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STRONG CONVERGENCE OF AN M-STEP PICARD-LIKE PROCESS TO A COMMON FIXED POINT OF A FINITE FAMILY OF LIPSCHITZIAN HEMICONTRACTIVE MAPS

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Abstract

Let K be a closed convex nonempty subset of a Hilbert space H and let the set of the common fixed points of a finite family of Lipschitzian hemicontractive maps on K be nonempty. Sufficient conditions for the strong convergence of the sequence of successive approximations generated by an M-step Picard-like process to a common fixed point of the family are proved.

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1. Introduction

Let H be a Hilbert space and let K be a nonempty subset of H. K is said to be (*sequentially*) *compact* if every closed bounded sequence in K has a subsequence that converges in K. K is said to be *boundedly compact* if every bounded subset of K is compact. In finite dimensional spaces, closed subsets are boundedly compact. Given a subset S of K, we shall denote by $co(S)$ and $ccl(S)$ the convex hull and the closed convex hull of S, respectively. If K is boundedly compact convex and S is bounded, then $co(S)$ and hence $ccl(S)$ are compact convex subsets of K.

A map $T : K \to E$ is said to be *semi-compact* if for any bounded sequence $\{x_n\} \subset K$ such that $\| x_n - Tx_n \| \to 0$ as $n \to \infty,$ there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that x_{n_j} converges strongly to some $x^* \in K$ as $j \to \infty.$ The map T is said to be *demi-compact* at $z \in E$ if for any bounded sequence $\{x_n\} \subset K$ such that $\| x_n - Tx_n \| \to z$ as $n \to \infty,$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and a point $p \in K$ such that x_{n_j} converges strongly to p as $j \to \infty.$ (Observe that if T is additionally continuous, then $p - Tp = z$.) A nonlinear map $T : K \to E$ is said to be *completely continuous* if it maps bounded sets into relatively compact sets. T is said to be *Lipschitzian* if $\exists L \geq 0$ such that

$$\| Tx - Ty \| \leq L \| x - y \|, \quad \forall x, y \in K. \tag{1}$$

If $L = 1,$ then T is called *nonexpansive* and if $L < 1,$ then the mapping T is called a *contraction*. The mapping T with domain $D(T)$ and the range $R(T)$ in H is called *pseudocontractive* if $\forall x, y \in D(T),$

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + \| (I - T)x - (I - T)y \|^2. \tag{2}$$

If (2) holds for all $x \in D(T)$ and $y \in F(T)$ (fixed point set of T), then T is said to be *hemicontractive*.

The class of pseudocontractive maps has been extensively studied (see, e.g., [1-6] and the references therein). It is clear that the important class of nonexpansive mappings is a subclass of the class of pseudocontractive maps.

In [4], Ishikawa introduced a new iteration method and proved that it converges strongly to a fixed point of a Lipschitz pseudocontractive map defined on a compact convex subset of a Hilbert space. In fact, he proved the following result.

Theorem 1 (Ishikawa [4])**.** *Let K be a compact convex nonempty subset of a Hilbert space and let* $T : K \to K$ *be a Lipschitz pseudocontractive map. Let* $\{\alpha_n\}, \{\beta_n\}$ *be real sequences satisfying the following conditions*:

1. $0 < \alpha_n \leq \beta_n < 1.$

2. $\lim_{n\to\infty} \beta_n = 0.$

3. $\Sigma\alpha_n\beta_n = \infty.$

Then starting with an arbitrary $x_0 \in K$, *the sequence* $\{x_n\}$ *of successive approximations defined by*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n; \quad n \geq 0, \tag{3}$$

converges strongly to a fixed point of T.

