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ON THE LOCAL AND GLOBAL PRINCIPLE FOR SYSTEMS OF RATIONAL HOMOGENEOUS FORMS IN A FINITE NUMBER OF VARIABLES

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Abstract

In this paper, we prove that the Hasse principle for any system of rational cubic forms, any system of rational homogeneous forms of degree at least 3 in an arbitrary number of variables is equivalent to the Hasse principle of certain systems of rational quadratic forms. This shows, in particular, the Hasse principle for any system of rational cubic forms or rational homogeneous forms of degree at least 3 in an arbitrary number of variables holds if and only if the intersection of the nonempty sets of nontrivial rational solutions of each quadratic form of the associated system of quadratic forms is nonempty.

1. Introduction

In Diophantine number theory, one is concerned with the fundamental question of whether a homogeneous polynomial

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$$P(x_1, x_2, x_3, \ldots, x_n) = 0 \tag{1.1}$$

with rational coefficients has a nontrivial rational solution. Answering this question, in general, is very difficult and not always possible. This is the central theme of Hilbert's Tenth Problem [5].

1.1. The Hasse principle

If a nontrivial rational solution to (1.1) exists, then we call it a *global solution* since $\mathbb{Q}$ is a global field. Since $\mathbb{Q}$ is a subfield of each of its completion $\mathbb{Q}_p$'s, where p is a finite or infinite prime, this solution is also a nontrivial solution in each $\mathbb{Q}_p$. The Hasse principle says that the *reverse direction also holds*. That is, if equation (1.1) has a nontrivial solution in each completion $\mathbb{Q}_p$, called a *local solution* since $\mathbb{Q}_p$ is a local field, then it also has a solution in $\mathbb{Q}$. If the polynomial in (1.1) has this property, then we say that *it satisfies the Hasse principle*. Not every homogeneous polynomial satisfies the Hasse principle. One of the best-known examples of such a polynomial is given by Selmer [8]:

$$3x^3 + 4y^3 + 5z^3 = 0. \tag{1.2}$$

For the Hasse principle for forms of degrees 3 and 4, see [2] and [6] for some results. Concerning systems of forms, the Hasse principle for systems of quadratic forms has been studied by Davenport [1], Heath-Brown [3] and others. For systems of cubic forms, there are currently no similar systematic studies. Some results are obtained for such systems such as those found in [4] and [9].

1.2. The Hasse Minkowski theorem

Minkowski established, around 1920, the Hasse principle for quadratic forms with rational coefficients.

Theorem 1.1. *Let* $F(x_1, x_2, x_3, ..., x_n) = 0$ *be a quadratic form with rational coefficients. Then* $F(x_1, x_2, x_3, ..., x_n) = 0$ *has a nontrivial local solution at all places if and only if* $F(x_1, x_2, x_3, ..., x_n) = 0$ *has a nontrivial global solution, i.e., a nontrivial rational solution. That is, the Hasse principle holds for* $F(x_1, x_2, x_3, ..., x_n) = 0$.

For cubic forms of two variables, Fujiwara [2] obtains the following result:

Theorem 1.2. *Let* $P(x_1, x_2) = 0$ *be a cubic form with rational coefficients, called a binary cubic form. Then the Hasse principle holds for* $P(x_1, x_2) = 0$.

For any system of binary rational cubic forms, the following result is known:

Theorem 1.3 [9]. *If*

$$S = \begin{cases} G_1(x, y) = 0 \\ G_2(x, y) = 0 \\ \vdots \\ G_m(x, y) = 0 \end{cases}$$

is a system of binary rational quadratic forms, then S satisfies the Hasse principle.

The argument for this theorem makes use of the result concerning quadratic forms in [7].

2. Result

In this paper, we prove that the Hasse principle for any rational cubic form or rational homogeneous form of degree at least 3 in an arbitrary number of variables is equivalent to the Hasse principle of a certain system of rational quadratic forms.

