JP Journal of Algebra, Number Theory and Applications
© 2016 Pushpa Publishing House, Allahabad, India
Published Online: January 2016
http://dx.doi.org/10.17654/NT038010001
Volume 38, Number 1, 2016, Pages 1-18 **ISSN: 0972-5555**

INVERSE LIMIT OF LOCAL HOMOLOGY

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Abstract

Let M be a module over a commutative ring R with non-zero identity. In this paper, for a pair of ideals $(\mathfrak{a}, I)$ of R, we introduce the inverse limit of local homology $\mathfrak{F}_i^{\mathfrak{a}, I}(M)$ as follows: $\mathfrak{F}_i^{\mathfrak{a}, I}(M) = \varprojlim_{n \in \mathbb{N}} H_i^I(M/\mathfrak{a}^n M)$ for all $i \geq 0$. Also, we study some of its properties and analyze their structure, the vanishing, non-vanishing and Artinianness of $\mathfrak{F}_i^{\mathfrak{a}, I}(M)$. Moreover, there are some exact sequences concerning the inverse limit of local homology, among them a variant of Mayer-Vietoris sequence.

1. Introduction

Throughout this paper, assume that R is a commutative ring with non-zero identity, I and $\mathfrak{a}$ are ideals of R and M is an R-module. In [6], we have

Received: June 14, 2015; Revised: July 28, 2015; Accepted: August 20, 2015
2010 Mathematics Subject Classification: 13D45, 16E30, 13J99.
Keywords and phrases: inverse limit; local homology; local cohomology; vanishing of inverse limit of local homology, linearly compact module.
Work partially supported by CNPq-Brazil - Grant 245872/2012-4 and FAPESP-Brazil - Grant 2012/20304-1.
Communicated by K. K. Azad

that Grothendieck introduced the definition of local cohomology module as the module $H_I^i(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}_R^i(R/I^t, M)$ that is called the *ith local cohomology module* of M with respect to I. In [4], Cuong and Nam defined the local homology modules $H_i^I(M)$ with respect to I by

$$H_i^I(M) = \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/I^t, M).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenlees and May [8] for an Artinian R-module M. For basic results about local homology, we refer the reader to [4, 5] and [21]; for local cohomology, see [3].

In this paper, we introduce the inverse limit of local homology denoted by $\mathfrak{F}_i^{\mathfrak{a}, I}(M) = \varprojlim_{n\in\mathbb{N}} H_i^I(M/\mathfrak{a}^n M)$. This definition is an attempt to introduce something similar the formal local cohomology defined in [18] and [9].

The organization of the paper is as follows. In the next section, we study some basic and main properties and show that under certain conditions, the inverse limit of local homology is reduced to local homology module with respect to an ideal. Also, we obtain some long exact sequences, for example a Mayer-Vietoris sequence.

In Section 3, we provide some conditions for that inverse limit of local homology become an Artinian R-module. Section 4 is devoted to study the vanishing and non-vanishing of a such module.

2. Some Properties and Exact Sequences

Definition 2.1 [4, Definition 3.1]**.** Let M be an R-module. The ith local homology module $H_i^I(M)$ of M with respect to I is defined by

$$H_i^I(M) = \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/I^t, M).$$

Now, consider the family of local homology modules $\{H_i^I(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$, there is a natural homomorphism

$$\phi_{n,k} : H_i^I(M/\mathfrak{a}^n M) \to H_i^I(M/\mathfrak{a}^k M),$$

for any $0 < k \leq n$.

Thus, we have two families of *R*-modules $\{H_i^I(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$, and of *R*-homomorphisms $\{\phi_{n,k} : H_i^I(M/\mathfrak{a}^n M) \to H_i^I(M/\mathfrak{a}^k M) | k \leq n, k, n \in \mathbb{N}\}$.

These families form an inverse system. Their inverse limit that is given by $\varprojlim_{n\in\mathbb{N}} H_i^I(M/\mathfrak{a}^n M)$ is called the *inverse limit of local homology*, and will be denoted by $\mathfrak{F}_i^{\mathfrak{a},I}(M)$.

Now, let *M* be an *R*-module and let *N* be a submodule of *M*. For $m \in M$, we define a subset of $M : m + N = \{m + n | n \in N\}$. A subset of *M* is said to be a *coset* of *N* if there exists $m \in M$ such that it is equal to $m + N$ [14, Definition 5]. Moreover, $x \in m + N$ if and only if there exists *n* such that $n \in N$ and $x = m + n$.

Consider that the ring *R* is Noetherian and has a topological structure. Let us recall the concept of linearly compact modules by terminology of Macdonald [11, Definition 3.1]. Let *M* be a topological *R*-module. A nucleus of *M* is a neighbourhood of the zero element of *M*, and a nuclear base of *M* is a base for the nuclei of *M*. *M* is Hausdorff if and only if the intersection of all the nuclei of *M* is 0. It is said to be *linearly topologized* if *M* has a *nuclear* base Σ consisting of submodules. A Hausdorff linearly topologized *R*-module *M* is said to be *linearly compact* if *M* has the following property: if $\mathfrak{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in *M* which has the finite intersection property, then the cosets in $\mathfrak{F}$ have a non-empty intersection. It should be noted that an Artinian *R*-module is linearly compact with the discrete topology [7, Theorem 2.1]. A Hausdorff linearly topologized *R*-module *M* is called *semidiscrete* if every submodule of *M* is closed. For an R-module M and a submodule $N \subseteq M$, we define the set $(N :_M \mathfrak{a}) = \{m \in M \mid \mathfrak{a}m \subseteq N\}$. Observe that $(N :_M \mathfrak{a})$ is a submodule of M and that $N \subseteq (N :_M \mathfrak{a})$. For an R-module M, the $\mathfrak{a}$-torsion of M is defined by

