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MATHEMATICAL ANALYSIS OF MELODIES: SYMMETRY

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Abstract

A plane graph, called an *M*-graph, is attached to every melody. The main purpose of the paper is to obtain a complete classification of melodies of arbitrary length whose *M*-graphs have lines of symmetry. A crucial role is played by a matrix pencil, attached to the set of those melodies, whose fiber at a point is revealed to hide a twelve-tone row composed by Webern.

0. Introduction

In this paper, we investigate symmetry of melodies. More precisely, we attach a plane graph, called an "*M-graph*", to each melody by the algorithm introduced in [1], and consider when the *M*-graph has a line of symmetry. For example, the *M*-graph of the twelve-tone row used in String Quartet op. 28 by Webern has a marvelous symmetry as is observed in Figure 1 below. We will give a complete classification of melodies with symmetric *M*-graph. Some partial results in this direction are obtained in the previous paper by the

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author [2] under the assumption that melodies consist of mutually distinct tones. In order to remove the assumption, we reformulate our problem so that we can study an arbitrary set of points in the plane, and establish a simple linear-algebraic criterion for the set to have symmetry. Thereafter, we apply the criterion to the *M*-graphs of melodies. As a result, we find that the problem is translated into the one about the distribution of the ranks of a certain matrix pencil parametrized by the projective line $\mathbf{P}^1$, and detect that Webern's melody and the like are hidden in the fiber at $(1, 1) \in \mathbf{P}^1$.

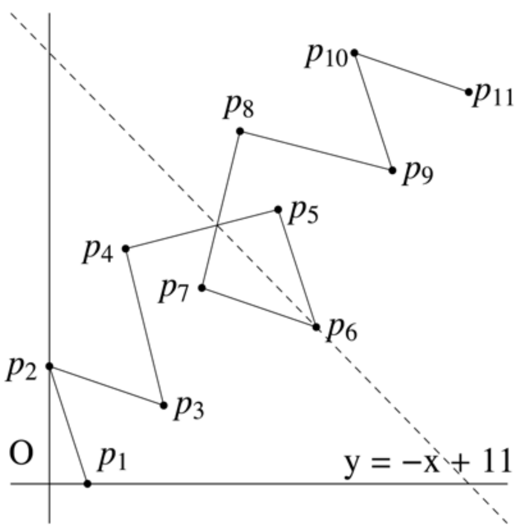

Figure 1. Webern: String Quartet op. 28.

Our classification shows that one cannot surpass Webern in creating melodies with symmetric *M*-graph if we restrict our attention to those of length twelve. More precisely, we prove that if a melody $\mathbf{a} = (a_1, ..., a_N)$ of an arbitrary length *N*, which is expressed as a sequence of integers, has a line of symmetry, then it necessarily is (1) periodic with period four or (2) symmetric itself in the sense that $a_i = a_{N+1-i}$ holds for any $i \in [1, N]$ or (3) antisymmetric in the sense that $a_i = -a_{N+1-i} + k$ holds for any $i \in [1, N]$ with a fixed constant *k*. The melody by Webern falls into the category (3) with $N = 12$ and $k = 11$. However, our result might be somewhat disappointing for anyone who tries to create a new symmetric melody of length twelve, it can create as many melodies of arbitrary length with symmetry as one hopes. In any case, our result certifies the ingenuity of Webern who could not have envisaged our classification.

The plan of this paper is as follows: Section 1 deals with symmetry of general plane graphs. A simple criterion for a graph to have a line of symmetry is formulated in terms of a certain matrix composed of the coordinates of vertices of the graph. In Section 2, we recall the definition of the *M*-graph of a melody. Thereafter, we examine the cases of melodies of lengths 4, 5 and 12, applying the general criterion obtained in the previous section. Our analysis of the case of length 12 will enable one to understand the process employed in Section 3 where we deal with symmetries of melodies of arbitrary length. Here we need to divide our argument into two parts according to the parity of the length. In Section 4, we apply our results to obtain several good-looking symmetric melodies. The reader is invited to take a look at several *M*-graphs illustrated in the last pages, which have amusing symmetries.

1. Symmetry of Points in the Plane

Let $N \geq 3$ be an integer and let $p_i = (x_i, y_i)$, $1 \leq i \leq N$, be N points in the real plane $\mathbb{R}^2$. Let ℓ be a line and let $r_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection map with respect to ℓ. We call ℓ the *line of symmetry* of the ordered set $\mathbf{p} = (p_1, ..., p_N)$, if the condition

$$r_\ell(p_i) = p_{N+1-i}, \quad 1 \leq i \leq N \tag{1.1}$$

holds. Note that the situation differs slightly according to the parity of *N*. When *N* is even, the condition (1.1) holds if and only if both of the following two conditions hold:

(E1) The midpoint of the segment $p_i p_{N+1-i}$ lies on ℓ for any $i \in \left[1, \frac{N}{2}\right]$.

(E2) The segment $p_i p_{N+1-i}$ intersects transversally with ℓ for any $i \in \left[1, \frac{N}{2}\right]$.

On the other hand, when N is odd, the condition (1.1) holds if and only if all of the following three conditions hold:

(O1) The midpoint of the segment $p_i p_{N+1-i}$ lies on ℓ for any $i \in \left[1, \frac{N-1}{2}\right]$.

(O2) The point $p_{(N+1)/2}$ lies on ℓ.

(O3) The segment $p_i p_{N+1-i}$ intersects transversally with ℓ for any $i \in \left[1, \frac{N-1}{2}\right]$.

We can express these conditions simply in terms of a certain matrix. In order to simplify our description, we put

$$x_i^+ = \frac{x_i + x_{N+1-i}}{2}, \quad x_i^- = \frac{x_i - x_{N+1-i}}{2},$$

$$y_i^+ = \frac{y_i + y_{N+1-i}}{2}, \quad y_i^- = \frac{y_i - y_{N+1-i}}{2}$$

for any $i \in [1, N]$.

When $N = 2m$ with $m \geq 2$, there exists a line ℓ such that the condition (E1) holds if and only if the inequality

$$rank \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1^+ & x_2^+ & \cdots & x_m^+ \\ y_1^+ & y_2^+ & \cdots & y_m^+ \end{pmatrix} \leq 2 \tag{1.2}$$

holds, for (E1) is equivalent to the existence of a triple $(a, b, c) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ such that

$$a + b\frac{x_i + x_{2m+1-i}}{2} + c\frac{y_i + y_{2m+1-i}}{2} = 0$$

holds for any $i \in [1, m]$, namely, all of the midpoints

$$\left(\frac{x_i + x_{2m+1-i}}{2}, \frac{y_i + y_{2m+1-i}}{2} \right)$$

of $p_i p_{2m+1-i}$ $(i \in [1, m])$ lie on one and the same line $\ell_{(a, b, c)} : a + bx + cy = 0$. Moreover, if the condition (E2) is met for $\ell = \ell_{(a, b\, c)}$, then we must have

$$c\, \frac{x_i - x_{2m+1-i}}{2} - b\, \frac{y_i - y_{2m+1-i}}{2} = 0$$

for any $i \in [1, m]$. Therefore, both conditions (E1) and (E2) hold for some line if and only if the inequality

$$rank \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ x_1^+ & \cdots & x_m^+ & -y_1^- & \cdots & -y_m^- \\ y_1^+ & \cdots & y_m^+ & x_1^- & \cdots & x_m^- \end{pmatrix} \le 2 \qquad (1.3)$$

holds.

When $N = 2m + 1$, both conditions (O1) and (O2) hold for some line if and only if

$$rank \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \frac{x_1 + x_{2m+1}}{2} & \cdots & \frac{x_m + x_{m+2}}{2} & x_{m+1} \\ \frac{y_1 + y_{2m+1}}{2} & \cdots & \frac{y_m + y_{m+2}}{2} & y_{m+1} \end{pmatrix} \le 2.$$

Accordingly, all of the three conditions (O1), (O2) and (O3) hold for some line if and only if the inequality

$$rank \begin{pmatrix} 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ x_1^+ & \cdots & x_m^+ & x_{m+1} & -y_1^- & \cdots & -y_m^- \\ y_1^+ & \cdots & y_m^+ & y_{m+1} & x_1^1 & \cdots & x_m^- \end{pmatrix} \le 2 \qquad (1.4)$$

holds. Furthermore, in each case, the coefficient (a, b, c) of the line ℓ of symmetry is given by a basis of the kernel of the linear map $\mathbb{R}^3 \to \mathbb{R}^n$, which is defined by the right multiplication of the matrix in (1.3) or (1.4). Note here that $\mathbb{R}^N$ is regarded as a vector space of row vectors of length *N*. We record these facts as a proposition for later use:

Proposition 1.1. *Let N be an integer* ≥ 3. *For a given ordered set* $\mathbf{p} = (p_1, \ldots, p_N)$ *of N points in* $\mathbb{R}^2$ *with* $p_i = (x_i, y_i)$, $1 \leq i \leq N$, *let*

$$x_i^+ = \frac{x_i + x_{N+1-i}}{2}, \quad x_i^- = \frac{x_i - x_{N+1-i}}{2},$$

$$y_i^+ = \frac{y_i + y_{N+1-i}}{2}, \quad y_i^- = \frac{y_i - y_{N+1-i}}{2},$$

for $i \leq \left\lfloor \frac{N}{2} \right\rfloor$. *When N is even, we put* $N = 2m$ *and let*

$$M_N = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ x_1^+ & \cdots & x_m^+ & -y_1^- & \cdots & -y_m^- \\ y_1^+ & \cdots & y_m^+ & x_1^- & \cdots & x_m^- \end{pmatrix}. \tag{1.5}$$

When N is odd, we put $N = 2m + 1$ *and let*

$$M_N = \begin{pmatrix} 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ x_1^+ & \cdots & x_m^+ & x_{m+1} & -y_1^- & \cdots & -y_m^- \\ y_1^+ & \cdots & y_m^+ & y_{m+1} & x_1^1 & \cdots & x_m^- \end{pmatrix}. \tag{1.6}$$

Then **p** *has a line of symmetry if and only if* $\operatorname{rank} M_N \leq 2$. *Furthermore, the coefficient* (a, b, c) *of the line* $\ell : a + bx + cy = 0$ *of symmetry is given by a basis of the kernel of the linear map* $\mathbb{R}^3 \to \mathbb{R}^N$ *defined by the right multiplication of the matrix* M_N.

We illustrate this by a few examples.

Example 1.1. Let $p_1 = (5, 4)$, $p_2 = (2, 5)$, $p_3 = (-6, 1)$, $p_4 = (-7, -2)$. In this case, we have

$$x_1^+ = -1, \quad x_1^- = 6,$$

$$y_1^+ = 1, \quad y_1^- = 3,$$

$$x_2^+ = -2, \quad x_2^- = 4,$$

$$y_2^+ = 3, \quad y_2^- = 2.$$

Hence, the matrix M_4 in (1.4) becomes

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -2 & -3 & -2 \\ 1 & 3 & 6 & 4 \end{pmatrix}$$

and it is of rank equal to 2. Furthermore, by means of elementary column transformations, it becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{pmatrix},$$

hence the kernel of the associated linear map is spanned by $(a, b, c) = (1, 2, 1)$. Therefore, the line of symmetry of $\mathbf{p} = (p_1, p_2, p_3, p_4)$ is given by $1 + 2x + y = 0$ or equivalently by $y = -2x - 1$.

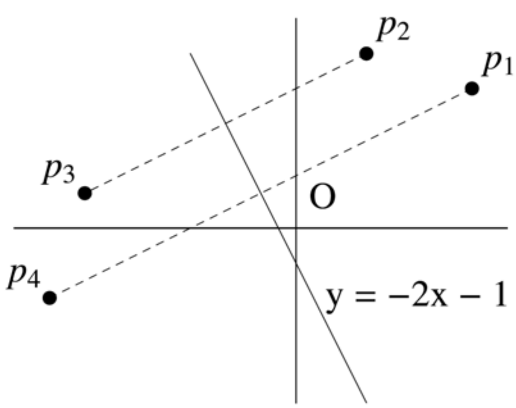

Figure 2. Symmetric point set of length four.

Example 1.2. Let $p_1 = (-3, 0)$, $p_2 = (-2, 2)$, $p_3 = (1, 3)$, $p_4 = (2, 0)$, $p_5 = (1, -2)$. In this case, we have

$$x_1^+ = -1, \quad x_1^- = -2,$$

$$y_1^+ = -1, \quad y_1^- = 1,$$

$$x_2^+ = 0, \quad x_2^- = -2$$

$$y_2^+ = 1, \quad y_2^- = 1.$$

Hence, the matrix M_5 in (1.5) becomes

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 3 & -2 & -2 \end{pmatrix}$$

and it is of rank equal to 2. Furthermore, by means of elementary column transformations, it becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}$$

and hence the kernel of the associated linear map is spanned by $(a, b, c) = (-1, -2, 1)$. Therefore, the line of symmetry of $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$ is defined by $-1 - 2x + y = 0$ or equivalently by $y = 2x + 1$.

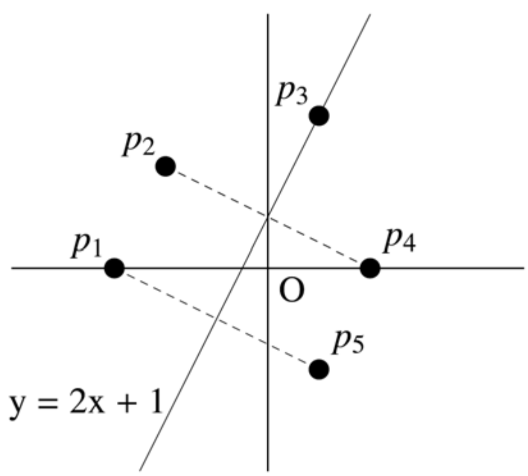

Figure 3. Symmetric point set of length five.

2. Symmetry of *M*-graphs

In [1], a method of visualization of a melody is introduced. It attaches a plane graph to a given melody and the authors employ the graph to investigate how the characteristic of a melody is reflected to its shape. In this section, we recall the definition and study when the graph has symmetry in the sense of the previous section for small N.

In order to express a melody by a definite sequence of integers, we let $C4$ (middle C) correspond to 0, $C\#4$ to 1, and so on. In this way, we can associate a sequence of integers with each melody. For example, the melody "$C4$, $D4$, $F4$, $E4$", which is the main theme of the fourth movement of the Jupiter symphony by Mozart, corresponds to the sequence "0, 2, 5, 4". From now on, we identify a melody of finite length with the sequence of integers of finite length, which is constructed by this rule. Furthermore, to any sequence $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ of integers, we attach a sequence of points $\mathbf{p} = (p_1, p_2, \ldots, p_{N-1})$ with $p_i \in \mathbf{R}^2$ $(1 \le i \le N-1)$ by the following rule:

$$p_1 = (a_1, a_2),\ p_2 = (a_2, a_3), \ldots,\ p_{N-1} = (a_{N-1}, a_N).$$

Let $G(\mathbf{a}) = (V(\mathbf{a}), E(\mathbf{a}))$ be the directed graph with the set of vertices

$$V(\mathbf{a}) = (p_1, p_2, \ldots, p_{N-1})$$

and the set of edges

$$E(\mathbf{a}) = \{(p_1, p_2), (p_2, p_3), \ldots, (p_{N-2}, p_{N-1})\}.$$

We call $G(\mathbf{a})$ the *M-graph* associated to the melody **a** ("*M*" stands for <u>m</u>elody). However, since we focus only on the ordered set of its vertices in the present paper, we set

$$M(\mathbf{a}) = (p_1, p_2, \ldots, p_{N-1})$$

and call it *the M-graph of the melody* **a** by abuse of language.