Theorem 2.1. *Let*

$$\mathcal{S} = \begin{cases} G_1(x_1, x_2, ..., x_n) = 0 \\ G_2(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ G_m(x_1, x_2, ..., x_n) = 0 \end{cases}$$

be a system of rational homogeneous cubic forms in n variables. Then there exists a system of finitely many rational quadratic forms $\mathcal{T}$ *such that the Hasse principle for* $\mathcal{S}$ *is equivalent to the Hasse principle for* $\mathcal{T}$.

If all forms in S are replaced by rational homogeneous forms of degree at least 3 *in n variables, then the same statement also holds.*

This result shows that the Hasse principle for any system of rational cubic forms or rational homogeneous forms of degree at least 3 in an arbitrary number of variables holds if and only if the intersection of the nonempty sets of nontrivial rational solutions of each quadratic form of the associated system of quadratic forms is nonempty. This provides in particular an intuitive way for seeing why it is more difficult for the Hasse principle for forms of degrees higher than two to hold.

3. Proof of Result

Proof. (Proof of Theorem 2.1)

(I) All forms in $\mathcal{S}$ are cubic forms:

For each i in $\{1, ..., m\}$, define

$$F_{ij}(x_1, x_2, ..., x_n) = x_j G_i(x_1, x_2, ..., x_n) \tag{3.1}$$

for $j = 1, ..., n$. Thus, we obtain the system

$$\mathcal{L} = \begin{cases} F_{11}(x_1, x_2, \ldots, x_n) = 0 \\ F_{12}(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ F_{1n}(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ F_{21}(x_1, x_2, \ldots, x_n) = 0 \\ F_{22}(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ F_{2n}(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ F_{m1}(x_1, x_2, \ldots, x_n) = 0 \\ F_{m2}(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ F_{mn}(x_1, x_2, \ldots, x_n) = 0. \end{cases} \tag{3.2}$$

Next, let us define the following variables:

$$X_{u,v} := x_u x_v \tag{3.3}$$

for $u, v \in \{1, \ldots, n\}$. Then

$$X_{u,v}^2 - X_{u,u} X_{v,v} = 0 \tag{3.4}$$

for $u, v \in \{1, \ldots, n\}$ and

$$X_{u,v} X_{s,t} - X_{u,s} X_{v,t} = 0 \tag{3.5}$$

as well as

$$X_{u,v} X_{s,t} - X_{u,t} X_{v,s} = 0 \tag{3.6}$$

for all $u, v, s, t \in \{1, \ldots, n\}$. Now let us denote the system

$$\begin{cases} X_{u,v}^2 - X_{u,u}X_{v,v} = 0 \\ X_{u,s}^2 - X_{u,u}X_{s,s} = 0 \\ X_{u,t}^2 - X_{u,u}X_{t,t} = 0 \\ X_{v,s}^2 - X_{v,v}X_{s,s} = 0 \\ X_{v,t}^2 - X_{v,v}X_{t,t} = 0 \\ X_{s,t}^2 - X_{s,s}X_{t,t} = 0 \\ X_{u,v}X_{s,t} - X_{u,s}X_{v,t} = 0 \\ X_{u,v}X_{s,t} - X_{u,t}X_{v,s} = 0 \end{cases} \tag{3.7}$$

by

$$\overline{X}_{[uvst]} = 0$$

and the system

$$\{\overline{X}_{[uvst]} = 0\}_{\{[uvst]|u,v,s,t\in\{1,\ldots,n\}\}} \tag{3.8}$$

by

$$\boldsymbol{X}_{1,\ldots,n} = 0,$$

where $\{[uvst]|u, v, s, t \in \{1, ..., n\}\}$ denotes the collection of all distinct unordered quadruples $[uvst]$ with u, v, s, t being in $\{1, ..., n\}$. There are $\binom{n}{4}$ such collections of distinct unordered quadruples. Thus, for each $[uvst]$, system $\overline{X}_{[uvst]} = 0$ consists of $8\binom{n}{4}$ rational quadratic forms.

