JP Journal of Algebra, Number Theory and Applications

Published Online: January 2016
http://dx.doi.org/10.17654/NT038010019
Volume 38, Number 1, 2016, Pages 19-28 **ISSN: 0972-5555**

ON THE DIOPHANTINE EQUATION $4q^x + 7^y = z^{2m}$

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Abstract

Let q be a prime number and m be a non-negative integer. In this paper, we show that all non-negative integer solutions of the Diophantine equation $4q^x + 7^y = z^{2m}$ are the following:

$$(q, x, y, z) = \begin{cases} (2, 1, 0, 3) \text{ and } (2, 3, 2, 9); & \text{if } m = 1, \\ (2, 3, 2, 3); & \text{if } m = 2. \end{cases}$$

1. Introduction

Solving Diophantine equations of the form $2^x + p^y = z^2$, where p is prime, has been widely studied by many mathematicians. For example, Acu [1] proved in 2007 that $(x, y, z) = (3, 0, 3), (2, 1, 3)$ are the only two non-negative solutions for the case $p = 5$ whereas it was shown in 2011 by Suvarnamani et al. [10] that there is no non-negative solution (x, y, z) when x is even, for the case $p = 7, 11$.

Received: July 21, 2015; Revised: August 24, 2015; Accepted: September 9, 2015
2010 Mathematics Subject Classification: 11D61.
Keywords and phrases: exponential Diophantine equation.
Communicated by Istvan Gaal

In 2012, the author [2] studied the Diophantine equations $4^x + p^y = z^2$, where p is prime and Sroysang [5-7] studied the Diophantine equations $8^x + 19^y = z^2$, $31^x + 32^y = z^2$ and $32^x + 49^y = z^2$.

In 2013, Sroysang [4, 8] studied the Diophantine equations $7^x + 8^y = z^2$ and $23^x + 32^y = z^2$.

Later, in 2014, Sroysang [9] studied the Diophantine equations $8^x + 13^y = z^2$.

Inspired by all references, our aim is to find all possible non-negative solutions to the Diophantine equation $4q^x + 7^y = z^{2m}$, where q is prime and m, x, y, z are non-negative integers.

2. Main Results

In this study, we use the statement on Catalan's conjecture, that the only solution in integers $a > 1$, $b > 1$, $x > 1$, $y > 1$ of the equation $a^x - b^y = 1$ is $(a, b, x, y) = (3, 2, 2, 3)$. (Consult [3] for more details.)

Throughout this paper, q denotes a prime number. First, we give the following lemmas which will be used in the subsequent study.

Lemma 1. *The Diophantine equation* $q^x - 7^y = 1$ *has only two non-negative integer solutions*, *namely*, $(q, x, y) = (2, 1, 0)$ *and* $(q, x, y) = (2, 3, 1)$.

Proof. By Catalan's conjecture, $q^x - 7^y = 1$ has no solution only when $x > 1$ and $y > 1$. It suffices to consider only the case $x \leq 1$ or case $y \leq 1$. Thus, we consider the following:

Case $x = 0$**.** We have $1 - 7^y = 1$. So $7^y = 0$, which is impossible.

Case $x = 1$**.** We have $q = 1 + 7^y$. This implies that q is even. Since q is prime, $q = 2$. It follows that $y = 0$. In this case, we have $(q, x, y) = (2, 1, 0)$.

Case $y = 0$**.** We have $q^x = 2$. So $q = 2$ and $x = 1$.

Case $y = 1$**.** We have $q^x = 8$. So $q = 2$ and $x = 3$. Thus, $(q, x, y) = (2, 3, 1)$.

Finally, one can easily check that $(q, x, y) = (2, 1, 0)$ and $(q, x, y) = (2, 3, 1)$ are solutions of the Diophantine equation $q^x - 7^y = 1$. This finishes the proof. □

Lemma 2. *The Diophantine equation* $z^2 - 7^y = 1$ *has no non-negative integer solution*.