We are interested in the problem to determine whether or not the *M*-graph of a melody has a line of symmetry. We examine a few cases with small N. When $N = 3$, however, the M-graph becomes a segment and has evidently a line of symmetry. For this reason, we begin with the simplest nontrivial case when $N = 4$.

2.1. The case when $N = 4$

Let $\mathbf{a} = (a_1, a_2, a_3, a_4)$ be a melody of length four, and let $p_1 = (a_1, a_2)$, $p_2 = (a_2, a_3)$, $p_3 = (a_3, a_4)$ so that $M(\mathbf{a}) = (p_1, p_2, p_3)$. In this case, the coordinates $x_1^{\pm}$, $y_1^{\pm}$ are given by

$$x_1^+ = \frac{a_1 + a_3}{2}, \quad x_1^- = \frac{a_1 - a_3}{2},$$

$$y_1^+ = \frac{a_2 + a_4}{2}, \quad y_1^- = \frac{a_2 - a_4}{2}$$

and the matrix M_3 in (1.6) becomes

$$M_3 = \begin{pmatrix} 1 & 1 & 0 \\ x_1^+ & x_2 & -y_1^- \\ y_1^+ & y_2 & x_1^- \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ \frac{a_1 + a_3}{2} & a_2 & -\frac{a_2 - a_4}{2} \\ \frac{a_2 + a_4}{2} & a_3 & \frac{a_1 - a_3}{2} \end{pmatrix}. \quad (2.1)$$

The last matrix can be reduced to a simpler form by a number of elementary transformations. Here and throughout the paper, we come to employ elementary row (resp. column) transformations many times, so it is worth to introduce the notation for them:

Definition 2.1. For any $c \in \mathbf{R}$, we denote the elementary transformation, which adds $c \times$ (the ith row) to the jth row by $E_{row}(i, j; c)$ and the transformation, which multiplies c to the ith row by $E_{row}(i; c)$. The corresponding column transformations are denoted by $E_{column}(i, j; c)$ and $E_{column}(i; c)$, respectively.

Now applying $E_{row}(2; 2)$ and $E_{row}(3; 2)$ to the rightmost matrix in (2.1), we see that the matrix M_3 is similar to

$$\begin{pmatrix} 1 & 1 & 0 \\ a_1 + a_3 & 2a_2 & -a_2 + a_4 \\ a_2 + a_4 & 2a_3 & a_1 - a_3 \end{pmatrix}.$$

Furthermore, applying $E_{column}(2, 1; -1)$ and $E_{column}(3, 1; -1)$ successively to this matrix, it becomes

$$\begin{pmatrix} 0 & 1 & 0 \\ a_1 - a_2 + a_3 - a_4 & 2a_2 & -a_2 + a_4 \\ -a_1 + a_2 - a_3 + a_4 & 2a_3 & a_1 - a_3 \end{pmatrix}.$$

Finally, by $E_{row}(2, 3; 1)$, this matrix is transformed to

$$\begin{pmatrix} 0 & 1 & 0 \\ a_1 - a_2 + a_3 - a_4 & 2a_2 & -a_2 + a_4 \\ 0 & 2a_2 + 2a_3 & a_1 - a_2 - a_3 + a_4 \end{pmatrix}.$$

Therefore, we see that $rank\ M_3 \leq 2$ if and only if

$$(a_1 - a_2 + a_3 - a_4)(a_1 - a_2 - a_3 + a_4) = 0.$$

Hence, we obtain the following:

Proposition 2.1. *For a melody* $\mathbf{a} = (a_1, \ldots, a_4)$ *of length four, the M-graph* $M(\mathbf{a})$ *has a line of symmetry if and only if one of the following conditions holds*:

(1) $a_1 - a_2 + a_3 - a_4 = 0$,

(2) $a_1 - a_2 - a_3 + a_4 = 0$.

Furthermore, if the condition (1) *is met, then the two segments* $p_1 p_2$ *and* $p_2 p_3$ *are transversal to each other and have the same length*.

Proof. Only the last assertion needs to be proved. It follows from (1) that $a_3 - a_4 = -(a_1 - a_2)$. Hence,

$$\begin{aligned}(p_1 - p_2) \cdot (p_2 - p_3) &= (a_1 - a_2)(a_2 - a_3) + (a_2 - a_3)(a_3 - a_4) \\ &= (a_1 - a_2)(a_2 - a_3) - (a_2 - a_3)(a_1 - a_2) \\ &= 0.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&(p_1 - p_2) \cdot (p_1 - p_2) - (p_2 - p_3) \cdot (p_2 - p_3) \\ &= ((a_1 - a_2)^2 + (a_2 - a_3)^2) - ((a_2 - a_3)^2 + (a_3 - a_4)^2) \\ &= ((a_1 - a_2)^2 + (a_2 - a_3)^2) - ((a_2 - a_3)^2 + (a_1 - a_2)^2) \\ &= 0.\end{aligned}$$

This completes the proof. □

Remark. The reason why we add the last statement to the proposition is that we will come across to the condition (1) several times when we deal with melodies of arbitrary length, and that we will need to know the shape of the first three points in their *M*-graph.

2.2. The case when $N = 5$

Now we examine the case when $N = 5$. Let $\mathbf{a} = (a_1, ..., a_5)$ be a melody of length five, and let $p_i = (a_i, a_{i+1})$, $1 \le i \le 4$ so that $M(\mathbf{a}) = (p_1, ..., p_4)$. In this case, the coordinates $x_i^{\pm}$, $y_i^{\pm}$ $(i = 1, 2)$ are given by

$$x_1^+ = \frac{a_1 + a_4}{2}, \quad x_1^- = \frac{a_1 - a_4}{2},$$

$$y_1^+ = \frac{a_2 + a_5}{2}, \quad y_1^- = \frac{a_2 - a_5}{2},$$

$$x_2^+ = \frac{a_2 + a_3}{2}, \quad x_2^- = \frac{a_2 - a_3}{2},$$

$$y_2^+ = \frac{a_3 + a_4}{2}, \quad y_2^- = \frac{a_3 - a_4}{2}$$

and the matrix M_4 in (1.5) becomes

$$M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \frac{a_1 + a_4}{2} & \frac{a_2 + a_3}{2} & -\frac{a_2 - a_5}{2} & -\frac{a_3 - a_4}{2} \\ \frac{a_2 + a_5}{2} & \frac{a_3 + a_4}{2} & \frac{a_1 - a_4}{2} & \frac{a_2 - a_3}{2} \end{pmatrix}.$$

This time we indicate successive transformations at the right side of the transposed matrices. Furthermore, we employ the notation $A \sim B$ which means that A and B are similar:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ \frac{a_1 + a_4}{2} & \frac{a_2 + a_3}{2} & -\frac{a_2 - a_5}{2} & -\frac{a_3 - a_4}{2} \\ \frac{a_2 + a_5}{2} & \frac{a_3 + a_4}{2} & \frac{a_1 - a_4}{2} & \frac{a_2 - a_3}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ a_1 + a_4 & a_2 + a_3 & -a_2 + a_5 & -a_3 + a_4 \\ a_2 + a_5 & a_3 + a_4 & a_1 - a_4 & a_2 - a_3 \end{pmatrix}$$

($\Leftarrow$ by $E_{row}(2; 2)$ and $E_{row}(3; 2)$)

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 + a_4 & -a_1 + a_2 + a_3 - a_4 & -a_2 + a_5 & -a_3 + a_4 \\ a_2 + a_5 & -a_2 + a_3 + a_4 - a_5 & a_1 - a_4 & a_2 - a_3 \end{pmatrix}$$

($\Leftarrow$ by $E_{column}(1, 2; -1)$)

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a_1 + a_2 + a_3 - a_4 & -a_2 + a_5 & -a_3 + a_4 \\ 0 & -a_2 + a_3 + a_4 - a_5 & a_1 - a_4 & a_2 - a_3 \end{pmatrix}$$

($\Leftarrow$ by $E_{row}(1, 2; -a_1 - a_4)$ and $E_{row}(1, 3; -a_2 - a_5)$)

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a_1 + a_5 & -a_2 + a_5 & -a_3 + a_4 \\ 0 & a_1 - a_5 & a_1 - a_4 & a_2 - a_3 \end{pmatrix}$$

($\Leftarrow$ by $E_{column}(3, 2; 1)$ and $E_{column}(4, 2; 1)$)

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a_1 + a_5 & -a_2 + a_5 & -a_3 + a_4 \\ 0 & 0 & a_1 - a_2 - a_4 + a_5 & a_2 - 2a_3 + a_4 \end{pmatrix}$$

($\Leftarrow$ by $E_{row}(2, 3; 1)$).

Therefore, when $-a_1 + a_5 \neq 0$, we see that $rank\, M_4 \leq 2$ if and only if every entry in the last row vanishes, namely, $a_1 + a_5 = a_2 + a_4 = 2a_3$. On the other hand, when $-a_1 + a_5 = 0$, we see that $rank\, M_4 \leq 2$ if and only if the determinant of the 2×2-matrix composed of the south-east entries is equal to zero. We can compute its determinant keeping in mind that $a_5 = a_1$ as follows:

$$\det\begin{pmatrix} -a_2 + a_5 & -a_3 + a_4 \\ a_1 - a_2 - a_4 + a_5 & a_2 - 2a_3 + a_4 \end{pmatrix}$$

$$= \det\begin{pmatrix} a_1 - a_2 & -a_3 + a_4 \\ 2a_1 - a_2 - a_4 & a_2 - 2a_3 + a_4 \end{pmatrix}$$

$$= \det\begin{pmatrix} a_1 - a_2 & -a_3 + a_4 \\ a_1 - a_4 & a_2 - a_3 \end{pmatrix}$$

($\Leftarrow$ by $E_{row}(1, 2; -1)$)

$$= \det\begin{pmatrix} a_1 - a_2 + a_3 - a_4 & -a_3 + a_4 \\ a_1 - a_2 + a_3 - a_4 & a_2 - a_3 \end{pmatrix}$$

($\Leftarrow$ by $E_{column}(2, 1; -1)$)

$$= \det\begin{pmatrix} a_1 - a_2 + a_3 - a_4 & -a_3 + a_4 \\ 0 & a_2 - a_4 \end{pmatrix}$$

$$(\Leftarrow \text{ by } E_{row}(1,\ 2;\ -1))$$

$$= (a_1 - a_2 + a_3 - a_4)(a_2 - a_4).$$

Therefore, we obtain the following:

Proposition 2.2. *For a melody* $\mathbf{a} = (a_1, ..., a_5)$ *of length five, the M-graph* $M(\mathbf{a})$ *has a line of symmetry if and only if one of the following conditions holds*:

(1) $a_1 + a_5 = a_2 + a_4 = 2a_3$,

(2) $a_1 = a_5$ *and* $a_2 = a_4$,

(3) $a_1 = a_5$ *and* $a_1 + a_3 = a_2 + a_4$.

Remark 1. As is stated in Proposition 2.1, in case of (3), the quadrilateral $p_1 p_2 p_3 p_4$ constitutes a square.

Remark 2. When we deal with symmetry of melodies of arbitrary length in Section 3, we must assume that $N \geq 6$. This is another reason why we choose the cases when $N = 4$ or $N = 5$ to begin our description in these two Subsections 2.1 and 2.2.

2.3. The case when $N = 12$

In this subsection, we examine the case when $N = 12$. The reason why we choose this case is two-fold. Firstly, it is directly connected with the twelve-tone method and may be of useful for actual composition. Secondly, it helps us to understand what kind of elementary transformations should be employed in general.

Let $\mathbf{a} = (a_1, ..., a_{12})$ be a melody of length twelve, and let $p_i = (a_i, a_{i+1})$ for any $i \in [1, 11]$ so that $M(\mathbf{a}) = (p_1, ..., p_{11})$. In this case, the coordinates x_i^+, y_i^+, x_i^-, y_i^- $(1 \leq i \leq 5)$ are given by

$$x_1^+ = \frac{a_1 + a_{11}}{2}, \quad x_1^- = \frac{a_1 - a_{11}}{2},$$

$$y_1^+ = \frac{a_2 + a_{12}}{2}, \quad y_1^- = \frac{a_2 - a_{12}}{2},$$

$$x_2^+ = \frac{a_2 + a_{10}}{2}, \quad x_2^- = \frac{a_2 - a_{10}}{2},$$

$$y_2^+ = \frac{a_3 + a_{11}}{2}, \quad y_2^- = \frac{a_3 - a_{11}}{2},$$

$$x_3^+ = \frac{a_3 + a_9}{2}, \quad x_3^- = \frac{a_3 - a_9}{2},$$

$$y_3^+ = \frac{a_4 + a_{10}}{2}, \quad y_3^- = \frac{a_4 - a_{10}}{2},$$

$$x_4^+ = \frac{a_4 + a_8}{2}, \quad x_4^- = \frac{a_4 - a_8}{2},$$

$$y_4^+ = \frac{a_5 + a_9}{2}, \quad y_4^- = \frac{a_5 - a_9}{2},$$

$$x_5^+ = \frac{a_5 + a_7}{2}, \quad x_5^- = \frac{a_5 - a_7}{2},$$

$$y_5^+ = \frac{a_6 + a_8}{2}, \quad y_5^- = \frac{a_6 - a_8}{2}$$

and

$$x_6 = a_6, \quad y_6 = a_7.$$

Therefore, the matrix M_{11} is given by

$$M_{11} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_1^+ & x_2^+ & x_3^+ & x_4^+ & x_5^+ & a_6 & -y_1^- & -y_2^- & -y_3^- & -y_4^- & -y_5^- \\ y_1^+ & y_2^+ & y_3^+ & y_4^+ & y_5^+ & a_7 & x_1^- & x_2^- & x_3^- & x_4^- & x_5^- \end{pmatrix}.$$

Applying $E_{row}(2; 2)$ and $E_{row}(3; 2)$ to M_{11}, it becomes

$$M^1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1Y & 2X & 39 & 48 & 57 & 66 & \overline{2}Z & \overline{3}Y & \overline{4}X & \overline{5}9 & \overline{6}8 \\ 2Z & 3Y & 4X & 59 & 68 & 77 & 1\overline{Y} & 2\overline{X} & 3\overline{9} & 4\overline{8} & 5\overline{7} \end{pmatrix},$$

where "X, Y, Z" stand for "10, 11, 12", respectively, and we employ the notation, which abbreviates $-a_5 + a_9$ as $\bar{5}9$, for example. By applying $E_{column}(5, 6; -1)$, $E_{column}(4, 5; -1)$, ..., $E_{column}(1, 2; -1)$ successively to M^1, we see that M^1 is similar to the following matrix:

$$M^1 \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1Y & \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{3}48\bar{9} & \bar{4}57\bar{8} & \bar{5}66\bar{7} & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ 2Z & \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & \bar{4}59\bar{X} & \bar{5}68\bar{9} & \bar{6}77\bar{8} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}.$$

At this point, we notice that the rank of M^1 depends only on the following matrix M^2, which is obtained from the last matrix by deleting its first row and column:

$$M^2 = \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{3}48\bar{9} & \bar{4}57\bar{8} & \bar{5}66\bar{7} & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & \bar{4}59\bar{X} & \bar{5}68\bar{9} & \bar{6}77\bar{8} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}. \tag{2.2}$$

