JP Journal of Fixed Point Theory and Applications
© 2015 Pushpa Publishing House, Allahabad, India
Published Online: December 2015
http://dx.doi.org/10.17654/JPFPTADec2015_129_155
Volume 10, Number 3, 2015, Pages 129-155 **ISSN: 0973-4228**

A FIXED POINT OPTIMIZATION ALGORITHM FOR THE EQUILIBRIUM PROBLEM OVER THE FIXED POINT SET AND ITS APPLICATIONS

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Abstract

In this paper, we discuss the equilibrium problem for a continuous bifunction over the fixed point set of a firmly nonexpansive mapping. We then present an iterative algorithm, which uses the firmly nonexpansive mapping at each iteration, for solving the problem. The algorithm is quite simple and it does not require monotonicity and Lipschitz-type condition on the equilibrium function. At the end of the paper, we present a numerical example and an application to the power control in CDMA data networks.

1. Introduction

In recent years, equilibrium problem (EP) is an important subject that recently has been considered in many research papers. It is well known that various classes of optimization, variational inequality, Kakutani fixed point,

Received: October 14, 2015; Revised: November 18, 2015; Accepted: December 15, 2015
2010 Mathematics Subject Classification: 65K10, 65K15, 90C25, 90C33.
Keywords and phrases: equilibrium problem, Nash equilibrium, variational inequality problem, fixed point problem, firmly nonexpansive mapping, CDMA network.
Communicated by K. K. Azad

Nash equilibrium in noncooperative game theory and minimax problems can be formulated as an equilibrium problem of the form (EP) [5].

The typical form of equilibrium problems is formulated by means of Ky Fan's inequality due to Ky Fan's contribution to this field and is given as [5]:

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C, \qquad EP(f, C)$$

where C is a nonempty closed convex subset in $\mathbb{R}^n$ and $f : C \times C \to \mathbb{R}$ is a bifunction such that $f(x, x) = 0$ for all $x \in C$. The set of solutions of $EP(f, C)$ is denoted by $Sol(f, C)$.

If $f(x, y) = \langle F(x), y - x \rangle$, where F is a mapping from C to C, then problem $EP(f, C)$ becomes the following variational inequality:

$$\text{find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0 \text{ for all } y \in C. \qquad VI(F, C)$$

The set of solutions of $VI(F, C)$ is denoted by $Sol(F, C)$.

It is well-known that x^* is solution of $VI(F, C)$ if and only if it is the fixed point of the mapping $Pr_C(I - \lambda F)$, that is, $x^* = Pr_C(x - \lambda F(x^*))$, where $\lambda > 0$ and Pr_C is Euclidean projector on C. Under the assumptions that F is strongly monotone and Lipschitz continuous, the mapping $Pr_C(I - \lambda F)$ is strictly contractive over C, hence the sequence $\{x^k\}_{k \in \mathbb{N}}$ generated by the projected gradient algorithm

$$\begin{cases} x^0 \in C, \\ x^{k+1} = Pr_C(x^k - \lambda F(x^k)), \end{cases}$$

converges to the unique solution x^* of $VI(F, C)$ [51].

If F is monotone and Lipschitz, then the projected gradient algorithm may not be convergent. For example, suppose $C = \mathbb{R}^2$ and F is a rotation with $\frac{\pi}{2}$ angle. It is obvious that F is monotone and Lipschitz. However, since $\| x^{k+1} \| \geq \| x^k \|$ for all k, the sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by the projected gradient algorithm does not converge to the origin - the unique solution of $VI(F, C)$.

In order to deal with this situation, Korpelevich introduced in [21] an extragradient algorithm:

$$\begin{cases} x^0 \in C, \\ y^k = Pr_C(x^k - \lambda F(x^k)), \\ x^{k+1} = Pr_C(x^k - \lambda F(y^k)). \end{cases}$$

Under the assumptions that F is L-Lipschitz and monotone, $\lambda \in \left(0, \frac{1}{L}\right)$, the sequences $\{x^k\}_{k\in\mathbb{N}}$ and $\{y^k\}_{k\in\mathbb{N}}$ converge to the same point $x^* \in Sol(F, C)$.

This extragradient algorithm has been extended to equilibrium problem in [29]:

$$\begin{cases} x^0 \in C, \\ y^k = \operatorname{argmin}\left\{\lambda f(x^k, y) + \frac{1}{2}\| y - x^k \| : y \in C\right\}, \\ x^{k+1} = \operatorname{argmin}\left\{\lambda f(y^k, y) + \frac{1}{2}\| y - x^k \| : y \in C\right\}. \end{cases}$$

Under the assumptions that f is pseudomonotone and Lipschitz-type continuous, the authors showed that the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges to a solution of $EP(f, C)$.

To avoid the Lipschitzian condition, the Armijo-backtracking linesearch has been introduced in [45] to solve $VI(F, C)$. The authors used a hyperplane separating x^k from the solution set. Then the new iterate x^{k+1} is the projection of x^k onto this hyperplane. This method is also extended for pseudomonotone equilibrium problems in [1].

Since all the above methods require monotonicity or pseudomonotonicity of function *f*, a natural question arises: *Is it possible to solve equilibrium problems without the monotone and Lipschitz conditions on f.*

To answer this question, we introduce an algorithm to the following equilibrium problem over the fixed point set: given a continuous function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying $f(x, x) = 0$ for all $x \in \mathbb{R}^n$ and a firmly nonexpansive mapping $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$\text{find a point } x^* \in Fix(T) \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in Fix(T), \tag{$EP(f, Fix(T))$}$$

where $Fix(T) = \{x \in \mathbb{R}^n : Tx = x\}$. The solution set of this problem is denoted by $Sol(f, Fix(T))$. We note that, with $T = Pr_C$, the problem $EP(f, Fix(T))$ becomes problem $EP(f, C)$. Moreover, in many cases, we deal with the equilibrium problems, of which constraint set C is implicitly given. Then, the basic method cannot be applied effectively.