This result has been generalized in several ways by many authors, see, e.g., Qihou [5] who proved a convergence theorem in the case where *T* is a Lipschitz hemicontraction. Chidume and Moore [1] who proved the case where *T* is a continuous hemicontraction and for iteration process with errors, Ghosh and Debnath [3] who proved the convergence of a Picard-like process to a fixed point of a quasi-nonexpansive map. Recently, Moore and Nnubia [6] proved a convergence theorem for a Picard-like process to a point in the common fixed point set of a finite family of Lipschitzian hemicontractions.

Our purpose in this paper is to construct an M-step Picard-like iteration process which converges strongly to a common fixed point of a finite family of Lipschitz hemicontractive self maps of a closed convex nonempty subset of a Hilbert space.

We need the following lemma in this work:

Lemma 1.1 [4]**.** *For any* x, y, z *in a Hilbert space H and a real number* $\lambda \in [0, 1]$,

$$\| \lambda x + (1-\lambda) y - z \|^2 = \lambda \| x - z \|^2 + (1-\lambda) \| y - z \|^2 - \lambda(1-\lambda) \| x - y \|^2. \quad (4)$$

Lemma 1.2. *Let K be a closed convex nonempty subset of a Banach space E and let* $T_i : K \to K$, *where* $i \in I = \{1, 2, ..., m\}$ *be a finite family of continuous nonlinear maps in K such that* $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ *and let* $\{x_n\}_{n\geq 1}$ *be a sequence in K satisfying*

1. $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0, \forall i \in I$,

2. $\exists n_* \in \mathbb{N}$ *such that* $\| x_{n+1} - x^* \| \leq (1 + \tau_n) \| x_n - x^* \| + \nu_n, \forall n \geq n_*$, *where* $\sum_{n\geq 0} \nu_n < \infty$ *and* $\sum_{n\geq 0} \tau_n < \infty$. *Then*

1. $\lim_{n\to\infty} \| x_n - x^* \|$ *exists and* $\{x_n\}$ *is bounded.*

2. $\{x_n\}$ *converges strongly to a common fixed point of* T_i*'s if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

3. $\{x_n\}$ *converges strongly to a common fixed point of* T_i*'s if one of the* T_i*'s satisfies any of the following conditions*:

(a) *condition B*,

(b) *semi-compact*,

(c) *demi-compact at* $0 \in K$,

(d) *completely continuous*.

4. $\{x_n\}$ *converges strongly to a point of F if any of the following conditions holds*:

(a) *K is compact*.

(b) *K is boundedly compact*.

2. Main Result

2.1. Strong convergence theorem for finite family of Lipschitzian hemicontractive maps

We define the following auxiliary maps: $\alpha, \beta \in (0, 1)$ constants:

$$S_{i\beta} = (1-\beta)I + \beta T_i; \quad i \in \{1, 2, ..., N\}, \tag{5}$$

$$T_{i\alpha\beta} = (1-\alpha)I + \alpha T_i S_{i\beta}. \tag{6}$$

Theorem 2.1. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H and let* $\{T_i\}_{i=1}^{N}$ *be a finite family of Lipschitzian* (*with constant* $L_i > 0$) *hemicontractive maps from K into itself such that* $F = \bigcap_{i=1}^{N} F(T_i)$ *is not empty*. *Starting with an arbitrary* $x_0 \in K$, *define the iterative sequence* $\{x_n\}$ *by*

$$y_{n,0} = x_n,$$

$$y_{n,i} = T_{i\alpha\beta} y_{n,i-1},$$

$$y_{n,m} = x_{n+1}, \tag{7}$$

for $\beta = \dfrac{\sqrt{1+(1-c)L^2} - 1}{L^2}$, *where* $1 - 2\beta - \beta^2 L^2 = c$; $L = Max\{L_i : i = 1, ..., N\} < \infty$ *and for any* $\alpha \in (0, \beta)$. *Then*

1. $\lim_{n\to\infty} \| x_n - x^* \|$ *exists for* $x^* \in F$.

2. $\forall i \in \{1, 2, ..., N\}$, $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0$.