Starting with this matrix, we continue to apply several elementary transformations as follows:

$$M^2 \sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{3}48\bar{9} & \bar{4}57\bar{8} & \bar{4}66\bar{8} & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & \bar{4}59\bar{X} & \bar{5}68\bar{9} & \bar{5}77\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

($\Leftarrow$ by $E_{column}(4, 5; 1)$)

$$\sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{3}48\bar{9} & \bar{4}57\bar{8} & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & \bar{4}59\bar{X} & \bar{5}68\bar{9} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

($\Leftarrow$ by $E_{column}(10, 5; 1)$)

$$\sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{3}48\bar{9} & \bar{3}57\bar{9} & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & \bar{4}59\bar{X} & \bar{4}68\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

($\Leftarrow$ by $E_{column}(3, 4; 1)$)

$$\sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{3}48\bar{9} & \bar{3}7 & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & \bar{4}59\bar{X} & 6\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

$(\Leftarrow \text{by } E_{column}(9,\ 4;\ 1))$

$$\sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{2}48\bar{X} & \bar{3}7 & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & \bar{3}59\bar{Y} & 6\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

$(\Leftarrow \text{by } E_{column}(2,\ 3;\ 1))$

$$\sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{2}39\bar{X} & \bar{2}8 & \bar{3}7 & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{3}4X\bar{Y} & 5\bar{Y} & 6\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

$(\Leftarrow \text{by } E_{column}(8,\ 3;\ 1))$

$$\sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{1}39\bar{Y} & \bar{2}8 & \bar{3}7 & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & \bar{2}4X\bar{Z} & 5\bar{Y} & 6\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

$(\Leftarrow \text{by } E_{column}(1,\ 2;\ 1))$

$$\sim \begin{pmatrix} \bar{1}2X\bar{Y} & \bar{1}9 & \bar{2}8 & \bar{3}7 & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3Y\bar{Z} & 4\bar{Z} & 5\bar{Y} & 6\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

$(\Leftarrow \text{by } E_{column}(7,\ 2;\ 1))$

$$\sim \begin{pmatrix} 2\bar{9}X\bar{Y} & \bar{1}9 & \bar{2}8 & \bar{3}7 & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}3\bar{4}Y & 4\bar{Z} & 5\bar{Y} & 6\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

$(\Leftarrow \text{by } E_{column}(2,\ 1;\ -1))$

$$\sim \begin{pmatrix} 24\bar{9}\bar{Y} & \bar{1}9 & \bar{2}8 & \bar{3}7 & \bar{4}6 & \bar{2}Z & \bar{3}Y & \bar{4}X & \bar{5}9 & \bar{6}8 \\ \bar{2}\bar{4}9Y & 4\bar{Z} & 5\bar{Y} & 6\bar{X} & 7\bar{9} & 1\bar{Y} & 2\bar{X} & 3\bar{9} & 4\bar{8} & 5\bar{7} \end{pmatrix}$$

$(\Leftarrow \text{by } E_{column}(8,\ 1;\ -1)).$ (2.3)

Let M^3 denote the last matrix, and $M^3_{1\cdot}$ (resp. $M^3_{2\cdot}$) denote the first (resp. second) row of M^3. Then the rank of M^3 is smaller than or equal to one if and only if there exists a pair $(s, t) \in \mathbf{R}^2 - \{(0, 0)\}$ such that

$$sM_{1.}^3 + tM_{2.}^3 = \mathbf{0}, \tag{2.4}$$

where the right hand side denotes the zero vector of length ten. We divide our argument into two cases according as $t \neq 0$ or $t = 0$.

Case (A): $t \neq 0$**.** We may assume that $t = 1$. The coefficient matrix C of the simultaneous equation (2.4) with respect to $a_1, ..., a_{12}$ is found to be

$$C = \begin{pmatrix} 0 & s-1 & 0 & s-1 & 0 & 0 & 0 & 0 & -s+1 & 0 & -s+1 & 0 \\ -s & 0 & 0 & 1 & 0 & 0 & 0 & 0 & s & 0 & 0 & -1 \\ 0 & -s & 0 & 0 & 1 & 0 & 0 & s & 0 & 0 & -1 & 0 \\ 0 & 0 & -s & 0 & 0 & 1 & s & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -s & 0 & s & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & s \\ 0 & 1 & -s & 0 & 0 & 0 & 0 & 0 & 0 & -1 & s & 0 \\ 0 & 0 & 1 & -s & 0 & 0 & 0 & 0 & -1 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & -s & 0 & 0 & -1 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -s & -1 & s & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let c_i, $1 \leq i \leq 10$, denote the ith row of the matrix C. Then we see that

$$c_3 - c_6 = (-1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; s \;\; 0 \;\; 0 \;\; 0 \;\; -s),$$

$$c_2 - c_9 = (-s \;\; 0 \;\; 0 \;\; 0 \;\; s \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1)$$

and hence

$$c_3 - c_6 - s(c_2 - c_9) = (s^2 - 1 \;\; 0 \;\; 0 \;\; 0 \;\; -s^2 + 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)$$

$$= (s^2 - 1)(1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0).$$

Similarly, we have the following identities:

$$c_4 - c_7 - s(c_3 - c_{10}) = (s^2 - 1)(0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0),$$

$$c_5 - c_8 - sc_4 = (s^2 - 1)(0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0),$$

$$-c_9 + s(c_5 + c_{10}) = (s^2 - 1)(0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0),$$

$$c_5 - c_{10} + sc_9 = (s^2 - 1)(0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0),$$

$$-c_4 + s(c_5 - c_8) = (s^2 - 1)(0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0),$$

$$-c_3 + c_{10} + s(c_4 - c_7) = (s^2 - 1)(0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0),$$

$$-c_2 + c_9 + s(c_3 - c_6) = (s^2 - 1)(0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -1).$$

It follows that when $s^2 \neq 1$, we have

$$a_i = a_j, \text{ whenever } i \equiv j \pmod 4$$

for any $1 \leq i, \; j \leq 12$. Accordingly, by setting $a_5 = a_1, a_6 = a_2, \ldots,$ the equation $C^T(a_1 \;\; a_2 \;\; \cdots \;\; a_{12}) = \mathbf{0}$ is reduced to the equation

$$C'^T(a_1 \;\; a_2 \;\; a_3 \;\; a_4) = \mathbf{0},$$

where

$$C' = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -s & -1 & s \end{pmatrix}.$$

Namely, we arrive at the simultaneous equations

$$a_1 - a_2 + a_3 - a_4 = 0, \tag{2.5}$$

$$a_1 - sa_2 - a_3 + sa_4 = 0. \tag{2.6}$$

We notice that equation (2.5) implies that the four points $p_i = (a_i, a_{i+1})$, $1 \leq i \leq 4$, constitute a square by Proposition 2.1. By periodicity, we see that the sequence $\{p_i\}$ rotates along the square with vertices $p_1, \; p_2, \; p_3, \; p_4$. On the other hand, if $a_2 = a_4$, then equation (2.6) holds for any s under the condition that $a_1 = a_3$. In this case, (2.5) implies that $a_1 = a_2$ too, hence all of $a_i \; (1 \leq i \leq N)$ are one and the same, and the M-graph consists of only one point. Therefore, except for this trivial case, we have $a_2 \neq a_4$, and (2.6) implies that

$$s = \frac{a_1 - a_3}{a_2 - a_4}.$$

Next, we deal with the cases when $s^2 = 1$.

Case (A; 1): $s = 1$**.** In this case, the matrix C becomes

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

hence the equation $C^T(a_1 \;\; a_2 \;\; \cdots \;\; a_{12}) = \mathbf{0}$ is equivalent to the simultaneous equation

$$a_1 + a_{12} = a_2 + a_{11} = \cdots = a_6 + a_7.$$

Case (A; –1): $s = -1$**.** In this case, the matrix C becomes

$$C = \begin{pmatrix} 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the matrix is antisymmetric with respect to the transformation $a_i \mapsto a_{13-i}$ $(1 \le i \le 6)$, the equation $C^T(a_1 \;\; a_2 \;\; \cdots \;\; a_{12}) = \mathbf{0}$ is reduced to the equation $C'^T(b_1 \;\; b_2 \;\; \cdots \;\; b_6) = \mathbf{0}$, where $b_i = a_i - a_{13-i}$ $(1 \le i \le 6)$ and

$$C' = \begin{pmatrix} 0 & -2 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The equations corresponding to the ith row with $2 \le i \le 10$ hold if and only if

$$b_1 = -b_2 = b_3 = -b_4 = b_5 = -b_6.$$

But the first row forces b_2 to be equal to $-b_4$. It follows that

$$b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0,$$

which is equivalent to the condition that

$$a_i = a_{13-i}$$

holds for any $i \in [1, 6]$.

Case (B): $t = 0$. In this case, it follows from (2.4) that $M^3_{1.} = \mathbf{0}$, which is equivalent to the equalities

$$a_2 = a_4 = \cdots = a_{12},$$

$$a_1 = a_5 = a_9,$$

$$a_3 = a_7 = a_{11},$$

$$a_1 + a_3 = 2a_2.$$

We record what we obtain in this subsection as follows:

Proposition 2.3. *Let* $\mathbf{a} = (a_1, ..., a_{12})$ *be a melody of length twelve, and let* $p_i = (a_i, a_{i+1}),\ 1 \le i \le 11$ *so that* $M(\mathbf{a}) = (p_1, ..., p_{11})$. *The ordered set* $M(\mathbf{a})$ *of points has a line of symmetry if and only if one of the following conditions holds*:

(1) $a_i = a_{i+4}$ *for any* $i \in [1, 8]$, *and* $a_1 + a_3 = a_2 + a_4$.

(2) $a_1 + a_{12} = a_2 + a_{11} = \cdots = a_6 + a_7$.

(3) $a_1 = a_{12},\ a_2 = a_{11},\ ...,\ a_6 = a_7$.

(4) $a_2 = a_4 = \cdots = a_{12},\ \ a_1 = a_5 = a_9,\ \ a_3 = a_7 = a_{11}$ *and* $a_1 + a_3 = 2a_2$.

3. Symmetry of Melodies of Arbitrary Length

In this section, we generalize the argument in the previous section, especially that for the melodies of length twelve, and obtain a complete classification of melodies whose *M*-graphs have lines of symmetry. We divide our argument into two cases according to the parity of the length. In each case, our final classification result is stated at the end of respective subsection.

3.1. Melodies of length $N = 2m$

Let $\mathbf{a} = (a_1, ..., a_{2m})$ be a melody of length $2m$ with $m \ge 3$, and let $p_i = (a_i, a_{i+1}),\ 1 \le i \le 2m-1$ so that $M(\mathbf{a}) = (p_1, ..., p_{2m-1})$. In this case, the coordinates $x_i^+,\ y_i^+,\ x_i^-,\ y_i^-\ (1 \le i \le m-1)$ are given by

$$x_i^+ = \frac{a_i + a_{2m-i}}{2}, \quad x_1^- = \frac{a_i - a_{2m-i}}{2},$$

$$y_i^+ = \frac{a_{i+1} + a_{2m+1-i}}{2}, \quad y_i^- = \frac{a_{i+1} - a_{2m+1-i}}{2}$$

and

$$x_m = a_m, \quad y_m = a_{m+1}.$$

Therefore, the matrix M_{2m-1} is given by

$$M_{2m-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ x_1^+ & x_2^+ & \cdots & x_{m-1}^+ & a_m & -y_1^- & -y_2^- & \cdots & -y_{m-1}^- \\ y_1^+ & y_2^+ & \cdots & y_{m-1}^+ & a_{m+1} & x_1^- & x_2^- & \cdots & x_{m-1}^- \end{pmatrix}.$$

Apply $E_{row}(2; 2)$ and $E_{row}(3; 2)$ to M_{2m-1} and call the resulting matrix M^1. Its (i, j)-entry $(M^1)_{ij}$ is given by

$$(M^1)_{1j} = \begin{cases} 1, & 1 \le j \le m, \\ 0, & m+1 \le j \le 2m-1, \end{cases}$$

$$(M^1)_{2j} = \begin{cases} (j,\ 2m-j), & 1 \le j \le m, \\ (\overline{j-m+1},\ 3m+1-j), & m+1 \le j \le 2m-1 \end{cases}$$

and

$$(M^1)_{3j} = \begin{cases} (j+1,\ 2m+1-j), & 1 \le j \le m, \\ (j-m,\ \overline{3m-j}), & m+1 \le j \le 2m-1, \end{cases}$$

where we employ the notation, which abbreviates $a_1 - a_{2m-1}$ as $(1,\ \overline{2m-1})$, for example. By applying the elementary transformations

$$E_{column}(m-1,\ m;\ -1),\ E_{column}(m-2,\ m-1;\ -1),\ \ldots,\ E_{column}(1,\ 2;\ -1)$$

successively, we see that M^1 is equivalent to the matrix $M^{1'}$ whose (i, j)-entry $(M^{1'})_{ij}$ is given by

$$(M^{1'})_{1j} = \begin{cases} 1, & j = 1, \\ 0, & 2 \le j \le 2m-1, \end{cases} \tag{3.1}$$

$$(M^{1'})_{2j} = \begin{cases} (1,\ 2m-1), & j = 1, \\ (\overline{j-1},\ j,\ 2m-j,\ \overline{2m+1-j}), & 2 \le j \le m, \\ (\overline{j-m+1},\ 3m+1-j), & m+1 \le j \le 2m-1 \end{cases} \tag{3.2}$$

and

$$(M^{1'})_{3j} = \begin{cases} (2,\ 2m), & j = 1, \\ (\bar{j},\ j+1,\ 2m+1-j,\ \overline{2m+2-j}), & 2 \le j \le m, \\ (j-m,\ \overline{3m-j}), & m+1 \le j \le 2m-1. \end{cases} \tag{3.3}$$

It follows from (3.1) that the rank of $M^{1'}$ depends only on the $2 \times (2m-2)$ matrix M^2, which is obtained from $M^{1'}$ by deleting its first row and column. Its $(i,\ j)$-entry $(M^2)_{ij}$ is obtained simply by replacing j by $j+1$ in (3.2) and (3.3):

$$(M^2)_{1j} = \begin{cases} (\bar{j},\ j+1,\ 2m-j-1,\ \overline{2m-j}), & 1 \le j \le m-1, \\ (\overline{j-m+2},\ 3m-j), & m \le j \le 2m-2, \end{cases}$$

$$(M^2)_{2j} = \begin{cases} (\overline{j+1},\ j+2,\ 2m-j,\ \overline{2m+1-j}), & 1 \le j \le m-1, \\ (j-m+1,\ \overline{3m-j-1}), & m \le j \le 2m-2. \end{cases}$$

Apply $E_{column}(j-1,\ j;1)$ and $E_{column}(j+m-1,\ j;1)$ for each $j \in [2,\ m-1]$ to the matrix M^2, and we call the resulting matrix $M^{2'}$. Then the $(1,\ j)$-entry $M^{2'}_{1j}$ for $j \in [2,\ m-1]$ is computed to be

$$\begin{aligned} M^{2'}_{1j} &= (\bar{j},\ j+1,\ 2m-1-j,\ \overline{2m-j}) + (\overline{j-1},\ j,\ 2m-j,\ \overline{2m+1-j}) \\ &\quad + (\overline{j+1},\ 2m+1-j) \\ &= (\overline{j-1},\ 2m-1-j). \end{aligned}$$