Iiduka and Yamada proposed in [14] a subgradient-type method for solving the problem $(EP(f, Fix(T)))$:

Step 0. Choose $\varepsilon_1 \geq 0$, $\lambda_1 > 0$ and $x_1 \in \mathbb{R}^n$ arbitrarily, and let $\rho_1 := \| x_1 \|$ and $k = 1$.

Step 1. Given $x^k \in \mathbb{R}^n$ and $\rho_k \geq 0$, choose $\varepsilon_k \geq 0$ and $\lambda_k > 0$.

- Find a point $y^k \in K_k := \{x \in \mathbb{R}^n : \| x \| \leq \rho_k + 1\}$ which satisfies

$$f(x^k, y^k) \geq 0 \quad \text{and} \quad \max_{y \in K_k} f(y, x^k) \leq f(y^k, x^k) + \varepsilon_k.$$

- Choose $\xi_k \in \partial f(y^k, \cdot)(x^k)$ arbitrarily and compute

$$x^{k+1} = T(x^k - \lambda_k f(y^k, x^k)\xi_k) \quad \text{and} \quad \rho_{k+1} = \max\{\rho_k, \| x^{k+1} \|\}.$$

Step 2. Update $k := k + 1,$ and go to Step 1.

The convergence of this algorithm was proved under suitable assumptions. One of them is the boundedness of the sequence $\{\xi^k\}_{k\in\mathbb{N}}$.

In [18], Iiduka considered the variational inequality problem over the fixed point set: Given C is a nonempty closed convex subset in $\mathbb{R}^n$ and $F : C \to C$ is a continuous operator, $T : C \to C$ is a firmly nonexpansive mapping. The variational inequality problem over the fixed point set can be formulated as

$$\text{find a point } x^* \in Fix(T) \text{ such that } \langle F(x), y - x\rangle \geq 0 \text{ for all } y \in Fix(T).$$

The solution set of this problem is denoted by $VI(F, Fix(T))$. For solving this problem, Iiduka proposed a fixed point optimization algorithm:

Step 0. Choose $x^1 \in C,$ $\lambda_1 \in (0, \infty)$ and $\alpha_1 \in [0, 1)$ arbitrarily, and set $n := 1.$

Step 1. Given $x^k \in C,$ choose $\lambda_k \in (0, \infty),$ $\alpha_k \in [0, 1),$ and compute x^{k+1} as follows:

$$\begin{cases} y^k := T(x^k - \lambda_k F(x^k)), \\ x^{k+1} := Pr_C(\alpha_k x^k + (1 - \alpha_k) y^k). \end{cases} \tag{1}$$

Step 2. Update $n := n + 1,$ and go to Step 1.

To prove the convergence of this algorithm, the condition: $VI(F, Fix(T)) \subset \Omega := \{x \in Fix(T) : f(x^k, x) \leq 0, \forall k \geq k_0\}$ is needed.

The main goal of this paper is to extend the fixed point optimization algorithm for solving the problem $(EP(f, Fix(T)))$. The convergence of algorithm will be proved without the condition $VI(F, Fix(T)) \subset \Omega := \{x \in Fix(T) : f(x^k, x) \leq 0, \forall k \geq k_0\}$.

The rest of this paper is organized as follows. Section 2 briefly explains the necessary mathematical background. Section 3 presents the fixed point optimization algorithm and proves that it converges to a solution of problem $(EP(f, Fix(T)))$ under certain assumptions. Numerical results are provided in Section 4.

2. Mathematical Preliminaries

A function $g : \mathbb{R}^m \to \mathbb{R}$ is said to be τ-*Hölder continuous* if there exist $Q > 0$ and $\tau \in (0, 1]$ such that $|g(x) - g(y)| \leq Q\| x - y \|^{\tau}$ for all $x, y \in \mathbb{R}^m$. If $\tau = 1$, then g is said to be *Lipschitz continuous*. It is obvious that any Hölder continuous function is continuous.

A fixed point of mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, is a point, $x \in \mathbb{R}^n$, satisfying $T(x) = x$. The set $Fix(T) = \{x \in \mathbb{R}^n : T(x) = x\}$ called the *fixed point set* of T. A mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *nonexpansive* if $\| T(x) - T(y) \| \leq \| x - y \|$ for all $x, y \in \mathbb{R}^n$. Any nonexpansive mapping is also continuous. We summarize some properties of the fixed point set of a nonexpansive mapping in the following proposition:

Proposition 1 (See [11]). *Let C be a nonempty, closed convex subset of $\mathbb{R}^n$ and $T : C \to C$ be a nonexpansive mapping. Then*

(a) *$Fix(T)$ is closed and convex.*

(b) *If C is bounded, then $Fix(T)$ is nonempty.*

A mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *firmly nonexpansive* if $\| T(x) - T(y) \|^2 \leq \langle x - y, T(x) - T(y) \rangle$ for all $x, y \in \mathbb{R}^n$. Mapping T is firmly nonexpansive if and only if it can be formulated as $T = \frac{1}{2}I + \frac{1}{2}N$, where I is identity and N is some nonexpansive mapping. It is well-known that any firmly nonexpansive mapping is also nonexpansive.

Given a nonempty closed convex set C in $\mathbb{R}^n$. The metric projection onto C is defined as $Pr_C : \mathbb{R}^n \to C$, $Pr_C(x) = \text{argmin}\{\| x - y \| : y \in C\}$. The metric projection also can be defined by relation:

$$x^* \in C \text{ satisfied } x^* = Pr_C x \Leftrightarrow x^* \in C \text{ satisfied } \langle x - x^*, y - x^* \rangle \leq 0 \quad \text{for all } y \in C,$$

and therefore Pr_C is firmly nonexpansive with $Fix(Pr_C) = C$. We summarize some properties of the nonexpansive mapping in the following proposition:

Proposition 2 (See [51])**.** (a) *Let* $T_i : C \to C$ *be nonexpansive mappings* $(i = 1, 2, ..., m)$. *Then both* $T_m T_{m-1} \cdots T_1$ *and* $\sum_{i=1}^m w_i T_i$ *are also nonexpansive, where* $w_i \in [0, 1]$ *and* $\sum_{i=1}^m w_i = 1$.