Proposition 3.1. *Let F denote either* $\mathbb{Q}$ *or* $\mathbb{Q}_p$ *for some prime p finite or infinite. System* $\mathcal{S}$ *has a nontrivial solution in F if and only if the system*

$$\mathfrak{L} := \begin{cases} F_{11}(X_{1,1}, X_{2,1}, ..., X_{n,1}) = 0 \\ F_{12}(X_{1,2}, X_{2,2}, ..., X_{n,2}) = 0 \\ \vdots \\ F_{1n}(X_{1,n}, X_{2,n}, ..., X_{n,n}) = 0 \\ \vdots \\ F_{21}(X_{1,1}, X_{2,1}, ..., X_{n,1}) = 0 \\ F_{22}(X_{1,2}, X_{2,2}, ..., X_{n,2}) = 0 \\ \vdots \\ F_{2n}(X_{1,n}, X_{2,n}, ..., X_{n,n}) = 0 \\ \vdots \\ F_{m1}(X_{1,1}, X_{2,1}, ..., X_{n,1}) = 0 \\ F_{m2}(X_{1,2}, X_{2,2}, ..., X_{n,2}) = 0 \\ \vdots \\ F_{mn}(X_{1,n}, X_{2,n}, ..., X_{n,n}) = 0 \\ \boldsymbol{X}_{1,\dots,n} = 0 \end{cases} \tag{3.9}$$

of $mn + 8\binom{n}{4}$ *rational homogeneous quadratic forms has a nontrivial solution in* F.

Proof. By construction, the 'only if' direction is clear. Let us prove the if direction.

For $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$,

$$F_{ij}(X_{1,j}, X_{2,j}, ..., X_{n,j}) = \sum_{1 \le u \le n} a_{ujuj}^{(ij)} X_{uj}^2 + \sum_{1 \le u \ne v \le n} a_{ujvj}^{(ij)} X_{uj} X_{vj} \tag{3.10}$$

for $a_{ujuj}^{(ij)}$'s and $a_{ujvj}^{(ij)}$'s some elements of F for all u and v in $\{1, ..., n\}$.

Suppose system $\mathfrak{L}$ has a nontrivial solution in F, say,

$$(A_{1,1}, ..., A_{1,n}, ..., A_{n,1}, ..., A_{n,n}), \tag{3.11}$$

where A_{ij}'s are in F for all i, j and $A_{i_0 j_0} \neq 0$ for some $A_{i_0 j_0}$. Then we have

$$\begin{cases} A_{u,v}^2 - A_{u,u}A_{v,v} = 0 \\ A_{u,s}^2 - A_{u,u}A_{s,s} = 0 \\ A_{u,t}^2 - A_{u,u}A_{t,t} = 0 \\ A_{v,s}^2 - A_{v,v}A_{s,s} = 0 \\ A_{v,t}^2 - A_{v,v}A_{t,t} = 0 \\ A_{s,t}^2 - A_{s,s}A_{t,t} = 0 \\ A_{u,v}A_{s,t} - A_{u,s}A_{v,t} = 0 \\ X_{u,v}A_{s,t} - A_{u,t}A_{v,s} = 0 \end{cases} \tag{3.12}$$

by (3.7) and

$$0 = \sum_{1 \leq u \leq n} a_{ujuj}^{(ij)} A_{uj}^2 + \sum_{1 \leq u \neq v \leq n} a_{ujvj}^{(ij)} A_{uj} A_{vj} \tag{3.13}$$

for $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ as well as for all u and v in $\{1, ..., n\}$. It follows that