Proof. The Diophantine equation $z^2 - 7^y = 1$ can be rewritten as

$$7^y = z^2 - 1 = (z + 1)(z - 1).$$

There exist α, β such that $7^\alpha = z + 1$, $7^\beta = z - 1$, $\alpha > \beta$ and $\alpha + \beta = y$. Therefore,

$$7^\beta(7^{\alpha-\beta} - 1) = 7^\alpha - 7^\beta = (z + 1) - (z - 1) = 2.$$

This implies that $\beta = 0$ and $7^\alpha - 1 = 2$. Therefore, $7^\alpha = 3$ which is impossible.

Thus, the Diophantine equation $z^2 - 7^y = 1$ has no non-negative integer solution. This finishes the proof. □

Now we consider the Diophantine equation $4q^x + 7^y = z^2$.

Theorem 3. *The Diophantine equation*

$$4q^x + 7^y = z^2 \tag{1}$$

has only two non-negative integer solutions, namely, $(q, x, y, z) = (2, 1, 0, 3)$ *and* $(q, x, y, z) = (2, 3, 2, 9)$.

Proof. Note that z is an odd integer. Thus, $z^2 \equiv 1 \pmod 4$. It follows that

$$3^y \equiv 4q^x + 7^y = z^2 \equiv 1 \pmod 4.$$

This implies that y is even. Let $y = 2k$ for some $k \geq 0$. Since z is odd, $z + 7^k$ and $z - 7^k$ are even. Thus, equation (1) can be rewritten as

$$q^x = \frac{z^2 - 7^{2k}}{4} = \left(\frac{z + 7^k}{2}\right)\left(\frac{z - 7^k}{2}\right).$$

Then there are non-negative integers α, β such that $q^\alpha = \frac{z + 7^k}{2}$, $q^\beta = \frac{z - 7^k}{2}$, $\alpha > \beta$ and $\alpha + \beta = x$. Therefore,

$$q^\beta(q^{\alpha-\beta} - 1) = q^\alpha - q^\beta = \frac{z + 7^k}{2} - \frac{z - 7^k}{2} = 7^k. \tag{2}$$

If $k = 0$, then $q^\beta(q^{\alpha-\beta} - 1) = 1$. This implies that $\beta = 0$ and $q^\alpha - 1 = 1$. That is, $q = 2$ and $\alpha = 1$. In this case, we have $(q, x, y, z) = (2, 1, 0, 3)$.

Next, we consider the case $k > 0$.

If $\beta > 0$, then $q \mid 7^k$. Since q is prime, $q = 7$. Equation (2) can be rewritten as

$$7^\beta(7^{\alpha-\beta} - 1) = 7^k.$$

This implies that $k = \beta$ and $7^{\alpha-\beta} - 1 = 1$. Therefore, $7^{\alpha-\beta} = 2$ which is impossible. Thus, $\beta = 0$. Equation (2) can be rewritten as

$$q^{\alpha} - 1 = 7^{k}.$$

By Lemma 1 and $k > 0$, we have $(q, \alpha, k) = (2, 3, 1)$. Thus, $(q, x, y, z) = (2, 3, 2, 9)$.

It is easy to check that $(q, x, y, z) = (2, 1, 0, 3)$ and $(q, x, y, z) = (2, 3, 2, 9)$ are the solutions of $4q^x + 7^y = z^2$. This finishes the proof. □

Using Theorem 3, the following corollaries are easy to verify.

Corollary 4. *The Diophantine equation* $4q^x + 7^y = z^4$ *has only one non-negative integer solution, namely,* $(q, x, y, z) = (2, 3, 2, 3)$.

Corollary 5. *Let* $m \geq 3$. *The Diophantine equation* $4q^x + 7^y = z^{2m}$ *has no non-negative integer solution*.

It is clear that the Diophantine equation $4q^x + 7^y = 1$ has no non-negative integer solution. Thus, we can conclude the following theorem.