(b) *Let* $T_i : C \to C\,(i = 1, 2, ..., m)$ *be nonexpansive mappings satisfying* $\bigcap_{i=1}^m Fix(T_i) \neq \emptyset$. *Then* $Fix\left(\sum_{i=1}^m w_i T_i\right) = \bigcap_{i=1}^m Fix(T_i)$, *where* $w_i \in (0, 1]$ *and* $\sum_{i=1}^m w_i = 1$.

(c) $T : C \to C$ *is firmly nonexpansive if and only if* $2T - I$ *is nonexpansive*. *Moreover, for given firmly nonexpansive mappings* $T_i : C \to C\,(i = 1, 2, ..., m)$ *and* $w_i \geq 0$ *satisfying* $\sum_{i=1}^m w_i = 1$, $\sum_{i=1}^m w_i T_i$ *is firmly nonexpansive*.

We need the following technical lemma:

Lemma 1 (See [49])**.** *Suppose that* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *are sequences of nonnegative real numbers such that*

$$\alpha_{k+1} \le \alpha_k + \beta_k, \quad k \ge 1,$$

where $\sum_{k=1}^{\infty} \beta_k < \infty$. *Then the sequence* $\{\alpha_k\}$ *is convergent.*

3. Fixed Point Optimization Algorithm

Assumption 1. We assume

(A1) $C \subset \mathbb{R}^n$ is a nonempty, closed convex set.

(A2) $T : C \to C$ is a firmly nonexpansive mapping with $Fix(T) \neq \emptyset$.

(A3) $f : C \times C \to \mathbb{R}$ is a τ-Hölder continuous bifunction satisfying $f(x, x) = 0$ for all $x \in \mathbb{R}^n$. Function $f(x, \cdot)$ convex, $\forall x \in C$.

This paper discusses the following equilibrium problem over fixed point set:

Problem 1. Under Assumption 1, we are interested in

finding a point $x^* \in Fix(T)$ such that $f(x^*, y) \ge 0$ for all $y \in Fix(T)$.

For solving Problem 1, we investigate the asymptotic behavior of the sequence $\{x^k\}$ generated by the following algorithm:

Algorithm 1 (Fixed point optimization algorithm)

Step 0. Choose $x^1 \in C$, $\{\lambda_k\} \subset (0, \infty)$ and $\{\alpha_k\} \subset [0, 1)$ arbitrarily. Set $k := 1$.

Step 1. Given x^k, compute x^{k+1} as follows:

$$\begin{cases} y^k = \operatorname{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\| y - x^k \| : y \in C\right\}, \\ z^k = T(y^k), \\ x^{k+1} = \alpha_k x^k + (1 - \alpha_k) z^k. \end{cases}$$

Step 2. Update $k := k + 1,$ and go to Step 1.

Theorem 1. *Assume that*

(i) $Sol(f, C)$ *is nonempty*.

(ii) *There exists* $k_0 \in \mathbb{N}$ *such that the set* $\Omega := \{x \in Fix(T) : f(x^k, x) \leq 0, \forall k \geq k_0\}$ *is nonempty*.

(iii) *Sequences* $\{\alpha_k\}$ *and* $\{\lambda_k\}$ *satisfy* $\limsup_{k\to\infty} \alpha_k < 1,$ $\sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2-\tau}} < \infty.$

Then the sequences $\{x^k\}$ *and* $\{z^k\}$ *generated by Algorithm* 1 *have following properties*:

(a) *For every* $x \in \Omega,$ $\lim_{k\to\infty} \| x^k - x \|$ *exists*. *Two sequences* $\{x^k\}$ *and* $\{z^k\}$ *are bounded*.

(b) $\lim_{k\to\infty} \| x^k - z^k \| = 0$ *and* $\lim_{k\to\infty} \| x^k - T(x^k) \| = 0.$

(c) *If* $\| x^k - z^k \| = o(\lambda_k),$ *then* $\{x^k\}$ *converges to* $\hat{x} \in Sol(f, Fix(T)).$

Proof. (a) Since

$$y^k = \operatorname{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\| y - x^k \|^2 : y \in C\right\},$$

we have

$$0 \in \partial\left(\lambda_k f(x^k, \cdot) + \frac{1}{2}\| \cdot - x^k \|^2 + \delta_C(\cdot)\right)(y^k),$$

where δ_C is the index function onto C. There exist $w \in \partial f(x^k, \cdot)(y^k)$ and $v \in N_C(y^k)$ such that

$$0 = \lambda_k w + y^k - x^k + v.$$

We obtain

$$\langle v,\ x - y^k \rangle \le 0, \quad \forall x \in C$$

and hence

$$\langle x^k - y^k - \lambda_k w,\ x - y^k \rangle \le 0, \quad \forall x \in C.$$

From $w \in \partial f(x^k, \cdot)(y^k)$, it follows that

$$\begin{aligned}\lambda_k(f(x^k, x) - f(x^k, y^k)) &\ge \lambda_k \langle w,\ x - y^k \rangle \\ &\ge \langle x^k - y^k,\ x - y^k \rangle, \quad \forall x \in C.\end{aligned}$$

This implies that

$$\langle y^k - x^k,\ x^k - x \rangle \le \lambda_k(f(x^k, x) - f(x^k, y^k)) - \| y^k - x^k \|^2, \quad \forall x \in C. \tag{2}$$

Applying (2) with $x := x^k$, we have

$$\| x^k - y^k \|^2 \le -\lambda_k f(x^k, y^k) \le \lambda_k | f(x^k, y^k) |. \tag{3}$$

On the other hand, from τ-Hölder continuity of function *f*, it follows that there exist $Q > 0$ and $\tau \in (0, 1]$ such that

$$\begin{aligned}| f(x^k, y^k) | = | f(x^k, y^k) - f(x^k, x^k) | &\le Q\| (x^k, y^k) - (x^k, x^k) \|^\tau \\ &= Q\| x^k - y^k \|^\tau. \end{aligned}\tag{4}$$