Proof. Let $x^* \in \bigcap_{i=1}^{N} F(T_i)$ and $z_{n,i} = S_{i\beta} y_{n,i-1}, \forall i,$

$$
\begin{aligned}
\| z_{n,i} - x^* \|^2 &= \| S_{i\beta} y_{n,i-1} - x^* \|^2 = \| (1-\beta) y_{n,i-1} + \beta T_i y_{n,i-1} - x^* \|^2 \\
&= (1-\beta)\| y_{n,i-1} - x^* \|^2 + \beta \| T_i y_{n,i-1} - x^* \|^2 \\
&\quad - \beta(1-\beta)\| T_i y_{n,i-1} - y_{n,i-1} \|^2 \\
&\leq (1-\beta)\| y_{n,i-1} - x^* \|^2 + \beta(\| y_{n,i-1} - x^* \|^2 \\
&\quad + \| y_{n,i-1} - T_i Y_{n,i-1} \|^2) - \beta(1-\beta)\| T_i y_{n,i-1} - y_{n,i-1} \|^2 \\
&= \| y_{n,i-1} - x^* \|^2 + \beta^2 \| y_{n,i-1} - T_i y_{n,i-1} \|^2, \\
\| y_{n,i} - x^* \|^2 &= \| T_{i\alpha\beta} y_{n,i-1} - x^* \|^2 = \| (1-\alpha) y_{n,i-1} + \alpha T_i z_{n,i} - x^* \|^2 \\
&= (1-\alpha)\| y_{n,i-1} - x^* \|^2 + \alpha \| T_i z_{n,i} - x^* \|^2 \\
&\quad - \alpha(1-\alpha)\| T_i z_{n,i} - y_{n,i-1} \|^2 \\
&\leq (1-\alpha)\| y_{n,i-1} - x^* \|^2 + \alpha(\| z_{n,i} - x^* \|^2 + \| z_{n,i} - T_i z_{n,i} \|^2) \\
&\quad - \alpha(1-\alpha)\| T_i z_{n,i} - y_{n,i-1} \|^2 \\
&\leq (1-\alpha)\| y_{n,i-1} - x^* \|^2 + \alpha \| y_{n,i-1} - x^* \|^2 \\
&\quad + \alpha\beta^2 \| y_{n,i-1} - T_i y_{n,i-1} \|^2 \\
&\quad + \alpha \| z_{n,i} - T_i z_{n,i} \|^2 - \alpha(1-\alpha)\| T_i z_{n,i} - y_{n,i-1} \|^2, \\
\| z_{n,i} - T_i z_{n,i} \|^2 &= (1-\beta)\| y_{n,i-1} - T_i z_{n,i} \|^2 + \beta \| T_i y_{n,i-1} - T_i z_{n,i} \|^2 \\
&\quad - \beta(1-\beta)\| T_i y_{n,i-1} - y_{n,i-1} \|^2. \\
&\leq (1-\beta)\| y_{n,i-1} - T_i zn, i \|^2 \\
&\quad - \beta(1-\beta-\beta^2 L^2)\| y_{n,i-1} - T_i y_{n,i-1} \|^2.
\end{aligned}
$$

So,

$$\| y_{n,i} - x^* \|^2 \leq \| y_{n,i-1} - x^* \|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2) \| y_{n,i-1} - T_i y_{n,i-1} \|^2$$

$$- \alpha(\beta - \alpha) \| y_{n,i-1} - T_i z_{n,i} \|^2.$$

Thus,

$$\| y_{n,1} - x^* \|^2 \leq \| x_n - x^* \|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2) \| x_n - T_1 x_n \|^2$$

$$- \alpha(\beta - \alpha) \| x_n - T_1 z_{n,1} \|^2,$$

$$\| y_{n,2} - x^* \|^2$$

$$\leq \| y_{n,1} - x^* \|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2) \| y_{n,1} - T_2 y_{n,1} \|^2$$

$$- \alpha(\beta - \alpha) \| y_{n,1} - T_2 z_{n,2} \|^2$$

$$\leq \| x_n - x^* \|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2)(\| x_n - T_1 x_n \|^2 + \| y_{n,1} - T_2 y_{n,1} \|^2)$$

$$- \alpha(\beta - \alpha)(\| x_n - T_1 z_{n,1} \|^2 + \| y_{n,1} - T_2 z_{n,2} \|^2).$$