Similarly, we have

$$\begin{aligned} M^{2'}_{2j} &= (\overline{j+1},\ j+2,\ 2m-j,\ \overline{2m+1-j}) \\ &\quad + (\bar{j},\ j+1,\ 2m+1-j,\ \overline{2m+2-j}) + (j,\ \overline{2m-j}) \\ &= (j+2,\ \overline{2m+2-j}). \end{aligned}$$

Thus, we see that the whole entries of $M^{2'}$ are given by

$$M_{1j}^{2'} = \begin{cases} (\overline{1},\ 2,\ 2m-2,\ \overline{2m-1}), & j = 1, \\ (\overline{j-1},\ 2m-j-1), & 2 \le j \le m-1, \\ (\overline{j-m+2},\ 3m-j), & m \le j \le 2m-2 \end{cases}$$

and

$$M_{2j}^{2'} = \begin{cases} (\overline{2},\ 3,\ 2m-1,\ \overline{2m}), & j = 1, \\ (j+2,\ \overline{2m+2-j}), & 2 \le j \le m-1, \\ (j-m+1,\ \overline{3m-j-1}), & m \le j \le 2m-2. \end{cases}$$

When $m \ge 4$, we apply $E_{column}(2, 1; -1)$ and $E_{column}(m+2, 1; -1)$ to the matrix $M^{2'}$, which is a $2 \times (2m-2)$ matrix and has the $(m+2)$th column since $m+2 \le 2m-2$ for $m \ge 4$. Then the entries of the first column become as follows:

$$M_{11}^{2'} \Rightarrow (\overline{1},\ 2,\ 2m-2,\ \overline{2m-1}) - (\overline{1},\ 2m-3) - (\overline{4},\ 2m-2)$$

$$= (2,\ 4,\ \overline{2m-3},\ \overline{2m-1}),$$

$$M_{21}^{2'} \Rightarrow (\overline{2},\ 3,\ 2m-1,\ \overline{2m}) - (4,\ \overline{2m}) - (3,\ \overline{2m-3})$$

$$= (\overline{2},\ \overline{4},\ 2m-3,\ 2m-1).$$

On the other hand, when $m = 3$, we apply only $E_{column}(2, 1; -1)$ to the matrix $M^{2'}$, then we see that

$$M_{11}^{2'} \Rightarrow (\overline{1},\ 2,\ 4,\ \overline{5}) - (\overline{1},\ 3) = (2,\ 4,\ \overline{3},\ \overline{5}),$$

$$M_{21}^{2'} \Rightarrow (\overline{2},\ 3,\ 5,\ \overline{6}) - (4,\ \overline{6}) = (\overline{2},\ \overline{4},\ 3,\ 5).$$

Therefore, in any case, the entries of the resulting matrix, which we call M^3, are given by

$$M_{1j}^3 = \begin{cases} (2,\ 4,\ \overline{2m-3},\ \overline{2m-1}), & j = 1, \\ (\overline{j-1},\ 2m-j-1), & 2 \le j \le m-1, \\ (\overline{j-m+2},\ 3m-j), & m \le j \le 2m-2, \end{cases}$$

$$M_{2j}^3 = \begin{cases} (\overline{2},\ \overline{4},\ 2m-3,\ 2m-1), & j = 1, \\ (j+2,\ \overline{2m+2-j}), & 2 \le j \le m-1, \\ (j-m+1,\ \overline{3m-j-1}), & m \le j \le 2m-2. \end{cases}$$

Let $M_{1\cdot}^3$ (resp. $M_{2\cdot}^3$) denote the first (resp. second) row of M^3. Then the rank of M^3 is smaller than or equal to one if and only if there exists a pair $(s,\ t) \in \mathbf{R}^2 - \{(0,\ 0)\}$ such that

$$sM_{1\cdot}^3 + tM_{2\cdot}^3 = \mathbf{0}, \tag{3.4}$$

where the right hand side denotes the zero vector of length $2m-2$. Let c_i, $1 \le i \le 2m-2$, denote the ith component of the vector on the left hand side of (3.4). Then we have

$$c_1 = s(2,\ 4,\ \overline{2m-3},\ \overline{2m-1}) + t(\overline{2},\ \overline{4},\ 2m-3,\ 2m-1),$$

$$c_2 = s(\overline{1},\ 2m-3) + t(4,\ \overline{2m}),$$

$$c_3 = s(\overline{2},\ 2m-4) + t(5,\ \overline{2m-1}),$$

$$\dots$$

$$c_j = s(\overline{j-1},\ 2m-1-j) + t(j+2,\ \overline{2m+2-j}),\ (2 \le j \le m-1),$$

$$\dots$$

$$c_{m-1} = s(\overline{m-2},\ m) + t(m+1,\ \overline{m+3}),$$

$$c_m = s(\overline{2},\ 2m) + t(1,\ \overline{2m-1}),$$

$$c_{m+1} = s(\overline{3},\ 2m-1) + t(2,\ \overline{2m-2}),$$

$$\ldots$$

$$c_{m+k} = s(\overline{k+2}, 2m-k) + t(k+1, \overline{2m-1-k}), \ (0 \le k \le m-2),$$

$$\ldots$$

$$c_{2m-2} = s(\overline{m}, m+2) + t(m-1, \overline{m+1}).$$

We divide our argument into two cases according as $t \neq 0$ or $t = 0$.

Case (A): $t \neq 0$**.** We may assume that $t = 1$. Then for any $j \in [3, m-1]$, we have

$$\begin{aligned} c_j - c_{m-3+j} &= (s(\overline{j-1}, 2m-1-j) + (j+2, \overline{2m+2-j})) \\ &\quad - (s(\overline{j-1}, 2m+3-j) + (j-2, \overline{2m+2-j})) \\ &= -a_{j-2} + a_{j+2} + sa_{2m-1-j} - sa_{2m+3-j}. \end{aligned} \tag{3.5}$$

Similar vectors can be obtained for any $j \in [2, m-3]$ as

$$\begin{aligned} c_j - c_{m+1+j} &= (s(\overline{j-1}, 2m-1-j) + (j+2, \overline{2m+2-j})) \\ &\quad - (s(\overline{j+3}, 2m-1-j) + (j+2, \overline{2m-2-j})) \\ &= -sa_{j-1} + sa_{j+3} + a_{2m-2-j} - a_{2m+2-j}. \end{aligned}$$

By replacing *j* by $j-1$ here, we see that

$$c_{j-1} - c_{m+j} = -sa_{j-2} + sa_{j+2} + a_{2m-1-j} - a_{2m+3-j} \tag{3.6}$$

holds for any $j \in [3, m-2]$. Combining (3.5) and (3.6), we find that the equalities

$$(c_j - c_{m+1+j}) - s(c_{j-1} - c_{m+j}) = (s^2 - 1)(a_{j-2} - a_{j+2}),$$

$$-s(c_j - c_{m+1+j}) + (c_{j-1} - c_{m+j}) = (1 - s^2)(a_{2m-1-j} - a_{2m+3-j})$$

hold for any $j \in [3, m-2]$. It follows that when $s^2 \neq 1$, the equalities

$$a_{j-2} = a_{j+2}, \tag{3.7}$$

$$a_{2m-1-j} = a_{2m+3-j} \tag{3.8}$$

hold for any $j \in [3, m-2]$. It follows from (3.7) that

$$\begin{cases} a_1 = a_5, \\ a_2 = a_6, \\ \quad \cdots \\ a_{m-4} = a_m \end{cases} \tag{3.9}$$

and it follows from (3.8) that

$$\begin{cases} a_{m+1} = a_{m+5}, \\ a_{m+2} = a_{m+6}, \\ \quad \cdots \\ a_{2m-4} = a_{2m}. \end{cases} \tag{3.10}$$

In order to fill the gap, we employ the equality

$$c_{m-1} - c_{2m-4} = (-a_{m-3} + a_{m+1}) + s(a_m - a_{m+4}),$$

which comes from (3.5) with $j = m-1$ and the equality

$$\begin{aligned} c_{m-2} &= s(\overline{m-3}, m+1) + (m, \overline{m+4}) \\ &= s(-a_{m-3} + a_{m+1}) + a_m - a_{m+4}. \end{aligned}$$

These imply the following equalities:

$$c_{m-1} - c_{2m-4} - sc_{m-2} = (s^2 - 1)(a_{m-3} - a_{m+1}),$$

$$s(c_{m-1} - c_{2m-4}) - c_{m-2} = (s^2 - 1)(a_m - a_{m+4}).$$

Therefore, when $s^2 \neq 1$, we have

$$a_{m-3} = a_{m+1}, \tag{3.11}$$

$$a_m = a_{m+4}. \tag{3.12}$$

Furthermore, in order to fill the final gap, we use $c_{m-1} + c_{2m-2}$ and c_{2m-3}, which are expressed as

$$\begin{aligned} c_{m-1} + c_{2m-2} &= (s(\overline{m-2},\ m) + (m+1,\ \overline{m+3})) \\ &\quad + (s(\overline{m},\ m+2) + (m-1,\ \overline{m+1})) \\ &= s(-a_{m-2} + a_{m+2}) + (a_{m-1} - a_{m+3}), \\ c_{2m-3} &= s(-a_{m-1} + a_{m+3}) + (a_{m-2} - a_{m+2}). \end{aligned}$$

Therefore, we have

$$s(c_{m-1} + c_{2m-2}) + c_{2m-3} = (1 - s^2)(a_{m-2} - a_{m+2}),$$

$$(c_{m-1} + c_{2m-2}) + sc_{2m-3} = (1 - s^2)(a_{m-1} - a_{m+3}).$$

It follows that when $s^2 \neq 1$, we have

$$a_{m-2} = a_{m+2}, \tag{3.13}$$

$$a_{m-1} = a_{m+3}. \tag{3.14}$$

Combining (3.9)-(3.14), we obtain the following complete periodicity:

Proposition 3.1. *When* $s^2 \neq 1$, *the equality*

$$a_j = a_{j+4}$$

holds for any $j \in [1,\ 2m-4]$.

It follows from this proposition that the components c_j, $1 \leq j \leq 2m-2$, can be expressed solely by a_1, a_2, a_3, a_4. As for $c_1 = (s-1)(2,\ 4,\ \overline{2m-3},\ \overline{2m-1})$, we note that the set $\{2m-3,\ 2m-1\}$ coincides with the set $\{1,\ 3\}$ modulo four. Hence, it is expressed as

$$c_1 = (s-1)(-a_1 + a_2 - a_3 + a_4).$$

As for c_j with $j \in [2, m-1]$, it follows from the expression

$$c_j = s(\overline{j-1}, 2m-1-j) + (j+2, \overline{2m+2-j})$$

that the differences of the indices of both terms on the right hand side are equal to $2(m-j)$, which is congruent to 0 modulo 4 when $m \equiv j \pmod 2$. Hence, if $m \equiv j \pmod 2$, then c_j always vanishes. On the other hand, when $m \not\equiv j \pmod 2$, we see that

$$c_j = \begin{cases} \pm(a_1 - sa_2 - a_3 + sa_4), & \text{if } m \equiv 0 \pmod 2, \\ \pm(sa_1 + a_2 - sa_3 - a_4), & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

Next, we consider the case $j \in [m, 2m-2]$ so that

$$c_j = s(\overline{j-m+2}, 3m-j) + (j-m+1, \overline{3m-1-j}).$$

Note that the differences of the indices of both terms on the right hand side are equal to $4m-2(j+1)$, which is congruent to 0 if $j \equiv 1 \pmod 2$. Hence, in this case, c_j vanishes. On the other hand, when $j \equiv 0 \pmod 2$, we see that

$$c_j = \begin{cases} \pm(a_1 - sa_2 - a_3 + sa_4), & \text{if } m \equiv 0 \pmod 2, \\ \pm(sa_1 + a_2 - sa_3 - a_4), & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

Thus, the simultaneous equation $(c_1, c_2, \ldots, c_{2m-2}) = \mathbf{0}$ is reduced according to the parity of m to the following two equations:

When m is even,

$$\begin{cases} a_1 - a_2 + a_3 - a_4 = 0, \\ a_1 - sa_2 - a_3 + sa_4 = 0. \end{cases}$$

When m is odd,

$$\begin{cases} a_1 - a_2 + a_3 - a_4 = 0, \\ sa_1 + a_2 - sa_3 - a_4 = 0. \end{cases}$$

Therefore, when m is even, the simultaneous equation has the same form as the one obtained in Subsection 2.3, hence the sequence $\{p_i\}$ rotates along the square with vertices p_1, p_2, p_3, p_4, and except for the trivial case when $a_1 = \cdots = a_{2m}$, we have

$$s = \frac{a_1 - a_3}{a_2 - a_4}.$$

Even when m is odd, the simultaneous equation can be solved similarly and we see that the sequence $\{p_i\}$ rotates along the square with vertices p_1, p_2, p_3, p_4 too and except for the trivial case when $a_1 = \cdots = a_{2m}$, we have

$$s = \frac{a_4 - a_2}{a_1 - a_3}.$$

Next, we deal with the cases when $s^2 = 1$.

Case (A; 1): $s = 1$**.** In this case, let $b_k = a_k + a_{2m+1-k}$ for any $k \in [1, m]$. Then all of the components c_j, $j \in [1, 2m-2]$ can be expressed in terms of b_k as follows:

$$\begin{aligned}
c_1 &= 0,\\
c_2 &= -b_1 + b_4,\\
c_3 &= -b_2 + b_5,\\
&\cdots\\
c_{m-2} &= -b_{m-3} + b_m,\\
c_{m-1} &= -b_{m-2} + b_m,\\
c_m &= b_1 - b_2,\\
c_{m+1} &= b_2 - b_3,\\
&\cdots\\
c_{2m-2} &= b_{m-1} - b_m.
\end{aligned}$$

Therefore, the simultaneous equation $(c_1, c_2, ..., c_{2m-2}) = \mathbf{0}$ is equivalent to the conditions

$$b_1 = b_2 = \cdots = b_m,$$

namely,

$$a_1 + a_{2m} = a_2 + a_{2m-1} = \cdots = a_m + a_{m+1}.$$

Case (A; –1): $s = -1$**.** In this case, let $d_k = a_k - a_{2m+1-k}$ for any $k \in [1, m]$. Then all of the components c_j, $j \in [1, 2m-2]$ can be expressed in terms of d_k as follows:

$$c_1 = -2(d_2 + d_4),$$

$$c_2 = d_1 + d_4,$$

$$c_3 = d_2 + d_5,$$

$$\cdots$$

$$c_{m-2} = d_{m-3} + d_m,$$

$$c_{m-1} = d_{m-2} - d_m,$$

$$c_m = d_1 + d_2,$$

$$c_{m+1} = d_2 + d_3,$$

$$\cdots$$

$$c_{2m-2} = d_{m-1} + d_m.$$

The condition that $c_j = 0$ holds for any $j \in [2, 2m-2]$ is equivalent to the equalities

$$d_1 = -d_2 = d_3 = \cdots = (-1)^{m-1} d_m.$$

But the first condition $c_1 = 0$ forces d_2 to be equal to $-d_4$. It follows that

$$d_1 = \cdots = d_m = 0,$$

which is equivalent to the condition that

$$a_i = a_{2m+1-i}$$

holds for any $i \in [1, m]$.

