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SOME REMARKS ON CENTRALITY IN THE SUBTREE GRAPH OF A TREE

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Abstract

We consider the center problem in the subtree graph of a tree T. Let S_1 and S_2 be subtrees of a tree T. The subtree graph G_L has the vertex set of all subtrees of T and two subtrees S_1 and S_2 are joined by an edge, if S_2 is obtained from the subtree S_1 by adding/removing a single vertex. A subtree S of a tree T is a central subtree of T if S has the minimum eccentricity in the subtree graph. The graph center of G_L is the set of all central subtrees and a central subtree with the minimum number of vertices is a least central subtree of a tree T.

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We compare different centrality concepts in G_L. We show that in the subtree graph, the branch weight center and distance center coincide and are isomorphic to some hypercube. Moreover, the least subtree within these centers is unique.

1. Introduction

Let *G* be a finite, connected and undirected graph without loops and multiple edges. The set of vertices of *G* is denoted by *V* and the number of vertices of subgraph G_1 is denoted by $|G_1|$. The *distance* $d_G(u, v)$ of two vertices *u* and *v* in graph *G* is the length of the shortest path (i.e., geodesic) $u - v$ in *G*. A vertex *t* satisfying the equations $d_G(u, v) = d_G(u, t) + d_G(t, v)$, $d_G(u, w) = d_G(u, t) + d_G(t, w)$, $d_G(v, w) = d_G(v, t) + d_G(t, w)$ is called the *median* of the vertices *u*, *v* and *w*. If every triple *u*, *v*, *w* of vertices of a graph has a unique median, then the graph is called a *median graph*.

As is well known, the *eccentricity* $e(v)$ *of a vertex v* in a connected graph *G* with the vertex set *V* is the distance to a vertex farthest from *v*: $e(v) = \max\{d_G(u, v) | u \in V\}$. The *center* *C* of *G* consists of vertices with minimum eccentricity: $C = \{v | e(v) = \min\{e(u) | u \in V\}\}$ and the *radius* $\operatorname{rad} G$ of the graph *G* is its minimum eccentricity. The distance $d_G(u)$ of a vertex *u* is the sum $d_G(u) = \sum d_G(u, v)$, where *v* ranges over all vertices of *G*. The vertices of minimum distance constitute the distance center *D* of a graph. The vertices of the distance center are usually called *medians* of a graph. Since we are dealing with median graphs where the median concept has another meaning, we rename the vertices of this center concept by calling them '*vertices of a distance center*' or '*minimum distance vertices*'. See [3].

A vertex set $A \subset V$ in a graph *G* is called *convex* of *G* if *A* contains all vertices on any $u - v$ geodesic for every pair $u, v \in A$. The vertex sets *V* and $\varnothing$ are trivial convexes of *G*.

The following generalization of the branch weight is from the article [11]. A branch of a vertex *u* is convex not containing the vertex *u*. The branch weight $b(u)$ of a vertex *u* is the maximum number of vertices in a branch of *u*. The vertices having the minimum branch weight constitute the branch weight center *B* of *G*. Note that the definition of branch weight agrees with the usual definition of branch weight in the case of trees.

For two subtrees S_1 and S_2 of a tree *T*, the subforests induced by union and intersection of vertex sets of S_1 and S_2 are denoted by $S_1 \bigcup S_2$ and $S_1 \bigcap S_2$, respectively. The *subtree graph* G_L has the vertex set of all subtrees of *T* and two subtrees S_1 and S_2 are joined by an edge, if S_2 is obtained from the subtree S_1 by adding/removing a single vertex. We denote the *distance between two subtrees* S_1 *and* S_2 in G_L by $d_L(S_1, S_2)$. Let $e_L(S_1) = \max\{d_L(S_1, S) | S$ is a subtree of $T\}$ be the *L*-eccentricity of the subtree S_1. Clearly, $e_L(S_1)$ equals the eccentricity of the subtree S_1 in the graph G_L. The subtree S_1 is a *central subtree* of a tree *T* if it has the minimum *L*-eccentricity. A central subtree with the minimum number of vertices is a *least central subtree* of a tree *T*. If S_1 and S_2 are subtrees of a tree *T*, then the *distance* $d_T(S_1, S_2)$ *between* S_1 *and* S_2 *in T* is the length of the shortest path joining two vertices of S_1 and S_2 in *T*.

The basic theory of least central subtrees has been established in a series of articles [13, 6, 7, 5]. The joinsemilattice of subtrees $L(T)$ was introduced in [13]. The meet $S_1 \wedge S_2$ of subtrees S_1 and S_2 equals $S_1 \bigcap S_2$ whenever $S_1 \bigcap S_2 \neq \varnothing$ and the join $S_1 \vee S_2$ is the least subtree of *T* containing the subtrees S_1 and S_2. Clearly, the Hasse diagram graph of $L(T)$ equals the subtree graph G_L.