Combining (3) and (4), we have

$$\| x^k - y^k \| \le (Q\lambda_k)^{\frac{1}{2-\tau}}$$

and

$$\lambda_k | f(x^k, y^k) | \le (Q\lambda_k)^{\frac{2}{2-\tau}}.$$

Since $\sum_{k=1}^{\infty}\lambda_k^{\frac{1}{2-\tau}} < \infty$, it implies that $\lim_{k\to\infty}\| x^k - y^k \| = 0$ and $\sum_{k=1}^{\infty}\sqrt{\lambda_k | f(x^k, y^k)|} < \infty$. For all $x \in \Omega \subset Fix(T)$, we have

$$\begin{aligned}
\| x^{k+1} - x \|^2 &= \| a_k x^k + (1-\alpha_k)z^k - x \|^2 \\
&\le \alpha_k \| x^k - x \|^2 + (1-\alpha_k)\| T(y^k) - x \|^2 \\
&\le \alpha_k \| x^k - x \|^2 + (1-\alpha_k)\| y^k - x \|^2 \\
&\le \alpha_k \| x^k - x \|^2 + (1-\alpha_k)(\| y^k - x^k \|^2 \\
&\quad + \| x^k - x \|^2 + 2\langle y^k - x^k, x^k - x\rangle) \\
&\le \| x^k - x \|^2 + (1-\alpha_k)(\| y^k - x^k \|^2 \\
&\quad + 2(\lambda_k(f(x^k, x) - f(x^k, y^k)) - \| y^k - x^k \|^2)) \\
&= \| x^k - x \|^2 - (1-\alpha_k)\| y^k - x^k \|^2 \\
&\quad + 2\lambda_k(1-\alpha_k)(f(x^k, x) - f(x^k, y^k)) \\
&\le \| x^k - x \|^2 + 2\lambda_k | f(x^k, y^k)|.
\end{aligned}$$

Since $\sum_{k=1}^{\infty}\lambda_k | f(x^k, y^k)| < \infty$, by Lemma 1, we have $\lim_{k\to\infty}\| x^k - x \|$ exists for all $x \in \Omega$. This implies sequence $\{x^k\}$ is bounded. As T is firmly nonexpansive, it follows that $\{z^k\}$ is also bounded.

(b) By definition of firmly nonexpansive function, for all $x \in Fix(T)$, we have

$$\| z^k - x \|^2 = \| T(y^k) - T(x) \|^2 \le \langle y^k - x, z^k - x\rangle.$$

Applying $\langle a, b\rangle = \frac{1}{2}(a^2 + b^2 - (a-b)^2), \ \forall a, b \in C,$ we have

$$\| z^k - x \|^2 \le \frac{1}{2}(\| y^k - x \|^2 + \| z^k - x \|^2 - \| y^k - z^k \|^2).$$

Hence,

$$\begin{aligned}
&\| z^k - x \|^2 \\
&\le \| y^k - x \|^2 - \| y^k - z^k \|^2 \\
&= (\| y^k - x^k \|^2 + \| x^k - x \|^2 + 2\langle y^k - x^k, x^k - x\rangle) \\
&\quad - (\| y^k - x^k \|^2 + \| x^k - z^k \|^2 + 2\langle y^k - x^k, x^k - z^k\rangle) \\
&= \| x^k - x \|^2 - \| x^k - z^k \|^2 + 2\langle y^k - x^k, z^k - x\rangle, \quad \forall x \in Fix(T). \qquad (5)
\end{aligned}$$

Then we have

$$\begin{aligned}
&\| x^{k+1} - x \|^2 \\
&\le \alpha_k \| x^k - x \|^2 + (1-\alpha_k)\| z^k - x \|^2 \\
&\le \alpha_k \| x^k - x \|^2 + (1-\alpha_k)(\| x^k - x \|^2 - \| x^k - z^k \|^2 \\
&\quad + 2\langle y^k - x^k, z^k - x\rangle) \\
&= \| x^k - x \|^2 - (1-\alpha_k)(-\| x^k - z^k \|^2 + 2\langle y^k - x^k, z^k - x\rangle). \qquad (6)
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\alpha_k)\| x^k - z^k \|^2 \\
&\le \| x^k - x \|^2 - \| x^{k+1} - x \|^2 + 2(1-\alpha_k)\langle y^k - x^k, z^k - x\rangle.
\end{aligned}$$

Let $x \in \Omega$. From $\| x^k - y^k \| \to 0$, $\limsup_{k\to\infty} \alpha_k < 1$ and existence of $\lim_{k\to\infty}\| x^k - x \|$, it implies that $\| x^k - z^k \| \to 0$. Next, we have

$$\| z^k - T(x^k) \| = \| T(y^k) - T(x^k) \| \le \| y^k - x^k \|.$$

Since $\| y^k - x^k \| \to 0,$ we obtain that $\| z^k - T(x^k) \| \to 0.$ From

$$\| x^k - T(x^k) \| \le \| x^k - z^k \| + \| z^k - T(x^k) \|,$$

it implies that $\| x^k - T(x^k) \| \to 0.$

(c) The boundedness of $\{x^k\}$ guarantees the existence of a subsequence $\{x_i^k\}$ of $\{x^k\}$ such that $\lim_{i\to\infty} x^{k_i} = \hat{x}.$ From $0 = \lim_{i\to\infty}\| x^{k_i} - T(x^{k_i}) \| = \| \hat{x} - T(\hat{x}) \|,$ it implies that $\hat{x} \in Fix(T).$