Case (B): $t = 0$**.** In this case, the condition becomes as follows:

$$C_0 : \quad (2, 4, \overline{2m-3}, \overline{2m-1}) = 0$$

and

$$\begin{cases} C_1 : & (\overline{1}, 2m-3) = 0, \\ C_2 : & (\overline{2}, 2m-4) = 0, \\ \cdots & \cdots \\ C_{m-2} : & (\overline{m-2}, m) = 0, \\ \overline{C}_2 : & (\overline{2}, 2m) = 0, \\ \overline{C}_3 : & (\overline{3}, 2m-1) = 0, \\ \cdots & \cdots \\ \overline{C}_m : & (\overline{m}, m+2) = 0. \end{cases} \tag{3.15}$$

Here for ease of description below, we name each equation as is indicated. We note that the equations $C_1, ..., C_{m-2}$ can be expressed as

$$a_i = a_{2m-2-i} \text{ for any } i \in [1, 2m-3] \tag{3.16}$$

and that the equations $\overline{C}_2, ..., \overline{C}_m$ can be expressed as

$$a_i = a_{2m+2-i} \text{ for any } i \in [2, 2m]. \tag{3.17}$$

We divide our argument into two cases according to the parity of m.

Proposition 3.2. *When m is even, the simultaneous equation* (3.15) *holds if and only if*

$$a_2 = a_4 = \cdots = a_{2m},$$

$$a_1 = a_5 = \cdots = a_{2m-3},$$

$$a_3 = a_7 = \cdots = a_{2m-1}.$$

When m is odd, it holds if and only if

$$a_1 = a_3 = \cdots = a_{2m-1},$$

$$a_2 = a_6 = \cdots = a_{2m},$$

$$a_4 = a_8 = \cdots = a_{2m-2}.$$

Proof. In either case, we notice that

$$a_j = a_{j+4}, \quad 1 \le j \le 2m - 4.$$

This is because the equality (3.16) with $i = j$ provides us with the equality

$$a_j = a_{2m-2-j}$$

and the equality (3.17) with $i = 2m - 2 - j$ shows that

$$a_{2m-2-j} = a_{2m+2-(2m-2-j)} = a_{j+4}.$$

Furthermore, putting $i = m$ in (3.16) and (3.17), we have

$$a_m = a_{m-2} = a_{m+2}.$$

Therefore, the "only-if" part of the proposition is shown to hold. The "if-part" is deduced from the following observation. The number of equations in (3.15) is equal to $(m-2)+(m-1) = 2m-3$, which is three less than the number of unknowns. Hence, the dimension of the space of solutions of (3.15) is at least equal to three. Thus, the conditions given in the statement of the proposition are sufficient for the validity of (3.15). This completes the proof. □

Combining the equality $C_0 : a_2 + a_4 - a_{2m-3} - a_{2m-1} = 0$ and Proposition 3.2, we obtain the following:

Proposition 3.3. *When m is even, the simultaneous equation* (3.15) *together with* C_0 *holds if and only if*

$$a_2 = a_4 = \cdots = a_m,$$

$$a_1 = a_5 = \cdots = a_{2m-3},$$

$$a_3 = a_7 = \cdots = a_{2m-1},$$

$$a_1 + a_3 = 2a_2.$$

When m is odd, the simultaneous equation (3.15) *together with* C_0 *holds if and only if*

$$a_1 = a_3 = \cdots = a_{2m-1},$$

$$a_2 = a_6 = \cdots = a_{2m},$$

$$a_4 = a_8 = \cdots = a_{2m-2},$$

$$a_2 + a_4 = 2a_1.$$

Thereby, we complete our classification of melodies of even length whose *M*-graph has lines of symmetry. The result is as follows:

Theorem 3.1. *Let* $\mathbf{a} = (a_1, ..., a_{2m})$ *be a melody of even length* $2m$ *with* $m \geq 3$ *and let* $p_i = (a_i, a_{i+1}),\ 1 \leq i \leq 2m-1$ *so that* $M(\mathbf{a}) = (p_1, ..., p_{2m-1})$. *The ordered set* $M(\mathbf{a})$ *of points has a line of symmetry if and only if one of the following conditions holds*:

(1) $a_i = a_{i+4}$ *for any* $i \in [1, 2m-4]$ *and* $a_1 + a_3 = a_2 + a_4$.

(2) $a_1 + a_{2m} = a_2 + a_{2m-1} = \cdots = a_m + a_{m+1}$.

(3) $a_1 = a_{2m},\ a_2 = a_{2m-1}, ..., a_m = a_{m+1}$.

(4.1) *When m is even,*

$$a_2 = a_4 = \cdots = a_{2m},$$

$$a_1 = a_5 = \cdots = a_{2m-3},$$

$$a_3 = a_7 = a_{2m-1}$$

$$\text{and } a_1 + a_3 = 2a_2.$$

(4.2) *When m is odd,*

$$a_1 = a_3 = \cdots = a_{2m-1},$$

$$a_2 = a_6 = \cdots = a_{2m},$$

$$a_4 = a_8 = a_{2m-2}$$

$$\textit{and } a_2 + a_4 = 2a_1.$$

3.2. General case when $N = 2m+1$

When the length of a melody is odd, the process of column-reducing is similar to the one employed in the even case. Here arises, however, a difficulty, which prevents us from obtaining the periodicity for the fiber at generic $(s, t) \in \mathbf{P}^1$. We need additional consideration to reach our conclusion.

Let $\mathbf{a} = (a_1, \ldots, a_{2m+1})$ be a melody of length $2m+1$ with $m \geq 3$, and let $p_i = (a_i, a_{i+1})$ for any $i \in [1, 2m]$ so that $M(\mathbf{a}) = (p_1, \ldots, p_{2m})$. In this case, the coordinates x_i^+, y_i^+, x_i^-, y_i^- for $i \in [1, m]$ are given by

$$x_i^+ = \frac{a_i + a_{2m+1-i}}{2}, \quad x_1^- = \frac{a_i - a_{2m+1-i}}{2},$$

$$y_i^+ = \frac{a_{i+1} + a_{2m+2-i}}{2}, \quad y_i^- = \frac{a_{i+1} - a_{2m+2-i}}{2}$$

and the matrix M_{2m} is given by

$$M_{2m} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ x_1^+ & x_2^+ & \cdots & x_m^+ & -y_1^- & -y_2^- & \cdots & -y_m^- \\ y_1^+ & y_2^+ & \cdots & y_m^+ & x_1^- & x_2^- & \cdots & x_m^- \end{pmatrix}.$$

Apply $E_{row}(2; 2)$ and $E_{row}(3; 2)$ to M_{2m} and call the resulting matrix M^1. The (i, j)-entry $(M^1)_{ij}$ is given by

$$(M^1)_{1j} = \begin{cases} 1, & 1 \leq j \leq m, \\ 0, & m+1 \leq j \leq 2m, \end{cases}$$

$$(M^1)_{2j} = \begin{cases} (j,\ 2m+1-j), & 1 \le j \le m, \\ (\overline{j-m+1},\ 3m+2-j), & m+1 \le j \le 2m \end{cases}$$

and

$$(M^1)_{3j} = \begin{cases} (j+1,\ 2m+2-j), & 1 \le j \le m, \\ (j-m,\ \overline{3m+1-j}), & m+1 \le j \le 2m. \end{cases}$$

By applying the transformations

$$E_{column}(m-1,\ m;\ -1),\ E_{column}(m-2,\ m-1;\ -1),\ \ldots,\ E_{column}(1,\ 2;\ -1)$$

successively, we see that M^1 is equivalent to the matrix $M^{1'}$ whose $(i,\ j)$-entry $(M^{1'})_{ij}$ is given by

$$(M^{1'})_{1j} = \begin{cases} 1, & j = 1, \\ 0, & 2 \le j \le 2m, \end{cases}$$

$$(M^{1'})_{2j} = \begin{cases} (1,\ 2m), & j = 1, \\ (\overline{j-1},\ j,\ 2m+1-j,\ \overline{2m+2-j}), & 2 \le j \le m, \\ (\overline{j-m+1},\ 3m+2-j), & m+1 \le j \le 2m \end{cases}$$

and

$$(M^{1'})_{3j} = \begin{cases} (2,\ 2m+1), & j = 1, \\ (\bar{j},\ j+1,\ 2m+2-j,\ \overline{2m+3-j}), & 2 \le j \le m, \\ (j-m,\ \overline{3m+1-j}), & m+1 \le j \le 2m. \end{cases}$$

At this point, we notice that the rank of $M^{1'}$ depends only on the $2 \times (2m-1)$ matrix M^2, which is obtained from $M^{1'}$ by deleting its first row and the first column. Its $(i,\ j)$-entry $(M^2)_{ij}$ is given as follows:

$$(M^2)_{1j} = \begin{cases} (\bar{j},\ j+1,\ 2m-j,\ \overline{2m-j+1}), & 1 \le j \le m-1, \\ (\overline{j-m+2},\ 3m+1-j), & m \le j \le 2m-1, \end{cases}$$

$$(M^2)_{2j} = \begin{cases} (\overline{j+1},\ j+2,\ 2m+1-j,\ \overline{2m+2-j}), & 1 \le j \le m-1, \\ (j-m+1,\ \overline{3m-j}), & m \le j \le 2m-1. \end{cases}$$

Apply $E_{column}(j-1,\ j;1)$ and $E_{column}(j+m-1,\ j;1)$ for each $j \in [2,\ m-1]$ to the matrix M^2 and we call the resulting matrix $M^{2'}$. Then the $(1,\ j)$-entry $M^{2'}_{1j}$ for $j \in [2,\ m-1]$ is computed to be

$$\begin{aligned} M^{2'}_{1j} &= (\bar{j},\ j+1,\ 2m-j,\ \overline{2m+1-j}) \\ &\quad + (\overline{j-1},\ j,\ 2m+1-j,\ \overline{2m+2-j}) \\ &\quad + (\overline{j+1},\ 2m+2-j) \\ &= (\overline{j-1},\ 2m-j). \end{aligned}$$

Similarly, we have

$$\begin{aligned} M^{2'}_{2j} &= (\overline{j+1},\ j+2,\ 2m+1-j,\ \overline{2m+2-j}) \\ &\quad + (\bar{j},\ j+1,\ 2m+2-j,\ \overline{2m+3-j}) \\ &\quad + (j,\ \overline{2m+1-j}) \\ &= (j+2,\ \overline{2m+3-j}). \end{aligned}$$

Thus, we see that the entries of $M^{2'}$ are given by

$$M^{2'}_{1j} = \begin{cases} (\bar{1},\ 2,\ 2m-1,\ \overline{2m}), & j = 1, \\ (\overline{j-1},\ 2m-j), & 2 \le j \le m-1, \\ (\overline{j-m+2},\ 3m+1-j), & m \le j \le 2m-1 \end{cases}$$

and

$$M^{2'}_{2j} = \begin{cases} (\bar{2},\ 3,\ 2m,\ \overline{2m+1}), & j = 1, \\ (j+2,\ \overline{2m+3-j}), & 2 \le j \le m-1, \\ (j-m+1,\ \overline{3m-j}), & m \le j \le 2m-1. \end{cases}$$

Furthermore, applying $E_{column}(2, 1; -1)$ and $E_{column}(m+2, 1; -1)$ to the matrix $M^{2'}$, the entries of the first column become as follows:

$$M_{11}^{2'} \Rightarrow (\overline{1}, 2, 2m-1, \overline{2m}) - (\overline{1}, 2m-2) - (\overline{4}, 2m-1)$$

$$= (2, 4, \overline{2m-2}, \overline{2m}),$$

$$M_{21}^{2'} \Rightarrow (\overline{2}, 3, 2m, \overline{2m+1}) - (4, \overline{2m+1}) - (3, \overline{2m-2})$$

$$= (\overline{2}, \overline{4}, 2m-2, 2m).$$

Therefore, the entries of the resulting matrix, which we call M^3, are given by

$$M_{1j}^3 = \begin{cases} (2, 4, \overline{2m-2}, \overline{2m}), & j = 1, \\ (\overline{j-1}, 2m-j), & 2 \le j \le m-1, \\ (\overline{j-m+2}, 3m+1-j), & m \le j \le 2m-1, \end{cases}$$

$$M_{2j}^3 = \begin{cases} (\overline{2}, \overline{4}, 2m-2, 2m), & j = 1, \\ (j+2, \overline{2m+3-j}), & 2 \le j \le m-1, \\ (j-m+1, \overline{3m-j}), & m \le j \le 2m-1. \end{cases}$$

Let $M_{1.}^3$ (resp. $M_{2.}^3$) denote the first (resp. second) row of M^3. Then the rank of M^3 is smaller than or equal to one if and only if there exists a pair $(s, t) \in \mathbf{R}^2 - \{(0, 0)\}$ such that

$$sM_{1.}^3 + tM_{2.}^3 = \mathbf{0}, \tag{3.18}$$

where the right hand side denotes the zero vector of length $2m-1$. Let c_i, $1 \le i \le 2m-1$, denote the ith component of the vector on the left hand side of (3.18). Then we have

$$
\begin{cases}
c_1 = s(2,\ 4,\ \overline{2m-2},\ \overline{2m}) + t(\overline{2},\ \overline{4},\ 2m-2,\ 2m), \\
c_2 = s(\overline{1},\ 2m-2) + t(4,\ \overline{2m+1}), \\
c_3 = s(\overline{2},\ 2m-3) + t(5,\ \overline{2m}), \\
\cdots \\
c_j = s(\overline{j-1},\ 2m-j) + t(j+2,\ \overline{2m+3-j}),\ (2 \le j \le m-1), \\
\cdots \\
c_{m-1} = s(\overline{m-2},\ m+1) + t(m+1,\ \overline{m+4}), \\
c_m = s(\overline{2},\ 2m+1) + t(1,\ \overline{2m}), \\
c_{m+1} = s(\overline{3},\ 2m) + t(2,\ \overline{2m-1}), \\
\cdots \\
c_{m+k} = s(\overline{k+2},\ 2m+1-k) + t(k+1,\ \overline{2m-k}),\ (0 \le k \le m-1), \\
\cdots \\
c_{2m-1} = s(\overline{m+1},\ m+2) + t(m,\ \overline{m+1}).
\end{cases}
\tag{3.19}
$$

We divide our argument into two cases according as $t \neq 0$ or $t = 0$.