$$\left(\frac{A_{1,1}}{A_{i_0,j_0}}, ..., \frac{A_{1,n}}{A_{i_0,j_0}}, ..., \frac{A_{i_0,1}}{A_{i_0,j_0}}, ..., \frac{A_{i_0,j_0-1}}{A_{i_0,j_0}}, 1, \right.$$

$$\left. \frac{A_{i_0,j_0+1}}{A_{i_0,j_0}}, ..., \frac{A_{i_0,n}}{A_{i_0,j_0}}, ..., \frac{A_{n,1}}{A_{i_0,j_0}}, ..., \frac{A_{n,n}}{A_{i_0,j_0}}\right) \tag{3.14}$$

is also a nontrivial solution to system $\mathfrak{L}$. By (3.12), it follows that

$$A_{i_0,j_0}^2 = A_{i_0,i_0}A_{j_0,j_0} \tag{3.15}$$

and thus

$$A_{i_0,i_0} \neq 0 \quad \text{and} \quad A_{j_0,j_0} \neq 0. \tag{3.16}$$

Hence,

$$\left(\frac{A_{1,1}}{A_{i_0,i_0}}, \ldots, \frac{A_{1,n}}{A_{i_0,i_0}}, \ldots, \frac{A_{i_0,1}}{A_{i_0,i_0}}, \ldots, \frac{A_{i_0,i_0-1}}{A_{i_0,i_0}}, 1, \frac{A_{i_0,i_0+1}}{A_{i_0,i_0}},\right.$$

$$\left.\frac{A_{i_0,i_0+1}}{A_{i_0,i_0}}, \ldots, \frac{A_{i_0,n}}{A_{i_0,i_0}}, \ldots, \frac{A_{n,1}}{A_{i_0,i_0}}, \ldots, \frac{A_{n,n}}{A_{i_0,i_0}}\right) \tag{3.17}$$

and

$$\left(\frac{A_{1,1}}{A_{j_0,j_0}}, \ldots, \frac{A_{1,n}}{A_{j_0,j_0}}, \ldots, \frac{A_{j_0,1}}{A_{j_0,j_0}}, \ldots, \frac{A_{j_0,j_0-1}}{A_{j_0,j_0}}, 1, \frac{A_{j_0,j_0+1}}{A_{j_0,j_0}},\right.$$

$$\left.\frac{A_{j_0,j_0+1}}{A_{j_0,j_0}}, \ldots, \frac{A_{j_0,n}}{A_{j_0,j_0}}, \ldots, \frac{A_{n,1}}{A_{j_0,j_0}}, \ldots, \frac{A_{n,n}}{A_{j_0,j_0}}\right) \tag{3.18}$$

are also nontrivial solutions to system $\mathfrak{L}$ by (3.16). In particular, we have

$$\begin{cases} \left(\frac{A_{u,v}}{A_{i_0,i_0}}\right)^2 - \frac{A_{u,u}}{A_{i_0,i_0}}\frac{A_{v,v}}{A_{i_0,i_0}} = 0 \\ \left(\frac{A_{u,s}}{A_{i_0,i_0}}\right)^2 - \frac{A_{u,u}}{A_{i_0,i_0}}\frac{A_{s,s}}{A_{i_0,i_0}} = 0 \\ \left(\frac{A_{u,t}}{A_{i_0,i_0}}\right)^2 - \frac{A_{u,u}}{A_{i_0,i_0}}\frac{A_{t,t}}{A_{i_0,i_0}} = 0 \\ \left(\frac{A_{v,s}}{A_{i_0,i_0}}\right)^2 - \frac{A_{v,v}}{A_{i_0,i_0}}\frac{A_{s,s}}{A_{i_0,i_0}} = 0 \\ \left(\frac{A_{v,t}}{A_{i_0,i_0}}\right)^2 - \frac{A_{v,v}}{A_{i_0,i_0}}\frac{A_{t,t}}{A_{i_0,i_0}} = 0 \\ \left(\frac{A_{s,t}}{A_{i_0,i_0}}\right)^2 - \frac{A_{s,s}}{A_{i_0,i_0}}\frac{A_{t,t}}{A_{i_0,i_0}} = 0 \\ \frac{A_{u,v}}{A_{i_0,i_0}}\frac{A_{s,t}}{A_{i_0,i_0}} - \frac{A_{u,s}}{A_{i_0,i_0}}\frac{A_{v,t}}{A_{i_0,i_0}} = 0 \\ \frac{A_{u,v}}{A_{i_0,i_0}}\frac{A_{s,t}}{A_{i_0,i_0}} - \frac{A_{u,t}}{A_{i_0,i_0}}\frac{A_{v,s}}{A_{i_0,i_0}} = 0. \end{cases} \tag{3.19}$$