Next, we shall prove that $\hat{x} \in Sol(f, Fix(T)).$ It follows from (5) and (2) that for all $x \in Fix(T),$

$$\begin{aligned}
\| z^k - x \|^2 &\le \| x^k - x \|^2 - \| x^k - z^k \|^2 + 2\langle y^k - x^k, z^k - x^k \rangle \\
&\quad + 2\langle y^k - x^k, x^k - x \rangle \\
&\le \| x^k - x \|^2 + 2\langle y^k - x^k, z^k - x^k \rangle \\
&\quad + 2(\lambda_k (f(x^k, x) - f(x^k, y^k)) - \| y^k - x^k \|^2) \\
&\le \| x^k - x \|^2 + 2\langle y^k - x^k, z^k - x^k \rangle \\
&\quad + 2\lambda_k (f(x^k, x) - f(x^k, y^k)).
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &\le \| x^k - x \|^2 - \| z^k - x \|^2 + 2\langle y^k - x^k, z^k - x^k \rangle \\
&\quad + 2\lambda_k (f(x^k, x) - f(x^k, y^k)) \\
&\le (\| x^k - x \| + \| z^k - x \|^2)(\| x^k - z^k \|) + 2\langle y^k - x^k, z^k - x^k \rangle \\
&\quad + 2\lambda_k (f(x^k, x) - f(x^k, y^k)).
\end{aligned}$$

Since $\lambda_k > 0, \ \forall k \geq 1,$ we have

$$0 \leq (\| x^k - x \| + \| z^k - x \|) \frac{\| x^k - z^k \|}{\lambda_k} + 2 \frac{\langle y^k - x^k, z^k - x^k \rangle}{\lambda_k}$$

$$+ 2(f(x^k, x) - f(x^k, y^k)).$$

Let $k := k_i \to \infty.$ From assumption $\| x^k - z^k \| = o(\lambda_k),$ the boundedness of $\{x^k\}$ and $\{z^k\}$ and $f(x^k, y^k) \to 0,$ we have

$$0 \leq f(\hat{x}, x), \quad \forall x \in Fix(T).$$

Hence, $\hat{x} \in Sol(f, Fix(T)) \subset Fix(T).$

From (6), we have

$$\begin{aligned} \| x^{k+1} - x \|^2 &\leq \| x^k - x \|^2 \\ &\quad + (1 - \alpha_k)(-\| x^k - z^k \|^2 + 2\langle y^k - x^k, z^k - x \rangle) \\ &\leq \| x^k - x \|^2 + K\| y^k - x^k \| \\ &\leq \| x^k - x \|^2 + K\sqrt{\lambda_k | f(x^k, y^k)|}, \end{aligned}$$

where $K := \sup\{2\| z^k - x \| : k \geq 1\} < \infty.$ Since $\sum_{k=1}^{\infty} \sqrt{\lambda_k | f(x^k, y^k)|} < \infty,$ applying Lemma 1, we obtain that the limit $\lim_{k\to\infty}\| x^k - x \|$ exists $\forall x \in Fix(T).$ It implies that

$$\lim_{k\to\infty} \| x^k - \hat{x} \| = \lim_{i\to\infty} \| x^{k_i} - \hat{x} \| = 0.$$

That is, $x^k \to \hat{x} \in Sol(f, Fix(T)).$ □

Remark 1. When we choose $f(x, y) = \langle F(x), y - x \rangle,$ where $F : C \to C$ is a continuous operator, we have the fixed point optimization algorithm for the variational inequality problem over the fixed point set (1), which is proposed in [18]. However, not as in [18], the convergence of the algorithm is obtained without the condition:

$$VI(F,\ Fix(T)) \subset \Omega := \{x \in Fix(T) : \langle x^k - x,\ F(x^k)\rangle \geq 0,\ \forall k \geq k_0\}.$$

Remark 2. The condition $\| x^k - z^k \| = o(\lambda_k)$ is satisfied when we choose suitable parameters λ_k (see Example 1). Analogously to [18], the numerical results in Example 1 show that the condition $\| x^k - z^k \| = o(\lambda_k)$ is not satisfied with a fast diminishing constant sequence such as $\lambda_k = \frac{1}{k^\beta}$ $(\beta > 2)$. Hence, we will use a slowly diminishing constant sequence such as $\lambda_k = \frac{1}{k^\beta}$, $\beta \in (1, 2)$.

4. Numerical Examples

In this section, we present some examples to illustrate the proposed algorithm. Some comparisons are also reported. All the programming is implemented in MATLAB R2010b running on a PC with Intel®Core2™ Quad Processor Q9400 2.66Ghz 4GB RAM.

Example 1. We apply our Algorithm 1 to solve an equilibrium problem arising from the Cournot-Nash equilibrium model considered in [28]. Let $C = \mathbb{R}^7$, $f(x, y) = \langle Ax + By + c,\ y - x\rangle$, where

$$A = \begin{pmatrix} 3 & 1 & -2 & 3 & 4 & 2 & 0 \\ -1 & -4 & 3 & 0 & 2 & 4 & 2 \\ 3 & 1 & -3 & 2 & -2 & 2 & -3 \\ 1 & 1 & 2 & -4 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 & 3 & 2 & 3 \\ 1 & 3 & 2 & 0 & 1 & 3 & 1 \\ 2 & 1 & 3 & 0 & 1 & 2 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} -3 & 0 & 1 & -2 & 1 & 1 & 2 \\ 2 & 3 & -1 & 2 & 1 & 1 & 0 \\ 2 & 0 & 4 & -1 & 0 & 1 & 3 \\ -1 & 0 & 1 & -5 & 2 & 2 & -1 \\ 1 & 3 & 1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 1 & 2 & -1 & 4 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 & 4 \end{pmatrix}, \quad c = \begin{pmatrix} 5 \\ 7 \\ 9 \\ -8 \\ 6 \\ 10 \\ 9 \end{pmatrix}$$

and mapping $T : \mathbb{R}^7 \to \mathbb{R}^7$ defined by

$$T(x) = Pr_D(x),$$

where Pr_D is the metric projection onto

$$D = \{x \in \mathbb{R}^7 : \| x - (1, 0, 0, 0, 0, 0, 0) \| \le 1\}.$$

We note that f is not pseudomonotone and T is firmly nonexpansive mapping. Matrix B is positive definite, hence function $f(x, \cdot)$ is convex for all $x \in \mathbb{R}^7$. Choose $x^1 = (1, 2, 0, 1, 2, 0, 1)$, $\alpha_k = \alpha$, $\alpha \in (0, 1)$ and $\lambda_k = \frac{1}{k^\beta}$, $\beta > 1$. To check if the condition $\| x^k - y^k \| = o(\lambda_k)$ is satisfied, we shall investigate the asymptotic behavior of the sequence $u^k = \frac{\| x^k - z^k \|}{\lambda_k}$. It is seen from Figure 1 and Figure 2 that when $\alpha := 1/2$ and $\beta = 1.1, 1.2, 1.5,$ $(u^k)_{k \in \mathbb{N}}$ converges to 0 and when $\beta = 2.0, 2.3,$ the sequence $\{u^k\}_{k \in \mathbb{N}}$ does not converge to 0. Moreover, since $\| x^k \| \le 1$ and $\| y^k \| \le 1,$ we have

$$\begin{aligned} | f(x^k, y^k) | &= | \langle Ax^k + By^k + c, y^k - x^k \rangle | \\ &\le (\| A \| + \| B \| + \| c \|) \| y^k - x^k \|^1. \end{aligned}$$