Case (A): $t \neq 0$**.** We may assume that $t = 1$. Then for any $j \in [3,\ m-1]$, we have

$$
\begin{aligned}
c_j - c_{m-3+j} &= (s(\overline{j-1},\ 2m-j) + (j+2,\ \overline{2m+3-j})) \\
&\quad - (s(\overline{j-1},\ 2m+4-j) + (j-2,\ \overline{2m+3-j})) \\
&= -a_{j-2} + a_{j+2} + sa_{2m-j} - sa_{2m+4-j}.
\end{aligned}
\tag{3.20}
$$

Similar expressions can be obtained for any $j \in [2,\ m-2]$ as

$$
\begin{aligned}
c_j - c_{m+1+j} &= (s(\overline{j-1},\ 2m-j) + (j+2,\ \overline{2m+3-j})) \\
&\quad - (s(\overline{j+3},\ 2m-j) + (j+2,\ \overline{2m-1-j})) \\
&= -sa_{j-1} + sa_{j+3} + a_{2m-1-j} - a_{2m+3-j}.
\end{aligned}
$$

Therefore, by replacing j by $j-1$, we see that

$$c_{j-1} - c_{m+j} = -sa_{j-2} + sa_{j+2} + a_{2m-j} - a_{2m+4-j} \tag{3.21}$$

holds for any $j \in [3, m-1]$. It follows from (3.20) and (3.21) that for any $j \in [3, m-1]$, we have

$$(c_j - c_{m-3+j}) - s(c_{j-1} - c_{m+j}) = (s^2 - 1)(a_{j-2} - a_{j+2}),$$

$$-s(c_j - c_{m-3+j}) + (c_{j-1} - c_{m+j}) = (1 - s^2)(a_{2m-j} - a_{2m+4-j}).$$

Therefore, we see that when $s^2 \neq 1$, the equalities

$$a_{j-2} = a_{j+2}, \tag{3.22}$$

$$a_{2m-j} = a_{2m+4-j} \tag{3.23}$$

hold for any $j \in [3, m-1]$. It follows from (3.22) that

$$a_1 = a_5,$$

$$a_2 = a_6,$$

$$\cdots$$

$$a_{m-3} = a_{m+1}$$

and it follows from (3.23) that

$$a_{m+1} = a_{m+5},$$

$$a_{m+2} = a_{m+6},$$

$$\cdots$$

$$a_{2m-3} = a_{2m+1}.$$

However, we want to fill the gap, we notice that the same strategy as is used in the case when N is even, does not work. For this reason, we introduce the two kinds of alternating sums and auxiliary sums according to the parity of m:

Definition 3.1. When m is even, we let

$$d_m^{even} = \sum_{i=1}^{2m-1} (-1)^{i-1} c_i, \tag{3.24}$$

$$e_m^{even} = c_2 + c_m + c_{m+2} - c_{2m-3}. \tag{3.25}$$

When m is odd, we let

$$d_m^{odd} = \sum_{i=1}^{m} (-1)^{i-1} c_i + \sum_{i=m+1}^{2m-1} (-1)^i c_i, \tag{3.26}$$

$$e_m^{even} = c_2 - c_m + c_{m+2} + c_{2m-3}. \tag{3.27}$$

Then we can show the following:

Lemma 3.1. *Notation being as above, we have*

$$d_m^{even} + e_m^{even} = -a_{m-2} + sa_m + a_{m+2} - sa_{m+4}, \tag{3.28}$$

$$d_m^{odd} + e_m^{odd} = a_{m-2} - sa_m - a_{m+2} + sa_{m+4}. \tag{3.29}$$

Proof. Let C_m be a $(2m-1)\times(2m+1)$-matrix whose $(i,\ j)$-entry is the coefficient of a_j in c_i. Then it follows from (3.19) that its jth column A_j $(j \in [1,\ 2m+1])$ is specified by the following:

$$\begin{cases} A_1 = -se_2 + e_m, \\ A_2 = (s-1)e_1 - se_3 - se_m + e_{m+1}, \\ A_3 = -se_4 - se_{m+1} + e_{m+2}, \\ A_4 = (s-1)e_1 + e_2 - se_5 - se_{m+2} + e_{m+3}, \end{cases} \tag{3.30}$$

$$\begin{cases} A_5 = e_3 - se_6 - se_{m+3} + e_{m+4}, \\ A_6 = e_4 - se_7 - se_{m+4} + e_{m+5}, \\ \quad \cdots \\ A_{m-2} = e_{m-4} - se_{m-1} - se_{2m-4} + e_{2m-3}, \end{cases} \tag{3.31}$$

$$\begin{cases} A_{m-1} = e_{m-3} - se_{2m-3} + e_{2m-2}, \\ A_m = e_{m-2} - se_{2m-2} + e_{2m-1}, \end{cases} \tag{3.32}$$

$$A_{m+1} = (s+1)e_{m-1} - (s+1)e_{2m-1}, \tag{3.33}$$

$$\begin{cases} A_{m+2} = se_{m-2} - e_{2m-2} + se_{2m-1}, \\ A_{m+3} = se_{m-3} - e_{2m-3} + se_{2m-2}, \end{cases} \tag{3.34}$$

$$\begin{cases} A_{m+4} = se_{m-4} - e_{m-1} - e_{2m-4} + se_{2m-3}, \\ A_{m+5} = se_{m-5} - e_{m-2} - e_{2m-5} + se_{2m-4}, \\ \quad \cdots \\ A_{2m-3} = se_3 - e_6 - e_{m+3} + se_{m+4}, \end{cases} \tag{3.35}$$

$$\begin{cases} A_{2m-2} = (1-s)e_1 + se_2 - e_5 - e_{m+2} + se_{m+3}, \\ A_{2m-1} = -e_4 - e_{m+1} + se_{m+2}, \\ A_{2m} = (1-s)e_1 - e_3 - e_m + se_{m+1}, \\ A_{2m+1} = -e_2 + se_m. \end{cases} \tag{3.36}$$

First, we deal with the case when m is even. Let $[d_m^{even}, a_j]$, $1 \le j \le 2m+1$, denote the coefficient of a_j in d_m^{even}. The first equality in (3.31) shows that $[d_m^{even}, a_5] = 0$, since $m+3 \equiv 1 \pmod 2$, $m+4 \equiv 0 \pmod 2$, hence A_5 becomes equal to zero if we put $e_i = (-1)^{i-1}$ formally. In a similar vein, it follows from (3.31) that $[d_m^{even}, a_j] = 0$ for every $j \in [5, m-2]$. Arguing similarly with the equalities in (3.35), we see that every coefficient of $[d_m^{even}, a_j] = 0$ for any $j \in [m+4, 2m-3]$. For the remaining coefficients, it follows from (3.30), (3.32)-(3.34) and (3.36) that we have

$$[d_m^{even}, a_1] = s-1, \quad [d_m^{even}, a_2] = s, \quad [d_m^{even}, a_3] = -1, \quad [d_m^{even}, a_4] = s-1,$$

$$[d_m^{even}, a_{m-1}] = -s, \quad [d_m^{even}, a_m] = s, \quad [d_m^{even}, a_{m+1}] = 0,$$

$$[d_m^{even}, a_{m+2}] = 1, \quad [d_m^{even}, a_{m+3}] = -1,$$

$$[d_m^{even}, a_{2m-2}] = 1-s, \quad [d_m^{even}, a_{2m-1}] = -s,$$

$$[d_m^{even}, a_{2m}] = 1, \quad [d_m^{even}, a_{2m+1}] = 1-s. \tag{3.37}$$

On the other hand, it follows from the definition (3.27) that

$$\begin{aligned} e_m^{even} &= c_2 + c_m + c_{m+2} - c_{2m-3} \\ &= s(-a_1 + a_{2m-2}) + a_4 - a_{2m+1} \\ &\quad + s(-a_2 + a_{2m+1}) + a_1 - a_{2m} \\ &\quad + s(-a_4 + a_{2m-1}) + a_3 - a_{2m-2} \\ &\quad - (s(-a_{m-1} + a_{m+4}) + a_{m-2} - a_{m+3}) \\ &= (-s+1)a_1 - sa_2 + a_3 + (-s+1)a_4 \\ &\quad - a_{m-2} + sa_{m-1} + a_{m+3} - sa_{m+4} \\ &\quad + (s-1)a_{2m-2} + sa_{2m-1} - a_{2m} + (s-1)a_{2m+1}. \end{aligned}$$

Combining this with (3.37), we find that

$$d_m^{even} + e_m^{even} = -a_{m-2} + sa_m + a_{m+2} - sa_{m+4},$$

which is to be proved.

The case when m can be similarly dealt. The point is that the signs of the latter half of d_m^{odd} are opposite to those of d_m^{even} and hence the coefficients appearing in (3.31) and (3.35) vanish too as in the case m is even. □

Another useful equalities are as follows:

Lemma 3.2. *Notation being as above, we have*

$$c_{m-1} + c_{2m-1} = -sa_{m-2} + a_m + sa_{m+2} - a_{m+4}, \tag{3.38}$$

$$\begin{aligned} c_{2m-3} + c_{2m-2} &= a_{m-2} + (-s+1)a_{m-1} - sa_m \\ &\quad - a_{m+2} + (s-1)a_{m+3} + sa_{m+4}. \end{aligned} \tag{3.39}$$

Proof. It follows from (3.19) that

$$\begin{aligned} c_{m-1} + c_{2m-1} &= s(-a_{m-2} + a_{m+1}) + a_{m+1} - a_{m+4} \\ &\quad + s(-a_{m+1} + a_{m+2}) + a_m - a_{m+1} \\ &= -sa_{m-2} + a_m + sa_{m+2} - a_{m+4}. \end{aligned}$$

Similarly, it follows from (3.19) that

$$\begin{aligned} c_{2m-3} + c_{2m-2} &= s(-a_{m-1} + a_{m+4}) + a_{m-2} - a_{m+3} \\ &\quad + s(-a_m + a_{m+3}) + a_{m-1} - a_{m+2} \\ &= a_{m-2} + (-s+1)a_{m-1} - sa_m \\ &\quad - a_{m+2} + (s-1)a_{m+3} + sa_{m+4}, \end{aligned}$$

which is to be proved. □

Now we combine the two lemmas above and obtain the following:

Proposition 3.4. *When m is even, we have*

$$d_m^{even} + e_m^{even} - s(c_{m-1} + c_{2m-1}) = (s^2 - 1)(a_{m-2} - a_{m+2}),$$

$$s(d_m^{even} + e_m^{even}) - (c_{m-1} + c_{2m-1}) = (s^2 - 1)(a_m - a_{m+4}),$$

$$d_m^{even} + e_m^{even} + (c_{2m-3} + c_{2m-2}) = (-s+1)(a_{m-1} - a_{m+3}).$$

When m is odd, we have

$$d_m^{odd} + e_m^{odd} + s(c_{m-1} + c_{2m-1}) = (-s^2 + 1)(a_{m-2} - a_{m+2}),$$

$$s(d_m^{odd} + e_m^{odd}) + (c_{m-1} + c_{2m-1}) = (-s^2 + 1)(a_m - a_{m+4}),$$

$$d_m^{odd} + e_m^{odd} - (c_{2m-3} + c_{2m-2}) = (s-1)(a_{m-1} - a_{m+3}).$$

Proof. When m is even, it follows from (3.28) and (3.38) that

$$\begin{aligned} &d_m^{even} + e_m^{even} - s(c_{m-1} + c_{2m-1}) \\ &= (-a_{m-2} + sa_m + a_{m+2} - sa_{m+4}) - s(-sa_{m-2} + a_m + sa_{m+2} - a_{m+4}) \\ &= (s^2 - 1)(a_{m-2} - a_{m+2}) \end{aligned}$$

and that

$$s(d_m^{even} + e_m^{even}) - (c_{m-1} + c_{2m-1})$$

$$= s(-a_{m-2} + sa_m + a_{m+2} - sa_{m+4}) - (-sa_{m-2} + a_m + sa_{m+2} - a_{m+4})$$

$$= (s^2 - 1)(a_m - a_{m+4}).$$

Furthermore, it follows from (3.28) and (3.39) that

$$d_m^{even} + e_m^{even} + (c_{2m-3} + c_{2m-2})$$

$$= (-a_{m-2} + sa_m + a_{m+2} - sa_{m+4})$$

$$+ (a_{m-2} + (-s+1)a_{m-1} - sa_m - a_{m+2} + (s-1)a_{m+3} + sa_{m+4})$$

$$= (-s+1)(a_{m-1} - a_{m+3}).$$

When m is odd, it follows from (3.29) and (3.38) that

$$d_m^{odd} + e_m^{odd} + s(c_{m-1} + c_{2m-1})$$

$$= (a_{m-2} - sa_m - a_{m+2} + sa_{m+4}) + s(-sa_{m-2} + a_m + sa_{m+2} - a_{m+4})$$

$$= (-s^2 + 1)(a_{m-2} - a_{m+2})$$

and that

$$s(d_m^{odd} + e_m^{odd}) + (c_{m-1} + c_{2m-1})$$

$$= s(a_{m-2} - sa_m - a_{m+2} + sa_{m+4}) + (-sa_{m-2} + a_m + sa_{m+2} - a_{m+4})$$

$$= (-s^2 + 1)(a_m - a_{m+4}).$$

Furthermore, it follows from (3.29) and (3.39) that

$$d_m^{odd} + e_m^{odd} - (c_{2m-3} + c_{2m-2})$$

$$= (a_{m-2} - sa_m - a_{m+2} + sa_{m+4})$$

$$- (a_{m-2} + (-s+1)a_{m-1} - sa_m - a_{m+2} + (s-1)a_{m+3} + sa_{m+4})$$

$$= (s-1)(a_{m-1} - a_{m+3}).$$

This completes the proof. □

Combining (3.22), (3.23) with Proposition 3.4, we obtain the following:

Proposition 3.5. *When* $s^2 \neq 1$ *for any* $i \in [1, 2m-3]$, *we have*

$$a_i = a_{i+4}.$$

It follows from this proposition that the components c_j, $1 \leq j \leq 2m-1$, can be expressed solely by a_1, a_2, a_3, a_4. As for $c_1 = (s-1)(2, 4, \overline{2m-2}, \overline{2m})$, we note that the set $\{2m-2, 2m\}$ coincides with $\{2, 4\}$ modulo four. Hence, c_1 is equal to zero. As for c_j with $j \in [2, m-1]$, it follows from (3.19) that

$$c_j = s(\overline{j-1}, 2m-j) + (j+2, \overline{2m+3-j}).$$

Letting the value of pairs (m, j) run through $\mathbf{Z}_4^2$, we find that there are four cases into which c_j falls:

$$c_j = \begin{cases} \pm(a_2 - (s+1)a_3 + sa_4), & \text{if } (m, j) \equiv (0, 0), (0, 1), (2, 0), (2, 1) \pmod 4, \\ \pm((s+1)a_1 - sa_2 - a_4), & \text{if } (m, j) \equiv (0, 2), (0, 3), (2, 2), (2, 3) \pmod 4, \\ \pm(a_1 - (s+1)a_2 + sa_3), & \text{if } (m, j) \equiv (1, 0), (1, 3), (3, 0), (3, 3) \pmod 4, \\ \pm(sa_1 + a_3 - (s+1)a_4), & \text{if } (m, j) \equiv (1, 1), (1, 2), (3, 1), (3, 2) \pmod 4. \end{cases}$$

Next, we consider the case $j \in [m, 2m-1]$ so that

$$c_j = s(\overline{j-m+2}, 3m+1-j) + (j-m+1, \overline{3m-j}).$$

This time the shapes of c_j are classified as follows:

$$c_j = \begin{cases} \pm((s+1)a_1 - sa_2 - a_4), & \text{if } (m, j) \equiv (0, 0), (0, 3), (2, 1), (2, 2) \pmod 4, \\ \pm(a_2 - (s+1)a_3 + sa_4), & \text{if } (m, j) \equiv (0, 1), (0, 2), (2, 0), (2, 3) \pmod 4, \\ \pm(sa_1 + a_3 - (s+1)a_4), & \text{if } (m, j) \equiv (1, 0), (1, 3), (3, 1), (3, 2) \pmod 4, \\ \pm(a_1 - (s+1)a_2 + sa_3), & \text{if } (m, j) \equiv (1, 1), (1, 2), (3, 0), (3, 3) \pmod 4. \end{cases}$$

Thus, the simultaneous equation $(c_1, c_2, ..., c_{2m-2}) = \mathbf{0}$ is reduced according to the parity of m to the following two equations:

When m is even,

$$\begin{cases} (s+1)a_1 - sa_2 - a_4 = 0, \\ a_2 - (s+1)a_3 + sa_4 = 0. \end{cases} \tag{3.40}$$

When m is odd,

$$\begin{cases} a_1 - (s+1)a_2 + sa_3 = 0, \\ sa_1 + a_3 - (s+1)a_4 = 0. \end{cases}$$

In any case, we can show the following:

Lemma 3.3. *The points* $p_i = (a_i, a_{i+1})$, $1 \le i \le N-1$, *rotate along the square made of the first four points* $p_1, \ldots, p_4$.