As a result, it follows from (3.19) that the following statements hold:

(1) For each i in $\{1, ..., n\}$,

$$\frac{A_{i,i}}{A_{i_0 i_0}} = \left(\frac{A_{i,i_0}}{A_{i_0,i_0}}\right)^2 = \left(\frac{A_{i_0,i}}{A_{i_0,i_0}}\right)^2. \tag{3.20}$$

(2) For $i,\ j \in \{1, ..., n\}$,

$$\frac{A_{i,j}}{A_{i_0,i_0}} = \frac{A_{i,i_0}}{A_{i_0,i_0}}\frac{A_{i_0,j}}{A_{i_0,i_0}} = \frac{A_{i,i_0}}{A_{i_0,i_0}}\frac{A_{j,i_0}}{A_{i_0,i_0}} = \frac{A_{i_0,i}}{A_{i_0,i_0}}\frac{A_{i_0,j}}{A_{i_0,i_0}}$$

$$= \frac{A_{i_0,i}}{A_{i_0,i_0}}\frac{A_{j,i_0}}{A_{i_0,i_0}}. \tag{3.21}$$

From (3.9)-(3.10), it can be verified that the following equations hold for $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$:

$$0 = F_{ij}(A_{1,j},\ A_{2,j},\ ...,\ A_{n,j}) = F_{ij}\left(\frac{A_{1,j}}{A_{i_0,i_0}},\ \frac{A_{2,j}}{A_{i_0,i_0}},\ ...,\ \frac{A_{n,j}}{A_{i_0,i_0}}\right)$$

$$= \sum_{1\le u\le n} a_{ujuj}^{(ij)}\left(\frac{A_{uj}}{A_{i_0,i_0}}\right)^2 + \sum_{1\le u\ne v\le n} a_{ujvj}^{(ij)}\frac{A_{u,j}}{A_{i_0,i_0}}\frac{A_{v,j}}{A_{i_0,i_0}}. \tag{3.22}$$

In particular,

$$0 = F_{ii_0}(A_{1,i_0},\ A_{2,i_0},\ ...,\ A_{i_0-1,i_0},\ A_{i_0,i_0},\ A_{i_0+1,i_0},\ ...,\ A_{n,i_0})$$

$$= F_{ii_0}\left(\frac{A_{1,i_0}}{A_{i_0,i_0}},\ \frac{A_{2,i_0}}{A_{i_0,i_0}},\ ...,\ \frac{A_{i_0-1,i_0}}{A_{i_0,i_0}},\ 1,\ \frac{A_{i_0+1,i_0}}{A_{i_0,i_0}},\ ...,\ \frac{A_{n,i_0}}{A_{i_0,i_0}}\right)$$

$$= \sum_{1\le u\le n} a_{ui_0ui_0}^{(ij)}\left(\frac{A_{ui_0}}{A_{i_0,i_0}}\right)^2 + \sum_{1\le u\ne v\le n} a_{ui_0vi_0}^{(ij)}\frac{A_{u,i_0}}{A_{i_0,i_0}}\frac{A_{v,j}}{A_{i_0,i_0}}$$