Hence, $\forall i \in I$, we have

$$\| y_{n,i} - x^* \|^2 \leq \| x_n - x^* \|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2) \sum_{j=1}^{i} \| y_{n,j-1} - T_j y_{n,j-1} \|^2$$

$$- \alpha(\beta - \alpha) \sum_{j=1}^{i} \| y_{n,j-1} - T_j z_{n,j} \|^2.$$

Therefore,

$$\| x_{n+1} - x^* \|^2$$

$$\leq \| x_n - x^* \|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2) \sum_{j=1}^{m} \| y_{n,j-1} - T_j y_{n,j-1} \|^2$$

$$- \alpha(\beta - \alpha) \sum_{j=1}^{m} \| y_{n,j-1} - T_j z_{n,j} \|^2.$$

So, $\lim_{n\to\infty} \| x_n - x^* \|$ exists and

$$\lim_{n\to\infty} \| y_{n,\, j-1} - T_j y_{n,\, j-1} \| = 0 = \lim_{n\to\infty} \| y_{n,\, j-1} - T_j z_{n,\, j} \|, \quad \forall j \in I.$$

Now,

$$\begin{aligned}
\| x_n - T_j x_n \| &\le \| x_n - y_{n,\, j-1} \| + \| y_{n,\, j-1} - T_j y_{n,\, j-1} \| \\
&\quad + \| T_j y_{n,\, j-1} - T_j x_n \| \\
&\le (1+L)\| x_n - y_{n,\, j-1} \| + \| y_{n,\, j-1} - T_j y_{n,\, j-1} \|, \\
\| x_n - y_{n,\, j} \| &= \| (1-\alpha)(x_n - y_{n,\, j-1}) \| + \alpha \| (T_j z_{n,\, j} - x_n) \| \\
&\le (1-\alpha)\| y_{n,\, j-1} - x_n \| + \alpha \| x_n - T_j z_{n,\, j} \| \\
&\le \| y_{n,\, j-1} - x_n \| + \alpha \| y_{n,\, j-1} - T_j z_{n,\, j} \| \\
&\le \| y_{n,\, j-2} - x_n \| \\
&\quad + \alpha(\| y_{n,\, j-2} - T_{j-1} z_{n,\, j-1} \| + \| y_{n,\, j-1} - T_j z_{n,\, j} \|) \\
&\quad \vdots \\
&\le \| y_{n,0} - x_n \| + \alpha \sum_{i=1}^{j} (\| y_{n,i-1} - T_i z_{n,i} \|) \\
&= \alpha \sum_{i=1}^{j} (\| y_{n,i-1} - T_i z_{n,i} \|).
\end{aligned}$$

Hence, $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0, \ \forall i \in I,$ thus, completing the proof.

Theorem 2.2. *Let K, H, T_i's, F, $\{x_n\}$ be as in Theorem* 2.1. *Then* $\{x_n\}$ *converges strongly to a point of F if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0.$

Proof. The proof follows easily from Theorem 2.1 and Lemma 1.2 part 2.

Theorem 2.3. *Let K, H,* T_i*'s,* F*,* $\{x_n\}$ *be as in Theorem* 2.1. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* T_i*'s if one of the* T_i*'s satisfies any of the following conditions*:

1. *condition B*,

2. *semi-compact*,

3. *demi-compact at* $0 \in K$,

4. *completely continuous*.

Proof. The proof follows easily from Theorem 2.1 and Lemma 1.2 part 3.