Theorem 6. *Let m be a non-negative integer. All non-negative integer solutions of the Diophantine equation* $4q^x + 7^y = z^{2m}$ *are following*:

$$(q, x, y, z) = \begin{cases} (2, 1, 0, 3) \text{ and } (2, 3, 2, 9); & \text{if } m = 1, \\ (2, 3, 2, 3); & \text{if } m = 2. \end{cases}$$

Using Theorem 6, it is easy to show:

Example 7 (Theorem 3.1 [4])**.** The Diophantine equation $7^x + 8^y = z^2$ has only one non-negative integer solution, namely, $(x, y, z) = (0, 1, 3)$.

Example 8 (Theorem 3.1 [7])**.** The Diophantine equation $32^x + 49^y = z^2$ has only one non-negative integer solution, namely, $(x, y, z) = (1, 1, 9)$.

Example 9 (Theorem 2.1 [10])**.** The Diophantine equation $4^x + 7^y = z^2$ has no non-negative integer solution.

Proof. By Lemma 2, we know that the Diophantine equation $4^x + 7^y = z^2$ has no non-negative integer solution when $x = 0$. For $x \geq 1$, the Diophantine equation $4^x + 7^y = z^2$ can be rewritten as $4(2^{2x-2}) + 7^y = z^2$. Since $x \geq 1$, $2x - 2 \geq 0$. By Theorem 3, we have $(2x - 2, y, z) = (1, 0, 3)$ or $(2x - 2, y, z) = (3, 2, 9)$. Since x is an integer, $2x - 2 = 1$ and $2x - 2 = 3$ are impossible. Thus, the Diophantine equation $4^x + 7^y = z^2$ has no non-negative integer solution. □

The following statement is a consequence of Theorem 3.

Example 10. The Diophantine equation

$$196q^x + 7^y = z^2 \tag{3}$$

has only two non-negative integer solutions, namely, $(q, x, y, z) = (2, 1, 2, 21)$ and $(q, x, y, z) = (2, 3, 4, 63)$.

Proof. We know that $z \geq 3$ is an odd integer. Thus, $z + 1$ and $z - 1$ are even integers. So $\gcd\left(\frac{z+1}{2}, \frac{z-1}{2}\right) = 1$. We consider equation (3) into three cases:

Case 1. $y = 0$. Equation (3) can be rewritten as

$$196q^x = z^2 - 1 = (z + 1)(z - 1), \tag{4}$$

$$49q^x = \left(\frac{z+1}{2}\right)\left(\frac{z-1}{2}\right). \tag{5}$$

Since $\gcd\left(\frac{z-1}{2}, \frac{z+1}{2}\right) = 1$ and 7 is prime, $49 \left| \frac{z+1}{2} \right.$ or $49 \left| \frac{z-1}{2} \right.$ but not both.

Case $49 \left| \frac{z+1}{2} \right.$**.** Equation (5) can be rewritten as

$$q^x = \left(\frac{z+1}{98}\right)\left(\frac{z-1}{2}\right).$$

Then there are non-negative integers α, β such that $q^\alpha = \frac{z+1}{98}$, $q^\beta = \frac{z-1}{2}$ and $\alpha + \beta = x$. Since $z \geq 3$, it is easy to check that $\alpha < \beta$. One can see that

$$q^\alpha(q^{\beta-\alpha} - 49) = q^\beta - 49q^\alpha = \frac{z-1}{2} - \frac{z+1}{2} = -1.$$

This implies that $\alpha = 0$ and $q^\beta - 49 = -1$. Therefore, $q^\beta = 48$ which is impossible.

Case $49 \left| \frac{z-1}{2} \right.$**.** Equation (5) can be rewritten as

$$q^x = \left(\frac{z+1}{2}\right)\left(\frac{z-1}{98}\right).$$

Then there are non-negative integers α, β such that $q^\alpha = \frac{z+1}{2}$, $q^\beta = \frac{z-1}{98}$ and $\alpha + \beta = x$. It is easy to check that $\alpha > \beta$. One can see that

$$q^\beta(q^{\alpha-\beta} - 49) = q^\alpha - 49q^\beta = \frac{z+1}{2} - \frac{z-1}{2} = 1.$$

This implies that $\beta = 0$ and $q^\alpha - 49 = 1$. Therefore, $q^\alpha = 50$ which is impossible. In this case, the Diophantine equation (3) has no non-negative integer solution.