$$= G_i\left(\frac{A_{1,i_0}}{A_{i_0,i_0}},\ \frac{A_{2,i_0}}{A_{i_0,i_0}},\ ...,\ \frac{A_{i_0-1,i_0}}{A_{i_0,i_0}},\ 1,\ \frac{A_{i_0+1,i_0}}{A_{i_0,i_0}},\ ...,\ \frac{A_{n,i_0}}{A_{i_0,i_0}}\right) \tag{3.23}$$

for each i in $\{1, ..., n\}$ since

$$\frac{A_{ui_0}}{A_{i_0, i_0}} = \frac{A_{j, i_0}}{A_{i_0, i_0}} \frac{A_{i_0, i_0}}{A_{i_0, i_0}} \tag{3.24}$$

for each u in $\{1, ..., n\}$. Therefore,

$$\left(\frac{A_{1, i_0}}{A_{i_0, i_0}}, \frac{A_{2, i_0}}{A_{i_0, i_0}}, ..., \frac{A_{i_0 - 1, i_0}}{A_{i_0, i_0}}, 1, \frac{A_{i_0 + 1, i_0}}{A_{i_0, i_0}}, ..., \frac{A_{n, i_0}}{A_{i_0, i_0}} \right) \tag{3.25}$$

is a nontrivial solution in F of the system

$$\begin{cases} G_1(x_1, x_2, ..., x_n) = 0 \\ G_2(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ G_m(x_1, x_2, ..., x_n) = 0 \end{cases}$$

as needed. □

Therefore, the Hasse principle of any system of cubic forms $\mathcal{S}$ is equivalent to the Hasse principle of the associated system of quadratic forms $\mathcal{L}$.

(II) Forms in $\mathcal{S}$ are homogeneous forms of degree at least 3 in n variables:

Suppose that $\mathcal{S}$ has m forms and let d_i denote the homogeneous degree of form $G_i(x_1, ..., x_n)$ for $i = 1, ..., m$. Let d be the maximal of the d_i's. Let D be the least even positive integer such that $D \geq d$. Define system $\mathcal{L}$ as follows:

$$\mathcal{L} = \begin{cases} F_{11}(x_1, x_2, ..., x_n) = 0 \\ F_{12}(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ F_{1n}(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ F_{21}(x_1, x_2, ..., x_n) = 0 \\ F_{22}(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ F_{2n}(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ F_{m1}(x_1, x_2, ..., x_n) = 0 \\ F_{m2}(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ F_{mn}(x_1, x_2, ..., x_n) = 0, \end{cases} \tag{3.26}$$

where

$$F_{ij}(x_1, x_2, ..., x_n) = x_j^{D-d_i} G_i(x_1, x_2, ..., x_n). \tag{3.27}$$

Next, let us define the following variables: let I consist of all vectors of the form

$$\alpha = (\alpha_1, ..., \alpha_n), \tag{3.28}$$

where α_i's are nonnegative integers such that

$$\sum_{1 \le i \le n} \alpha_i = \frac{D}{2}. \tag{3.29}$$

For each α in I, let us define the following variables:

$$X_\alpha \coloneqq \prod_{1 \le i \le n} x_i^{\alpha_i}. \tag{3.30}$$

Then for each element X_α,

$$X_\alpha^2 - X_\beta X_\gamma = 0 \tag{3.31}$$

for all elements $\beta = (\beta_1, ..., \beta_n)$ and $\gamma = (\gamma_1, ..., \gamma_n)$ in I such that

$$2\alpha_i = \beta_i + \gamma_i \tag{3.32}$$

for each i in $\{1, ..., n\}$. For any pair of $\mu = (\mu_1, ..., \mu_n)$ and $\nu = (\nu_1, ..., \nu_n)$ in I,

$$X_\mu X_\nu - X_\eta X_\varepsilon = 0 \tag{3.33}$$

for all $\eta = (\eta_1, ..., \eta_n)$ and $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ in I such that