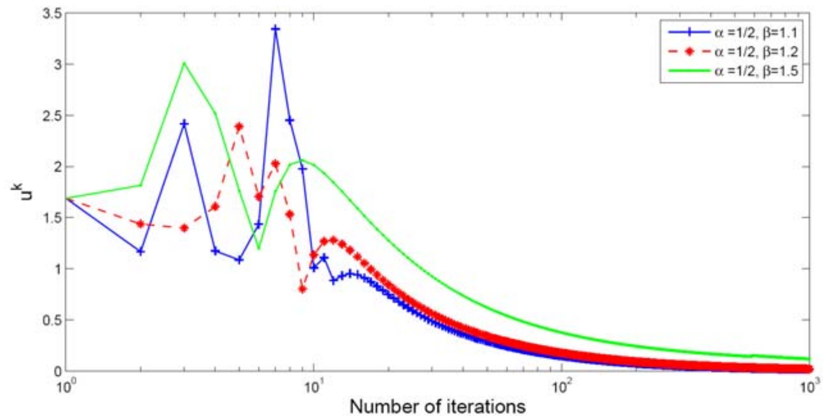

Figure 1. The condition $\| x^k - z^k \| = o(\lambda_k)$ is satisfied when $\alpha := 1/2$ and $\beta = 1.1, 1.2, 1.5.$

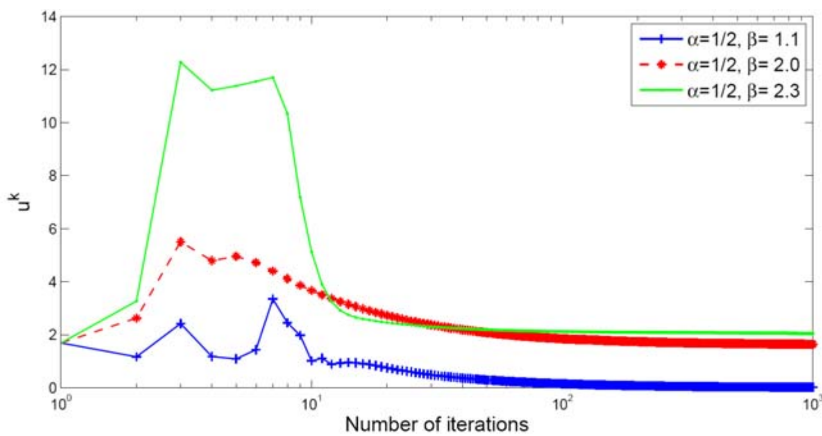

Figure 2. The condition $\| x^k - z^k \| = o(\lambda_k)$ is not satisfied when $\alpha := 1/2$ and $\beta = 2.0,\ 2.3.$

Choose $\lambda_k = \frac{1}{k^{1.1}}$, $\alpha_k = \frac{1}{2}$. From above argument, it implies that all conditions of Theorem 1 are satisfied. Applying Algorithm 1 for problem $EP(f, Fix(T))$, we have the result in Table 1. We use stopping criteria: $\| x^{k+1} - x^k \| \leq \varepsilon$ with $\varepsilon = 10^{-4}$.

Example 2. In this example, we will compare the performance of extragradient algorithm (ExtraGrad) in [29] and our Algorithm 1. Consider the equilibrium problem $EP(f, D)$ with *f* is given as in [1]

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad f(x, y) = \langle Ax + \chi^n(y + x) + \mu - \alpha,\ y - x \rangle,$$

where

$$A = \begin{pmatrix} 0 & \chi & \chi & \cdots & \chi \\ \chi & 0 & \chi & \cdots & \chi \\ \chi & \chi & 0 & \cdots & \chi \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \chi & \chi & \cdot & \cdots & 0 \end{pmatrix}, \quad \mu = (\mu_0, ..., \mu_0)^T, \quad \alpha = (\alpha_0, ..., \alpha_0)^T.$$

Table 1. Sequence $\{x^k\}_{k \in \mathbb{N}}$ converges to $x^* = (0.8530, -0.2444, -0.4909, 0.7094, -0.1133, -0.2882, -0.2801) \in Sol(f, Fix(T))$

Iter.	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	$\|x^{k+1} - x^k\|$
$k = 1$	1	2	0	1	2	0	1	1.6880
$k = 2$	0.8860	0.9624	0.2404	0.8279	0.9488	–0.0731	0.2504	0.5444
$k = 3$	0.9951	0.5601	0.1580	0.8728	0.6141	–0.0855	0.2100	0.7217
$k = 4$	1.1698	0.1519	–0.1328	0.7895	0.1756	–0.1399	0.0175	0.2554
$k = 5$	1.2806	0.0116	–0.1271	0.8228	0.0109	–0.1813	–0.0397	0.1850
$k = 6$	1.1363	0.0755	–0.0874	0.8984	–0.0311	–0.1673	–0.0317	0.2001
$k = 7$	1.2673	–0.0334	–0.1687	0.8811	–0.0436	–0.1228	0.0127	0.3933
$k = 8$	1.5777	–0.0054	–0.1185	0.6517	–0.0601	–0.1143	0.0581	0.2490
$k = 9$	1.4517	0.1127	–0.0914	0.7416	–0.1805	–0.1616	–0.0233	0.1762
…								
$k = 294$	0.8530	–0.2444	–0.4909	0.7094	–0.1133	–0.2882	–0.2801	$9.9444.10^{-5}$