Case 2. $y = 1$. Then the Diophantine equation (3) can be rewritten as $28q^x + 1 = 7k^2$, where $z = 7k$ for some $k \geq 1$. This implies that $7 \mid 1$. This is a contradiction.

Case 3. $y > 1$. Then the Diophantine equation (3) can be rewritten as $4q^x + 7^{y-2} = k^2$, where $z = 7k$ for some $k \geq 1$. By Theorem 3, we have $(q, x, y-2, k) = (2, 1, 0, 3)$ and $(q, x, y-2, k) = (2, 3, 2, 9)$ are only two non-negative integer solutions of the Diophantine equation $4q^x + 7^{y-2} = k^2$. Thus, $(q, x, y, z) = (2, 1, 2, 21)$ and $(q, x, y, z) = (2, 3, 4, 63)$.

It is easy to verity that $(q, x, y, z) = (2, 1, 2, 21)$ and $(q, x, y, z) = (2, 3, 4, 63)$ satisfy the equation $196q^x + 7^y = z^2$. This completes the proof. □

Using Example 10, the following example is easy to verify.

Example 11. Let $m \geq 2$ be an integer. Then the Diophantine equation $196q^x + 7^y = z^{2m}$ has no non-negative integer solution.

Example 12. The Diophantine equation $4q^{2x} + q^2 7^y = z^2$ has no non-negative integer solution.

Proof. We consider the Diophantine equation $4q^{2x} + q^2 7^y = z^2$ into two cases:

Case 1. $x = 0$. Suppose q is odd. This implies that z is odd. So $q^2 \equiv 1 \pmod{4}$ and $z^2 \equiv 1 \pmod{4}$. Thus,

$$3^y \equiv 4 + q^2 7^y = z^2 \equiv 1 \pmod{4}.$$

This implies that y is even. So let $y = 2k$ for some $k \geq 0$. So

$$4 = z^2 - q^2 7^{2k} = (z + q7^k)(z - q7^k).$$

Since $z + q7^k > z - q7^k > 0$, we have $z + q7^k = 4$ and $z - q7^k = 1$. It follows that $z = \frac{5}{2}$. This contradicts the fact that z is a non-negative integer.

Thus, q is even. Since q is prime, $q = 2$. Therefore, $4q^{2x} + q^2 7^y = z^2$ can be rewritten as $4 + 4(7^y) = z^2$ or $1 + 7^y = t^2$, where $z = 2t$ for some $t \geq 1$. By Lemma 2, the equation $1 + 7^y = t^2$ has no non-negative integer solution. This implies that the Diophantine equation $4q^{2x} + q^2 7^y = z^2$ has no non-negative integer solution.

Case 2. $x > 0$. The Diophantine equation $4q^{2x} + q^2 7^y = z^2$ can be rewritten as $4q^{2x-2} + 7^y = s^2$, where $z = sq$ for some $s \geq 1$. By Theorem 3, we have $(q, 2x-2, y, s) = (2, 1, 0, 3)$ or $(q, 2x-2, y, s) = (2, 3, 2, 9)$. We know that $2x - 2 = 1$ and $2x - 2 = 3$ are impossible. Thus, the Diophantine equation $4q^{2x} + q^2 7^y = z^2$ has no non-negative integer solution. □

Using Example 12, the following example is easy to verify.

Example 13. Let m be a non-negative integer. The Diophantine equation $4q^{2x} + q^2 7^y = z^{2m}$ has no non-negative integer solution.

3. Open Problems

Note that solving the Diophantine equation $4q^x + p^y = z^2$, where q, p are prime is still an open problem. The non-negative integer solutions of $4q^x + 17^y = z^2$ are not known, either.

Acknowledgement

This work is supported by Faculty of Science, Khon Kaen University, Thailand under Incubation Researcher Project.

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