The joinsemilattice has a certain algebraic structure with respect to meet and join operations. The subtrees (convex subalgebras/ideals) of a tree constitute the dual of a median semilattice in the sense of Sholander [14, 15].

The joinsemilattice $L(T)$ has a limited distributivity property (a median algebra property). This implies that if S_1 and S_2 are elements of $L(T)$ and if $S_1 \wedge S_2$ exists, the Hasse diagram graph of $L(T)$ contains the shortest path from S_1 to S_2 through $S_1 \vee S_2$ as well as through $S_1 \wedge S_2$. Similarly, if S_1 and S_2 are vertices of G_L (subtrees of T) such that $S_1 \cap S_2 \neq \emptyset$, then there exists a shortest path of G_L from S_1 to S_2 through $S_1 \cup S_2$ as well as through $S_1 \cap S_2$.

The underlying subtree graph G_L of a tree T is a median graph, [1, 2, 8-10]. Thus, every three subtrees S_1, S_2 and S_3 have a unique median subtree: a vertex $m(S_1, S_2, S_3)$ that belongs to shortest paths between each pair of S_1, S_2 and S_3.

2. Discrete Optimization Problems

The following lemma [13, Lemma 1] shows how one can calculate the distance $d_L(S_1, S_2)$ directly in the tree T without constructing the subtree graph G_L.

Lemma 1. *Let G_L be the subtree graph of a tree T. Let S_1 and S_2 be two subtrees of T. Then the distance between S_1 and S_2 in G_L is*

$$d_L(S_1, S_2) = \begin{cases} |S_1| + |S_2| + 2(d_T(S_1, S_2) - 1), & \textit{if } S_1 \cap S_2 = \emptyset, \\ |S_1 \cup S_2| - |S_1 \cap S_2| = |S_1| + |S_2| - 2|S_1 \cap S_2|, & \textit{if } S_1 \cap S_2 \neq \emptyset. \end{cases}$$

We consider the solutions of following three discrete optimization problems in G_L. A fourth approach has been introduced by Slater in [16]. See also [17]. We have not pursued this line in this short note.

The first discrete optimization problem reads

$$e_L(C_L) = \min_{S_1 \subseteq T} \max\{d_L(S_1, S) \,|\, S \text{ is a subtree of } T\}$$

subject to the additional constraint that among all subtrees satisfying the minimax criteria, only those subtrees which are minimal in size, are selected. Without the constraint, the solutions are vertices(subtrees) of the center of the subtree graph. Constrained solutions C_L are least central subtrees. The concept of a least central subtree in articles [13, 6, 7, 5] agrees with the definition in terms of solutions of the discrete optimization problem.

The second discrete optimization problem reads

$$d_L(D_L) = \min_S \left\{ \sum_{S_1 \subseteq T} d_L(S, S_1) \,|\, S \text{ is a subtree of } T \right\}$$

$$= \min_S \{ d_L(S) \,|\, S \text{ is a subtree of } T \}$$

subject to the additional constraint that among all subtrees satisfying the minimum criteria, only those subtrees which are minimal in size, are selected. Without the constraint, the solutions are vertices(subtrees) of the distance center of the subtree graph. Constrained solutions D_L are least central subtrees with respect to minimal distance sum criteria.

The third discrete optimization problem reads

$$b_L(B_L) = \min_S \{ b(S) \,|\, S \text{ is a subtree of } T \}$$

subject to the additional constraint that among all subtrees satisfying the minimum criteria, only those subtrees which are minimal in size, are selected. Without the constraint, the solutions are vertices(subtrees) of the branch weight center of the subtree graph. Constrained solutions B_L are least central subtrees with respect to minimal branch weight criteria.

3. The Distance Center and the Branch Weight Center

We first recall some results from article [12].

Theorem 1 [12, Theorem I]**.** *The branch weight center of a connected graph is convex*.

Theorem 2 [12, Theorems II-IV]**.** *Let G be a median graph. Then the distance center and the branch weight center coincide. Moreover, the vertices of the distance center induce a graph isomorphic to the Hasse diagram graph of a finite Boolean lattice.*

The proofs of these theorems are written in article [12]. We remark that Hasse diagram graphs of finite Boolean lattices are isomorphic to finite hypercubes Q_n, $n = 0, 1, 2, \ldots$. The graph Q_0 consists of a single vertex, Q_1 is a complete graph on two vertices, etc. By the properties of the subtree graph, we obtain the following general result.

Theorem 3. *Let G_L be a subtree graph of a tree T. Then the distance center and the branch weight center coincide. The distance center is isomorphic to a hypercube. The greatest vertex* (*subtree*), *say S_M, in the distance center is unique. The least vertex* (*subtree*), *say S_m, in the distance center is unique. We have the inclusion $S_m \subseteq S_M$, thus $d_L(S_m, S_M) = |S_M| - |S_m|$.*

Proof. The subtree graph G_L is a median graph. It follows from the results of [12] that distance center of the subtree graph is a hypercube. Moreover, the distance center and the branch weight center coincide and by Theorem 1, the distance center is convex.