$$\mu_i + \nu_i = \eta_i + \varepsilon_i \tag{3.34}$$

for each i in $\{1, ..., n\}$. Now let us represent the system consisting of all equations of the form (3.31) by

$$\boldsymbol{X}_{\alpha \in I} = 0 \tag{3.35}$$

and the system consisting of all equations of the form (3.33) by

$$\boldsymbol{X}_{(\mu, \nu) \in I \times I} = 0. \tag{3.36}$$

System $\mathcal{L}$ can be rewritten as:

$$\mathcal{L} = \begin{cases} F_{11}(\{X_\alpha\}_{\{\alpha \in I_{11} \subseteq I\}}) = 0 \\ F_{12}(\{X_\alpha\}_{\{\alpha \in I_{12} \subseteq I\}}) = 0 \\ \vdots \\ F_{1n}(\{X_\alpha\}_{\{\alpha \in I_{1n} \subseteq I\}}) = 0 \\ \vdots \\ F_{21}(\{X_\alpha\}_{\{\alpha \in I_{21} \subseteq I\}}) = 0 \\ F_{22}(\{X_\alpha\}_{\{\alpha \in I_{22} \subseteq I\}}) = 0 \\ \vdots \\ F_{2n}(\{X_\alpha\}_{\{\alpha \in I_{2n} \subseteq I\}}) = 0 \\ \vdots \\ F_{m1}(\{X_\alpha\}_{\{\alpha \in I_{m1} \subseteq I\}}) = 0 \\ F_{m2}(\{X_\alpha\}_{\{\alpha \in I_{m2} \subseteq I\}}) = 0 \\ \vdots \\ F_{mn}(\{X_\alpha\}_{\{\alpha \in I_{mn} \subseteq I\}}) = 0, \end{cases} \tag{3.37}$$

where I_{ij} is an appropriate subset, depending on (3.27), of I for each i in $\{1, \ldots, m\}$ and j in $\{1, \ldots, n\}$. By combining (3.35), (3.36) and (3.37), we have the following:

Proposition 3.2. *Let F denote either $\mathbb{Q}$ or $\mathbb{Q}_p$ for some prime p finite or infinite. Then system $\mathcal{S}$ has a nontrivial solution in F if and only if the system*

$$\mathfrak{L} = \begin{cases} F_{11}(\{X_\alpha\}_{\{\alpha \in I_{11} \subseteq I\}}) = 0 \\ F_{12}(\{X_\alpha\}_{\{\alpha \in I_{12} \subseteq I\}}) = 0 \\ \vdots \\ F_{1n}(\{X_\alpha\}_{\{\alpha \in I_{1n} \subseteq I\}}) = 0 \\ \vdots \\ F_{21}(\{X_\alpha\}_{\{\alpha \in I_{21} \subseteq I\}}) = 0 \\ F_{22}(\{X_\alpha\}_{\{\alpha \in I_{22} \subseteq I\}}) = 0 \\ \vdots \\ F_{2n}(\{X_\alpha\}_{\{\alpha \in I_{2n} \subseteq I\}}) = 0 \\ \vdots \\ F_{m1}(\{X_\alpha\}_{\{\alpha \in I_{m1} \subseteq I\}}) = 0 \\ F_{m2}(\{X_\alpha\}_{\{\alpha \in I_{m2} \subseteq I\}}) = 0 \\ \vdots \\ F_{mn}(\{X_\alpha\}_{\{\alpha \in I_{mn} \subseteq I\}}) = 0 \\ \boldsymbol{X}_{\alpha \in I} = 0 \\ \boldsymbol{X}_{(\mu, \nu) \in I \times I} = 0 \end{cases} \tag{3.38}$$

of homogeneous quadratic forms has a nontrivial solution in F.

Proof. A method similar to that found in the proof of Proposition 3.1 also works. □

Theorem 2.1 follows as a result. □

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