The feasible set is

$$D = \{x \in \mathbb{R}^n, \|x - (100, 1, 1, ..., 1)^T\| \leq 1\}.$$

In Algorithm 1, choose $C = \mathbb{R}^n$ and $T = Pr_C$. Then two problems $EP(f, D)$ and $EP(f, Fix(T))$ coincide. In both algorithms, we use the same starting points x^0, the stopping criteria $\| x^k - x^{k+1} \| < 10^{-5}$, $\chi = 3$, $\mu_0 = 3$, $\alpha_0 = 2$ and $n = 30$. In algorithm ExtraGrad, choose $\lambda = 0.01$ and in Algorithm 1, $\lambda_k = \frac{1}{1.1^k}$. The results are tabulated in Table 2. We can see that, in this example, the CPU time of Algorithm 1 is less than of ExtraGrad algorithm even though the ExtraGrad algorithm requires fewer iterations. This happens because there are two constrained convex programs need to be solved at each iteration in the ExtraGrad algorithm instead of one unconstrained convex program as in Algorithm 1.

Table 2. Comparison of algorithms in Example 2 with different starting points

	$x^0 = (1, 1, ..., 1)^T$		$x^0 = (1, 2, ..., n)^T$		$(2, 2, ..., 2)^T$	
	CPU times (s)	Iterations	CPU times (s)	Iterations	CPU times (s)	Iterations
ExtraGrad	25.8302	30	16.1573	20	14.9098	18
Algorithm 1	1.6714	100	0.7871	50	0.3878	26

Example 3. We will apply Algorithm 1 to the power-control problem for code-division multiple-access (CDMA) systems. We use the model, which was introduced in [15, 18]. Consider a network with n users. Let $I :=$ $1, 2, ..., n$ be the set of users and $\mathbf{p} := (p_1, p_2, ..., p_n)^T$ be the transmit power of users. Let $C_k := [P_k^{\min}, P_k^{\max}]$, where $P^{\max} > P^{\min} \geq 0$ and put

$$C := \prod_{k \in I} C_k.$$

The signal-to-interference-plus-noise ratio (SINR) of kth user can be expressed by a function of $\mathbf{p}$ as following: $\gamma_k : C \to \mathbb{R}$ for all

$$\mathbf{p} := (p_1,\ p_2, \ldots,\ k_n)^T \in C, \quad \gamma_k(\mathbf{p}) = \frac{p_k h_k^2}{\sigma^2 + \frac{1}{N}\sum_{j\neq k} p_j h_j^2},$$

where $h_k \in \mathbb{R}$ is the channel gain for the kth user, $\sigma^2 > 0$ is the noise power and $N > 0$ is processing gain.

Suppose that the utility of kth user is a function of $\mathbf{p}$:

$$U_k(\mathbf{p}) = \frac{L}{M} R_k g(\gamma_k(\mathbf{p})),$$

where L and M are the number of information bits and the total number of bits in a packet, respectively, R_k stands for the transmission rate for the kth user, and $g(\gamma) := (1 - e^{-\gamma})^M$ is the approximate packet success rate (PSR). Let

$$D := \bigcap_{k\in I} D_k, \text{ where } D_k := \{\mathbf{p} \in \mathbb{R}^n : \gamma_k(\mathbf{p}) \geq \delta_k\}(k \in I),$$

where $\delta_k > 0$ $(k \in I)$ is the required SINR for the kth user in the network. Let

$$f(\mathbf{p}, \mathbf{q}) = \sum_{k\in I} (U_k(\mathbf{p}) - U_k(\mathbf{p}_{\hat{k}}, q_k)) \tag{7}$$

for all $\mathbf{p}, \mathbf{q} \in C$, where $(\mathbf{p}_{\hat{k}}, q_k) := (p_1,\ p_2, \ldots,\ p_{k-1},\ q_k,\ p_{k+1}, \ldots,\ p_n)^T$ $\in C$. We have to choose the transmit power $\mathbf{p}^* \in C$ in order to maximize the utility of users. Moreover, each user must achieve the required SINR. That is, find $\mathbf{p}^* \in Sol(f, C \cap D)$.

However, the set $C \cap D$ can be empty, for example, when the noise σ^2 is large or one of the users is too far from base station. In order to avoid this drawback, consider the generalized convex feasible set [51], C_Φ, defined by

$$C_\Phi := \{\hat{\mathbf{p}} \in C : \Phi(\hat{\mathbf{p}}) = \min_{\mathbf{p}\in C} \Phi(\mathbf{p})\},$$

where

$$\Phi(\mathbf{p}) := \frac{1}{2}\sum_{k\in I} w_k d(\mathbf{p}, D_k)^2 \ (\mathbf{p} \in \mathbb{R}^n), \quad w_k \in (0, 1)(k \in I)$$

with $\sum_{k\in I} w_k = 1$ and $d(\mathbf{p}, D_k) := \min\{\| \mathbf{p} - \mathbf{q} \| : \mathbf{q} \in D_k\}(k \in I, \mathbf{p} \in \mathbb{R}^n)$. When $C \cap D \neq \emptyset$, we have $C_\Phi = C \cap D$. So C_Φ is a generalization of $C \cap D$. Since C_Φ is the set of all minimizers of Φ over C, it cannot be expressed explicitly. Hence, we cannot solve $EP(f, C_\Phi)$ directly. We define the mapping $N : \mathbb{R}^n \to \mathbb{R}^n$:

$$N(\mathbf{p}) := Pr_C\left[\sum_{k\in I} w_k Pr_{D_k}(\mathbf{p})\right] (\mathbf{p} \in \mathbb{R}^n), \tag{8}$$

where Pr_C is the metric projection onto C. Then N is nonexpansive and $Fix(N) = C_\Phi$. Let $T(\mathbf{p}) = \frac{1}{2}\mathbf{p} + \frac{1}{2}N(\mathbf{p})$. It can be seen that mapping T is firmly nonexpansive and $Fix(T) = Fix(N) = C_\Phi$. We will apply Algorithm 1 for problem $EP(f, Fix(T))$.