Next, we consider the joinsemilattice $L(T)$. Due to convexity of the distance center, the greatest vertex in the distance center must be unique. Otherwise the existence of two greatest central subtrees S_M and S'_M implies that the join $S_M \vee S'_M$ resides in the distance center, which contradicts the maximality of S_M.

If S is a vertex of a distance center, then $S \subseteq S_M$, otherwise $S_M \vee S$ belongs to the distance center contradicting the maximality of S_M. This implies that $d_L(S_M, S) = |S_M| - |S|$ for every vertex S of distance center. Clearly, S_m is the vertex of distance center farthest from S_M with distance

$d_L(S_M, S_m) = |S_M| - |S_m|$. Since distance center is a hypercube, S_m is unique. □

Corollary 1. *We have* $B_L = D_L = S_m$. *If* $S_m \neq S_M$, *then* $|G_L|$ *is an even number*.

Proof. If $S_m \neq S_M$, then there exist two adjacent vertices S_1 and S_2 in the distance center of the subtree graph G_L. Let $U_{i,j} = \{S \mid S$ is a subtree of T such that $d_L(S_i, S) < d_L(S_j, S)\}$, for $i, j = 1, 2$. By property 2.2 in [4], $d_L(S_1) - d_L(S_2) = |U_{2,1}| - |U_{1,2}|$. Since $d_L(S_1) = d_L(S_2)$, this implies that $|U_{1,2}| = |U_{2,1}|$ and since median graphs are bipartite, we have $|U_{1,2}| = |U_{2,1}| = \frac{1}{2}|G_L|$. □

Note that our result for uniqueness of least and greatest elements of the distance center in the joinsemilattice is true for general median lattices and semilattices (meetsemilattices and joinsemilattices). The proof remains essentially the same.

4. Comparison of Various Centers

We have proved that the distance center D and the branch weight center B of the subtree graph G_L coincide and are isomorphic to a hypercube. Moreover, the least element within these centers is unique. Our computations show that central hypercubes other than Q_0 and Q_1 and Q_2 are quite rare. In a complete computer search among subtree graphs for trees with $|T| \leq 14$, only one larger hypercube is found, namely, one Q_4 within subtree graphs for trees with $|T| = 12$. The simple example of a complete bipartite graph $K_{2,3}$ shows that the equality $D = B$ is not valid for general bipartite graphs.

The structure of the center of the subtree graph is more delicate. The path $P_5 = \langle\{v_1v_2v_3v_4v_5\}\rangle$ induced by five vertices shows that the center of the subtree graph need not be a connected subgraph of G_L. Here the center of the subtree graph of P_5 is $\langle\{v_3\}\rangle$, $\langle\{v_2v_3v_4\}\rangle$, $\langle\{v_1v_2v_3v_4v_5\}\rangle$. Furthermore, least central subtrees need not be unique. The tree with two paths $\langle\{v_1v_2v_3v_4v_5v_6v_7\}\rangle$ and $\langle\{v_8v_9v_4v_{10}v_{11}\}\rangle$ crossing at the vertex v_4 shows that there may be several least central subtrees in a tree: $e_L(\langle\{v_3, v_4, v_5, v_{10}\}\rangle) = 7 = e_L(\langle\{v_3, v_4, v_5, v_9\}\rangle)$. If several least central subtrees exist, then the set of least central subtrees cannot be a connected subgraph of G_L.

It seems that there is no inclusion between the least central subtree C_L and the least distance center subtree D_L in the graph of subtrees. We show this by examples. For the path P_5 induced by five vertices, the least central subtree is $C_L = \langle\{v_3\}\rangle$. The least distance center subtree is $D_L = \langle\{v_2, v_3, v_4\}\rangle$. Here C_L is a subtree of D_L. Another example consists of two paths $\langle\{v_1v_2v_3v_4v_5v_6\}\rangle$ and $\langle\{v_7v_3v_8\}\rangle$ crossing at vertex v_3. For the resulting tree of 8 vertices, the least central subtree is $C_L = \langle\{v_2, v_3, v_4, v_5\}\rangle$ and the least distance center subtree is $D_L = \langle\{v_2, v_3, v_4\}\rangle$. Here D_L is a subtree of C_L.

We are working in order to prove that the set of least central subtrees enjoys the so-called exchange property (swapping property). For any two least central subtrees C_L and C'_L $(C_L \neq C'_L)$, there exists a finite sequence of least central subtrees

$$C_L = C_L^1, C_L^2, ..., C_L^k = C'_L, \quad k \geq 2$$

such that for all $i = 2, ..., k$, the least central subtree C_L^i is obtained from the least central subtree C_L^{i-1} by removing a vertex and inserting a vertex. This means that the set of least central subtrees is closed with respect to the exchange operation.

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