Theorem 2.4. *Let K, H,* T_i*'s,* F*,* $\{x_n\}$ *be as in Theorem* 2.1. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* T_i*'s,* $\{x_n\}$ *converges strongly to a point of F if any of the following conditions hold*:

1. *K is compact*.

2. *K is boundedly compact*.

Proof. The proof follows easily from Theorem 2.1 and Lemma 1.2 part 4.

2.2. Strong convergence theorem for finite family of quasi-accretive operators

T is hemicontractive if and only if $A = I - T$ is quasi-accretive, so, we define the following auxiliary maps:

$$G_{i\beta} = I - \beta A_i; \quad i \in \{1, ..., m\}, \tag{8}$$

$$A_{i\alpha\beta} = I - \alpha\beta A_i - \alpha A_i G_{i\beta}. \tag{9}$$

Theorem 2.5. *Let H be a real Hilbert space and let* $A_i : H \to H$; $i \in \{1, 2, ..., m\}$ *be a finite family of* L_i*-Lipschitzian quasi-accretive maps such that the simultaneous nonlinear equations* $A_i x = 0$; $i \in \{1, 2, ..., m\}$ *have a solution* $x^* \in H$. *Starting with an arbitrary* $x_0 \in K$, *define the* *iterative sequence* $\{x_n\}$ *by*

$$y_{n,0} = x_n,$$

$$y_{n,i} = A_{i\alpha\beta} y_{n,i-1},$$

$$y_{n,m} = x_{n+1}, \tag{10}$$

for $\beta = \frac{\sqrt{1+(1-c)L^2} - 1}{L^2}$; $L = Max\{L_i : i = 1, ..., m\} < \infty$ *for some* $c \in (0, 1)$, $\alpha \in (0, \beta)$; *then*

1. $\lim_{n\to\infty} \| x_n - x^* \|$ *exists* $\forall x^* \in Z = \bigcap_{i=1}^{m} Z(A_i)$, *where* $Z(A) :=$ $\{w \in H \mid Aw = 0\}$ (*the zero set of* A),

2. $\lim_{n\to\infty} \| A_i x_n \| = 0$; $\forall i \in \{1, 2, ..., N\}$.

Proof. Let $T_i = I - A_i$. Then T_i is a Lipschitzian hemicontraction. Further,

$$S_{i\beta} = (1-\beta)I + \beta T_i = I - \beta(I - T_i) = I - \beta A_i = G_{i\beta},$$

$$T_{i\alpha\beta} = (1-\alpha)I + \alpha T_i S_{i\beta} = I - \alpha(I - G_{i\beta}) - \alpha A_i G_{i\beta}$$

$$= I - \alpha\beta A_i - \alpha A_i G_{i\beta} = A_{i\alpha\beta}.$$

Thus, Theorem 2.1 applies and we have the stated results.

As a result of Lemma 1.2, we have the following strong convergence results.

Theorem 2.6. *Let* K, H, A_i*'s*, Z, $\{x_n\}$ *be as in Theorem* 2.5. *Then* $\{x_n\}$ *converges strongly to a point of* Z *if and only if* $\liminf_{n\to\infty} d(x_n, Z) = 0$.

Theorem 2.7. *Let* K, H, A_i*'s*, Z, $\{x_n\}$ *be as in Theorem* 2.5. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* A_i*'s if one of the* A_i*'s satisfies any of the following conditions*:

1. *condition B*,

2. *semi-compact*,

3. *demi-compact at* $0 \in K$,

4. *completely continuous*.

Theorem 2.8. *Let* K, H, A_i*'s*, Z, $\{x_n\}$ *be as in Theorem* 2.5. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* A_i*'s*, $\{x_n\}$ *converges strongly to a point of Z if any of the following conditions holds*:

1. *K is compact*.

2. *K is boundedly compact*.

Our iterative process generalizes some of the existing ones, our result is subsequent to the result by the same authors in [6] and compliments that our theorems improve, generalize and extend several known results and our method of proof is of independent interest.

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