Proof. We deal only with the case when m is even, since the odd case can be treated similarly. It follows from (3.40) that

$$a_1 = \frac{s}{s+1} a_2 + \frac{1}{s+1} a_4,$$

$$a_3 = \frac{1}{s+1} a_2 + \frac{s}{s+1} a_4.$$

Therefore, we find that

$$\begin{aligned} p_1 - p_2 &= (a_1 - a_2, a_2 - a_3) \\ &= \frac{a_2 - a_4}{s+1}(-1, s), \\ p_2 - p_3 &= (a_2 - a_3, a_3 - a_4) \\ &= \frac{a_2 - a_4}{s+1}(s, 1), \\ p_3 - p_4 &= (a_3 - a_4, a_4 - a_1) \\ &= \frac{a_2 - a_4}{s+1}(1, -s), \\ p_4 - p_1 &= (a_4 - a_1, a_1 - a_2) \\ &= \frac{a_2 - a_4}{s+1}(-s, -1). \end{aligned}$$

Hence, the pair of adjacent sides $p_i - p_{i+1}$ and $p_{i+1} - p_{i+2}$ are transversal for $i \in [1, N-2]$. Furthermore, the square of the length of every side is computed to be

$$|p_i - p_{i+1}|^2 = \left(\frac{a_2 - a_4}{s+1}\right)^2 (s^2 + 1),$$

hence the proof is complete. □

Next, we deal with the cases when $s^2 = 1$.

Case (A; 1): $s = 1$**.** In this case, let $b_k = a_k + a_{2m+2-k}$ for any $k \in [1, m+1]$. Then all of the components c_j, $j \in [1, 2m-1]$ can be expressed in terms of b_k as follows:

$$\begin{aligned}
&c_1 = 0,\\
&c_2 = -b_1 + b_4,\\
&c_3 = -b_2 + b_5,\\
&\quad\cdots\\
&c_{m-2} = -b_{m-3} + b_m,\\
&c_{m-1} = -b_{m-2} + b_{m+1},\\
&c_m = b_1 - b_2,\\
&c_{m+1} = b_2 - b_3,\\
&\quad\cdots\\
&c_{2m-1} = b_m - b_{m+1}.
\end{aligned}$$

Therefore, the simultaneous equation $(c_1, c_2, ..., c_{2m-1}) = \mathbf{0}$ is equivalent to the conditions

$$b_1 = b_2 = \cdots = b_{m+1},$$

namely,

$$a_1 + a_{2m+1} = a_2 + a_{2m} = \cdots = a_m + a_{m+2} = 2a_{m+1}.$$

Case (A; –1): $s = -1$**.** In this case, let $d_k = a_k - a_{2m+2-k}$ for any $k \in [1, m]$. Then all of the components c_j, $j \in [1, 2m-1]$ can be expressed as follows:

$$c_1 = -2(d_2 + d_4),$$

$$c_2 = d_1 + d_4,$$

$$c_3 = d_2 + d_5,$$

$$\cdots$$

$$c_{m-2} = d_{m-3} + d_m,$$

$$c_{m-1} = d_{m-2},$$

$$c_m = d_1 + d_2,$$

$$c_{m+1} = d_2 + d_3$$

$$\cdots$$

$$c_{2m-2} = d_{m-1} + d_m,$$

$$c_{2m-1} = d_m.$$

Therefore, we have

$$d_1 = \cdots = d_m = 0,$$

which is equivalent to the condition that

$$a_i = a_{2m+2-i}$$

holds for any $i \in [1, m]$.

Case (B): $t = 0$**.** Letting $(s, t) = (1, 0)$ in (3.19), we have

$$C_0 : \quad (2,\ 4,\ \overline{2m-2},\ \overline{2m}) = 0$$

and

$$\begin{cases} C_1: & (\overline{1},\ 2m-2) = 0, \\ C_2: & (\overline{2},\ 2m-3) = 0, \\ \cdots & \cdots \\ C_{m-2}: & (\overline{m-2},\ m+1) = 0, \\ \overline{C}_2: & (\overline{2},\ 2m+1) = 0, \\ \overline{C}_3: & (\overline{3},\ 2m) = 0, \\ \cdots & \cdots \\ \overline{C}_{m+1}: & (\overline{m+1},\ m+2) = 0. \end{cases} \tag{3.41}$$

Here for ease of description below, we name each equation as indicated. We note that the equations $C_1, \ldots, C_{m-2}$ can be expressed as

$$a_i = a_{2m-1-i} \text{ for any } i \in [1,\ 2m-2] \backslash \{m-1,\ m\} \tag{3.42}$$

and that the equations $\overline{C}_2, \ldots, \overline{C}_{m+1}$ can be expressed as

$$a_i = a_{2m+3-i} \text{ for any } i \in [2,\ 2m+1]. \tag{3.43}$$

Therefore, for any $i \in [1,\ 2m-3] \backslash \{m-1,\ m\}$, we have $a_i = a_{2m-1-i} = a_{i+4}$, the latter equality is coming from (3.43). Thus, we have the periodicity

$$a_i = a_{i+4}, \quad i \in [1,\ 2m-3] \backslash \{m-1,\ m\}. \tag{3.44}$$

Furthermore, it follows from (3.43) that

$$a_{m-1} = a_{m+4}, \tag{3.45}$$

$$a_m = a_{m+3}, \tag{3.46}$$

which will be used frequently later. We want to fill the gap in (3.44), namely, to show that

$$a_{m-1} = a_{m+3}, \tag{3.47}$$

$$a_m = a_{m+4}. \tag{3.48}$$

Note that these two are equivalent under the conditions (3.45) and (3.46).

Therefore, we have only to show one of these equalities. We divide our argument into four cases according to the value of *m* modulo four.

Case I: $m \equiv 0 \pmod 4$**.** We put $m = 4k$. It follows from (3.44) that

$$a_2 = a_6 = \cdots = a_{4k-2} = a_{4k+2} = \cdots = a_{2m-2}.$$

Therefore, the equation C_0 implies that

$$a_4 = a_{2m}. \tag{3.49}$$

On the other hand, it follows from (3.44) that

$$a_4 = a_8 = \cdots = a_{4k}, \tag{3.50}$$

$$a_{4k+4} = a_{4k+8} = \cdots = a_{2m}. \tag{3.51}$$

Since the leftmost side of (3.50) and the rightmost side of (3.51) coincide by (3.49), we see that the rightmost side of (3.50) and the leftmost side of (3.51) coincide, namely, we have

$$a_m = a_{4k} = a_{4k+4} = a_{m+4},$$

which shows the validity of (3.48). Thus, we obtain the full periodicity, namely,

$$a_i = a_{i+4} \text{ for any } i \in [1, 2m-3]. \tag{3.52}$$

Case II: $m \equiv 1 \pmod 4$**.** We put $m = 4k + 1$. It follows from (3.44) that

$$a_2 = a_6 = \cdots = a_{4k-2} = a_{4k+2} = \cdots = a_{2m}.$$

Therefore, the equation C_0 implies that

$$a_4 = a_{2m-2}. \tag{3.53}$$

On the other hand, It follows from (3.46) and (3.44) that

$$a_m = a_{m+3} = a_{2m-2}. \tag{3.54}$$

Furthermore, it follows from (3.46) and (3.45) that

$$a_4 = a_{m-1} = a_{m+4}. \tag{3.55}$$

Hence, combining (3.54), (3.53) and (3.55), we obtain

$$a_m = a_{m+4}, \tag{3.56}$$

which is (3.48).

Case III: $m \equiv 2 \pmod 4$**.** We put $m = 4k + 2$. It follows from (3.44) that

$$a_4 = a_8 = \cdots = a_{4k} = a_{4k+4} = \cdots = a_{2m}.$$

Therefore, the equation C_0 implies that

$$a_2 = a_{2m-2}. \tag{3.57}$$

On the other hand, it follows from (3.44) that

$$a_2 = a_m, \tag{3.58}$$

$$a_{m+4} = a_{2m-2}. \tag{3.59}$$

Therefore, it follows from (3.57)-(3.59) that

$$a_m = a_{m+4}, \tag{3.60}$$

which is (3.48).

Case IV: $m \equiv 3 \pmod 4$**.** We put $m = 4k + 3$. It follows from (3.44) that

$$a_4 = a_8 = \cdots = a_{4k} = a_{4k+4} = \cdots = a_{2m-2}.$$

Therefore, the equation C_0 implies that

$$a_2 = a_{2m}. \tag{3.61}$$

On the other hand, it follows from (3.46) and (3.44) that

$$a_m = a_{m+3} = a_{2m}. \tag{3.62}$$

Furthermore, it follows from (3.44) and (3.45) that

$$a_2 = a_{m-1} = a_{m+4}. \tag{3.63}$$

Hence, combining (3.61)-(3.63), we obtain

$$a_m = a_{m+4},$$

which is (3.48).

Applying the above consideration, we obtain the following final specification, which divides the variables into two mutually equal ones:

Proposition 3.6. *The system of equations* C_0 *and* (3.41) *holds if and only if the periodicity*

$$a_i = a_{i+4}, \quad i \in [1, 2m-3]$$

and the following conditions hold:

$$a_1 = a_2, \quad a_3 = a_4, \textit{ when } m \equiv 0 \pmod 2,$$

$$a_1 = a_4, \quad a_2 = a_3, \textit{ when } m \equiv 1 \pmod 2.$$

Proof. Case I: $m \equiv 0 \pmod 2$**.** It follows from (3.43) that $a_2 = a_{2m+1}$. Combining this with the periodicity, we see that

$$a_1 = a_{2m+1} = a_2.$$

Furthermore, combining the equality $a_4 = a_{2m-1}$, which follows from (3.43), with the periodicity, we have

$$a_3 = a_{2m-1} = a_4.$$

Case II: $m \equiv 1 \pmod 2$**.** It follows from (3.43) that two equalities $a_4 = a_{2m-1}$, $a_3 = a_{2m}$ hold. These together with the periodicity imply that

$$a_1 = a_{2m-1} = a_4,$$

$$a_2 = a_{2m} = a_3.$$

Moreover, our system of equations

$$C_i \ (0 \le i \le m-2), \quad \overline{C}_j \ (2 \le j \le m+1)$$

consist of $2m-1$ equations for $2m+1$ unknowns a_j $(1 \le j \le 2m+1)$.

Hence, the dimension of the solution space must be greater than or equal to $(2m+1)-(2m-1)=2$. We, however, have already found that there are only two free variables in each case, and hence there cannot be further reduction of the number of free variables. This completes the proof. □

Thereby, we complete our classification of melodies of odd length whose M-graph has lines of symmetry. The result is as follows:

Theorem 3.2. *Let* $\mathbf{a}=(a_1, ..., a_{2m+1})$ *be a melody of odd length* $2m+1$ *with* $m \geq 3$ *and let* $p_i=(a_i, a_{i+1}), 1 \leq i \leq 2m$ *so that* $M(\mathbf{a})=(p_1, ..., p_{2m})$. *The ordered set* $M(\mathbf{a})$ *of points has a line of symmetry if and only if one of the following conditions holds*:

(1) $a_i = a_{i+4}$ *for any* $i \in [1, 2m-3]$ *and*

(1.1) $a_1 = \dfrac{sa_2 + a_4}{s+1}$, $a_3 = \dfrac{a_2 + sa_4}{s+1}$ *hold for some* $s \in \mathbf{R}\backslash\{\pm 1\}$, *when m is even*.

(1.2) $a_2 = \dfrac{a_1 + sa_3}{s+1}$, $a_4 = \dfrac{sa_1 + a_3}{s+1}$ *hold for some* $s \in \mathbf{R}\backslash\{\pm 1\}$, *when m is odd*.

(2) $a_1 + a_{2m+1} = a_2 + a_{2m} = \cdots = a_m + a_{m+2} = 2a_{m+1}$.

(3) $a_1 = a_{2m+1}, a_2 = a_{2m}, ..., a_m = a_{m+2}$.

(4) $a_i = a_{i+4}$ *for any* $i \in [1, 2m-3]$ *and*

(4.1) $a_1 = a_2, a_3 = a_4$, *when m is even*.

(4.2) $a_1 = a_4, a_2 = a_3$, *when m is odd*.

4. Examples of Symmetric Melodies

In this final section, we illustrate the results obtained in Theorems 3.1 and 3.2 by a few examples. The first subsection deals with melodies having period four. In the second and the third subsections, we treat melodies of lengths twelve and eleven, respectively, with amazing symmetries. In each figure, the dashed line signifies the line of symmetry of respective *M*-graphs.

4.1. Symmetric melodies with period four

The following graph is obtained by setting $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ in Theorem 3.1, (1) with $m = 4$:

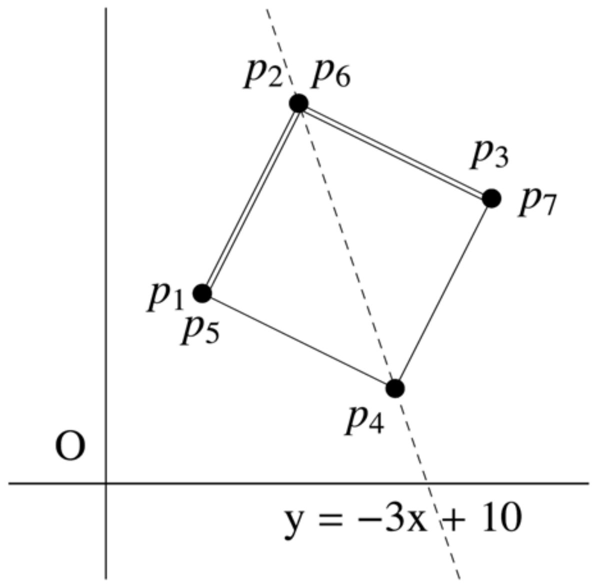

Figure 4. $M(1, 2, 4, 3, 1, 2, 4, 3)$: Theorem 3.1, (1) with $m = 4$.

