{R}/_{\sim})_{\alpha}^{-1} \subseteq u({}^{R}/_{\sim})_{\alpha}^{-1}$ similarly. Therefore, ${}^{R}/_{\sim}$ is fuzzy difunctional on ${}^{X}/_{\sim}$.

Theorem 3.10. *Let f be a balanced mapping from $X \times X$ into $Y \times Y$ and R be a fuzzy relation on Y. If R is fuzzy symmetric and transitive, then $f^{-1}(R)$ is fuzzy difunctional on X.*

Proof. Let $f(x, y) = (u, v)$. Then we have $f^{-1}(R)(x, y) = R(f(x, y))$ $= R(u, v) = R(v, u) = R(f(y, x)) = f^{-1}(R)(y, x)$. This means that $f^{-1}(R)$ is fuzzy symmetric. Next, let $f(x, y) = (u, v)$ for $z \in X$, since f is a balanced mapping, there exists $t_z \in Y$ such that $f(x, z) = (u, t_z)$ and $f(z, y) = (t_z, v)$. Then we have

$$(f^{-1}(R) \circ f^{-1}(R))(x, y) = \bigvee_z (R(f(x, z)) \wedge R(f(z, y)))$$

$$= \bigvee_z (R(u, t_z) \wedge R(t_z, y)), \text{ as } R \text{ is a balanced mapping}$$

$$\leq R \circ R(u, v)$$

$$\leq R(u, v), \text{ as } R \text{ is fuzzy transitive}$$

$$= f^{-1}(R)(x, y) \text{ for all } x, y \in X,$$

which yields $f^{-1}(R)$ is fuzzy transitive. Hence, $f^{-1}(R)$ is fuzzy difunctional.

Theorem 3.11. *Let f be a balanced mapping from* $X \times X$ *onto* $Y \times Y$ *and R be a fuzzy relation on X. If R is fuzzy symmetric and transitive, then* $f(R)$ *is fuzzy difunctional on Y.*

Proof. Let $(u, v) \in Y \times Y$ be given, we note that f is injective, hence f is bijective. Then there exists unique $(x, y) \in X \times X$ such that $f(x, y) =$ (u, v). Then we get

$$f(R)(u, v) = \bigvee_{f(t, s) = (u, v)} R(t, s) = R(x, y) = R(y, x) = f(R)(v, u),$$

this means that $f(R)$ is fuzzy symmetric.

Next, let $(u, v) \in Y \times Y$ and $f(x, y) = (u, v)$. Then we have

$$(f(R) \circ f(R))(u, v) = \bigvee_w (f(R)(u, w) \wedge f(R)(w, v))$$

$$= \bigvee_w \begin{pmatrix} R(x, t_w) & \wedge & R(t_w, y) \\ f(x, t_w) = (u, w) & & f(t_w, y) = (u, v) \end{pmatrix}$$

$$= R(x, t_w) \wedge R(t_w, y)$$

$$\leq R \circ R(x, y)$$

$$\leq R(x, y)$$

$$= f(R)(u, v) \text{ for all } u, v \in Y,$$

which yields $f(R)$ is fuzzy transitive, therefore $f(R)$ is fuzzy difunctional.

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