As in [18], we assume that $L = 100$, $M = 100$, $R_k = 10^4$ bits/second, $(k \in I)$, $N = 100$ and $\sigma^2 = 10 \times 10^{-14}$ watts. Suppose that, for all $k \in I$, $P_k^{\min} = 0.1$ watts and $P_k^{\max} = 1$ watts. The initial transmit power of all users is 0.1 watts and $h_k := \frac{0.3}{d_k^2}$, $(k \in I)$, where d_k is the distance from the kth user to the base station. Suppose $d_1 := 310$m, $d_2 := 460$m, $d_3 := 570$m, $d_4 := 660$m, $d_5 := 740$m, $d_6 := 810$m, $d_7 := 880$m, $d_8 := 940$m, $d_9 :=$ 1,000m and also $w_k = \frac{1}{9}\ (k \in I)$. The required SINR for the kth user is $\delta_k = 1\ (k \in I)$. Note that in this case, $C \cap D = \emptyset$ because $C \cap D_9 = \emptyset$.

It is obvious that function f is Lipschitz continuous on $C \times C$ and we can choose $\lambda_k = \frac{1}{(k+50)^{1.1}}$. The condition $\sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2-\tau}} < \infty$ is satisfied. We use $\alpha_k = \frac{1}{10}, \frac{1}{2}$. To check if convergence condition $\| x^n - z^n \| = o(\lambda_n)$ is satisfied, we consider the behavior of the sequence, $\{u_k\}_{k \geq 1}$, defined by

$$u_k := \frac{\| x^k - z^k \|}{\lambda_k}.$$

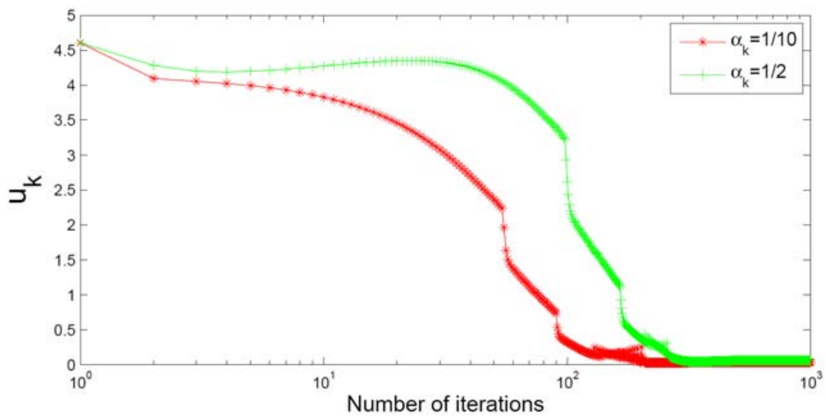

Figure 3. The condition $\| x^k - z^k \| = o(\lambda_k)$ is satisfied.

It is seen from Figure 3 that $\lim_{k \to \infty} u_k = 0$. That means the condition $\| x^n - z^n \| = o(\lambda_n)$ is satisfied. Choose $\lambda_k = \frac{1}{(k+50)^{1.1}}$, $\alpha_k = \frac{1}{10}$. Applying Algorithm 1 for problem $EP(f, Fix(T))$, we have the result in Figure 4. We use stopping criteria: $\| x^{k+1} - z^k \| \leq \varepsilon$ with $\varepsilon = 10^{-4}$.

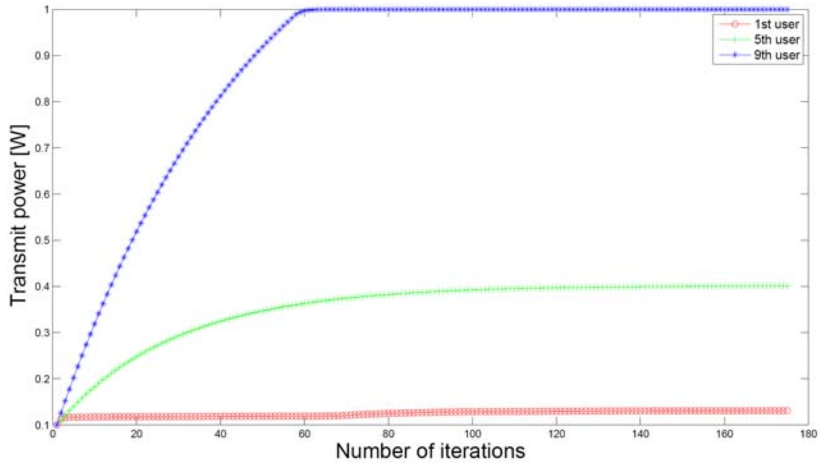

Figure 4. The transmit power of 1st user, 5th user and 9th user.

From Figure 4, we can see that the transmit power of the 1st user is low and the transmit power of the 9th user is high; in other words, transmitted powers are high when users are far from the base station. The algorithm stops after 175 iterations. The sequence $\{x^k\}$ converges to the solution x^* of $EP(f, Fix(T))$,

$$x^* = (0.1309, 0.1000, 0.1243, 0.2472, 0.4008, 0.5801, 0.8101, 1.0000, 1.0000).$$

5. Conclusion

In this paper, we have proposed the fixed point optimization algorithm for the equilibrium problem over fixed point set of firmly nonexpansive. The proposed algorithm does not require the monotonicity of bifunction. However, some convergence conditions are needed. The proposed problem can be applied for the equilibrium problem over set C, where C does not necessarily have explicit form. Finally, we have applied the algorithm to the power control problem for CDMA network and have presented the numerical examples for the transmit power. Numerical results have shown that with suitable choosing of parameters, the convergence conditions are satisfied and the proposed algorithm succeeds in approximating a solution of the proposed equilibrium problem.

Acknowledgement

The author thanks the anonymous referees for their valuable suggestions which let to the improvement of the manuscript.

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