The next two graphs are examples of Theorem 3.1, (4.1) with $m = 4$, where we set $a_1 = 1$, $a_2 = 2$, and of Theorem 3.1, (4.2) with $m = 3$, where $a_1 = 2$, $a_2 = 1$:

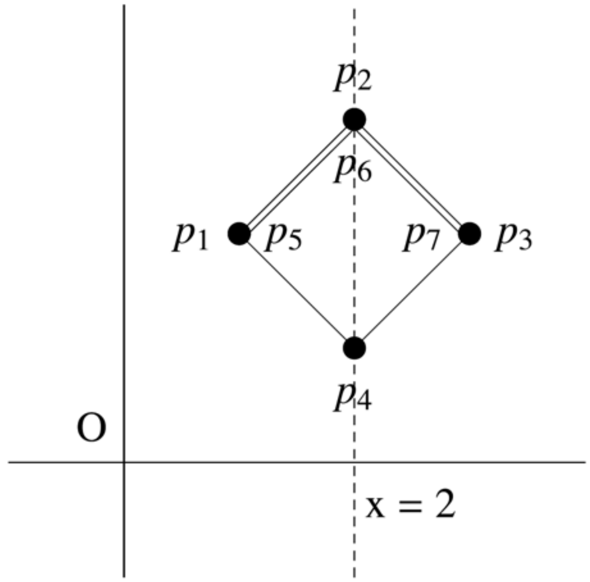

Figure 5. $M(1, 2, 3, 2, 1, 2, 3, 2)$: Theorem 3.1, (4.1) with $m = 4$.

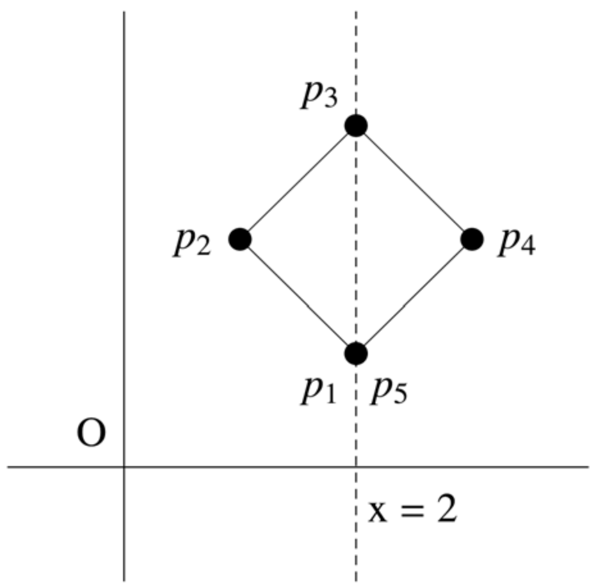

Figure 6. $M(2, 1, 2, 3, 2, 1)$: Theorem 3.1, (4.2) with $m = 3$.

The following two graphs are examples of Theorem 3.2, (1.1) with

$m = 4$, where we set $a_2 = 1$, $a_4 = 3$, $s = 2$, and of Theorem 3.2, (1.2) with $m = 3$, where $a_1 = 1$, $a_3 = 3$, $s = 2$:

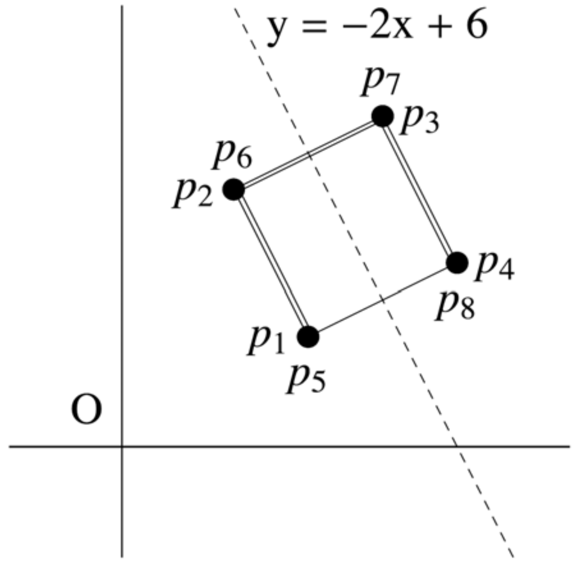

Figure 7. $M(5/3, 1, 7/3, 3, 5/3, 1, 7/3, 3, 5/3)$: Theorem 3.2, (1.1) with $m = 4$.

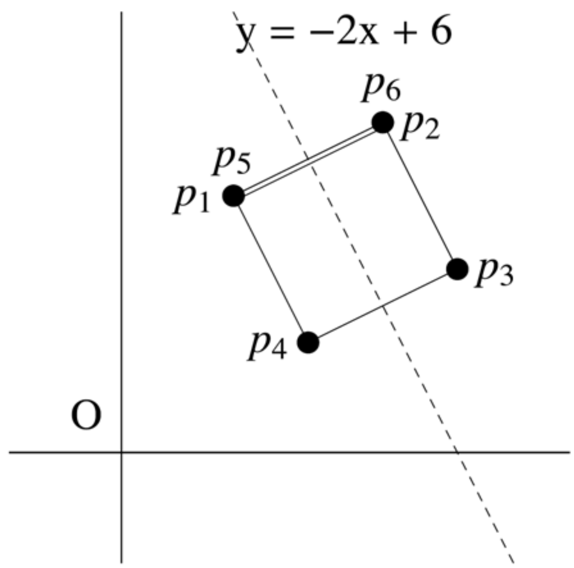

Figure 8. $M(1, 7/3, 3, 5/3, 1, 7/3, 3)$: Theorem 3.2, (1.2) with $m = 3$.

The next two graphs are examples of Theorem 3.2, (4.1) with $m = 4$, where we set $a_1 = 1$, $a_3 = 2$, and of Theorem 3.2, (4.2) with $m = 3$, where $a_1 = 1$, $a_2 = 2$:

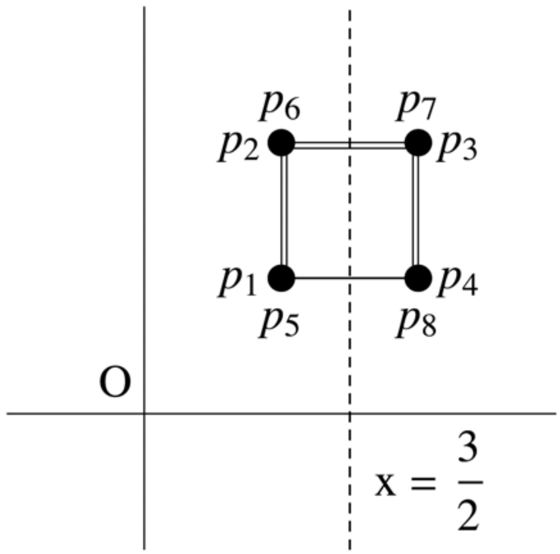

Figure 9. $M(1, 1, 2, 2, 1, 1, 2, 2, 1)$: Theorem 3.2, (4.1) with $m = 4$.

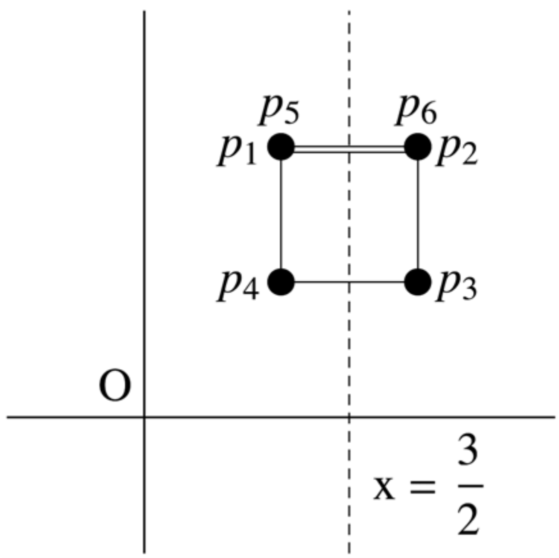

Figure 10. $M(1, 2, 2, 1, 1, 2, 2)$: Theorem 3.2, (4.2) with $m = 3$.

4.2. Symmetric melodies with length twelve

Firstly, we consider the melodies of length twelve. Since melodies which satisfy either condition specified in the cases (1), (4.1) and (4.2) of Theorem 3.1 are periodic with period four, their *M*-graphs go around some quadrangles and nothing interesting appears. For this reason, we focus on the melodies, which fall into the category (2) or (3) of Theorem 3.1. The first figure is the *M*-graph of the melody

$$\mathbf{a} = (0,\ 7,\ 3,\ 10,\ 5,\ 9,\ 2,\ 6,\ 1,\ 8,\ 4,\ 11),$$

in which no repeated tones appear. This satisfies the condition (2) of the theorem and the line of symmetry is defined by $y = -x + 11$:

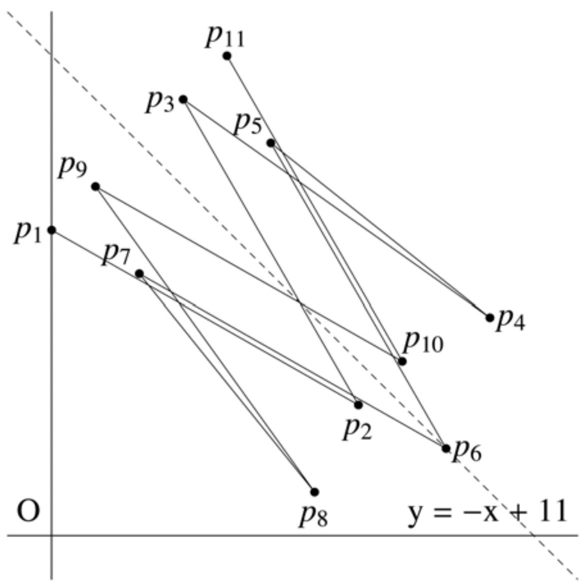

Figure 11. $M(0,\ 7,\ 3,\ 10,\ 5,\ 9,\ 2,\ 6,\ 1,\ 8,\ 4,\ 11)$.

The next figure shows the *M*-graph of the melody

$$\mathbf{a} = (1,\ 6,\ 2,\ 4,\ 5,\ 7,\ 7,\ 5,\ 4,\ 2,\ 6,\ 1),$$

which satisfies the condition (3) of the theorem:

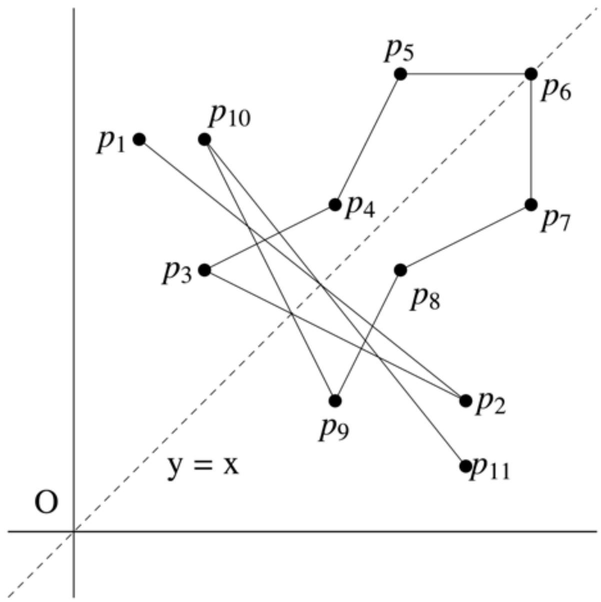

Figure 12. $M(1, 6, 2, 4, 5, 7, 7, 5, 4, 2, 6, 1)$.

This has a line of symmetry defined by $y = x$.

4.3. Symmetric melodies with length eleven

Here we illustrate a few symmetric melodies of length eleven. Since melodies which satisfy either condition specified in the cases (1), (4) of Theorem 3.2 are also periodic with period four, we focus on the melodies, which fall into the category (2) or (3) of Theorem 3.2. The first figure is the M-graph of the melody $\mathbf{a} = (0, 7, 1, 4, 2, 5, 8, 6, 9, 3, 10)$, in which no repeated tones appear. This satisfies the condition (2) of the theorem and the line of symmetry is defined by $y = -x + 10$:

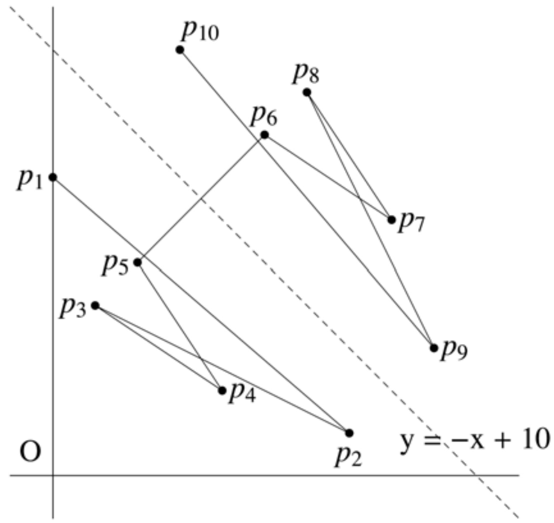

Figure 13. $M(0, 7, 1, 4, 2, 5, 8, 6, 9, 3, 10)$.

The next figure shows the M-graph of the melody

$$\mathbf{a} = (0, 2, 7, 5, 3, 8, 3, 5, 7, 2, 0),$$

which satisfies the condition (3) of the theorem:

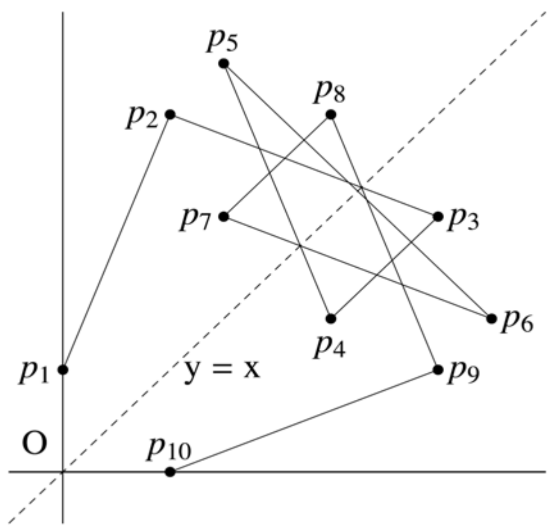

Figure 14. $M(0, 2, 7, 5, 3, 8, 3, 5, 7, 2, 0)$.

This has a line of symmetry defined by $y = x$.

We conclude this paper by the figure of a symmetric melody of length 16:

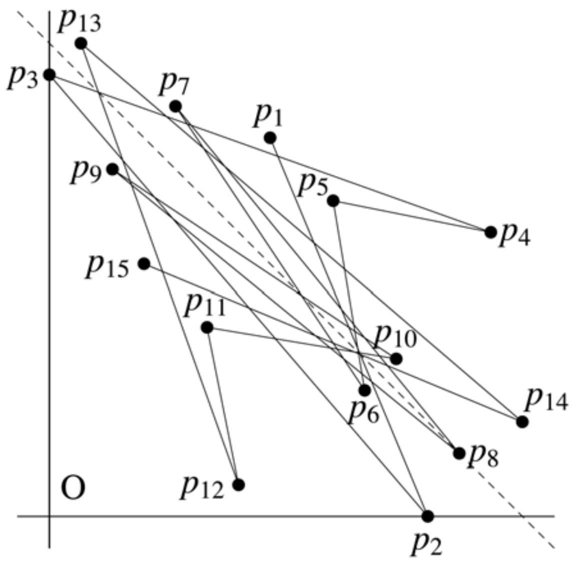

Figure 15. $M(7, 12, 0, 14, 9, 10, 4, 13, 2, 11, 5, 6, 1, 15, 3, 8)$.

References

[1] G. Gündüz and U. Gündüz, The mathematical analysis of the structure of some songs, Phys. A 357 (2005), 565-592.

[2] F. Hazama, Mathematical analysis of melodies: slope and discrete Fréchet distance, Far East J. Math. Sci. (FJMS) 97(5) (2015), 583-615.