$$\Gamma_{\mathfrak{a}}(M) := \bigcup_{n\in\mathbb{N}} (0 :_M \mathfrak{a}^n) = \{m \in M \mid \mathfrak{a}^n m = 0, \text{ for some integer } n \geq 1\}.$$

Observe that $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M.

Moreover, for a local ring $(R, \mathfrak{m})$ and Noetherian, we define the Matlis dual module $D(M) = \mathrm{Hom}_R(M, E)$ of M, where $E = E(R/\mathfrak{m})$ is the injective envelope of the residue field $R/\mathfrak{m}$.

Remark 2.2. The set of powers $\mathfrak{a}^n$ of ideal $\mathfrak{a}$ forms a base of open of the zero element of R. The topology defined in R is called $\mathfrak{a}$-*adic topology*. Similarly, given an R-module M, the set $\mathfrak{a}^n M$ is a base of open of the zero element of M defining the $\mathfrak{a}$-adic topology of M. The R-module M is said to be *complete* with respect to the $\mathfrak{a}$-adic topology if $\Lambda_{\mathfrak{a}}(M) \cong M$, where we use $\Lambda_{\mathfrak{a}}(M) = \varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M$ to denote the $\mathfrak{a}$-adic completion of M.

Proposition 2.3. *Suppose that R is a Noetherian ring. Let M be a linearly compact R-module. Then, for all $i \geq 0$, we have that*

$$\mathfrak{F}_i^{\mathfrak{a}, I}(M) = H_i^I(\varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M).$$

Proof. By definition, we have $\mathfrak{F}_i^{\mathfrak{a}, I}(M) = \varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/I^t, M/\mathfrak{a}^n M)$. As inverse limits commute (see [17, Theorem 2.26]), since that $\{M/\mathfrak{a}^n M\}_{n\in\mathbb{N}}$ is an inverse system of linearly compact modules (see [11, Properties 3.14, 3.3 and 3.5]), by [5, Lemma 2.7], we have that

$$\mathfrak{F}_i^{\mathfrak{a}, I}(M) = \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/I^t, \varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M) = H_i^I(\varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M). \qquad \square$$

Remark 2.4. Note that if M is complete with respect to the $\mathfrak{a}$-adic topology, then $\mathfrak{F}_i^{\mathfrak{a},I}(M) = H_i^I(M)$.

Proposition 2.5. *Let M be an R-module. Then, for all $i \geq 0$, the inverse limit of local homology $\mathfrak{F}_i^{\mathfrak{a},I}(M)$ is I-separated, i.e., $\bigcap_{s>0} I^s\mathfrak{F}_i^{\mathfrak{a},I}(M) = 0$.*

Proof. Note that for any inverse system of R-modules $\{M_t\}_{t\in\mathbb{N}}$, we have

$$I \varprojlim_{t\in\mathbb{N}} M_t \subseteq \varprojlim_{t\in\mathbb{N}} IM_t.$$

Thus,

$$\bigcap_{s>0} I^s\mathfrak{F}_i^{\mathfrak{a},I}(M) \cong \varprojlim_{s\in\mathbb{N}}(I^s \varprojlim_{n\in\mathbb{N}} H_i^I(M/\mathfrak{a}^n M)) \subseteq \varprojlim_{s\in\mathbb{N}} \varprojlim_{n\in\mathbb{N}} I^s H_i^I(M/\mathfrak{a}^n M).$$

As, by [17, Theorem 2.26], inverse limits commute, we have that

$$\bigcap_{s>0} I^s\mathfrak{F}_i^{\mathfrak{a},I}(M) \subseteq \varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}} \varprojlim_{s\in\mathbb{N}} I^s \mathrm{Tor}_i^R(R/I^t,\, M/\mathfrak{a}^n M) = 0,$$

since $I^s\mathrm{Tor}_i^R(R/I^t,\, M/\mathfrak{a}^n M) = 0$, for all $s \geq t$. □

Proposition 2.6. *Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring. Let M be an Artinian R-module. Then, for all $i \geq 0$, we have that*

$$D(\mathfrak{F}_i^{\mathfrak{a},I}(M)) \cong H_I^i(D(\Lambda_{\mathfrak{a}}(M))).$$

Proof. Since M is an Artinian R-module, the family $\{M/\mathfrak{a}^n M\}_{n\in\mathbb{N}}$ is an inverse system of Artinian R-modules and thus, we have that

$$\mathfrak{F}_i^{\mathfrak{a},I}(M) := \varprojlim_{n\in\mathbb{N}} H_i^I(M/\mathfrak{a}^n M) = H_i^I(\varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M) = H_i^I(\Lambda_{\mathfrak{a}}(M))$$

by Proposition 2.3. Now, by [10, Properties 27.2 and 27.3], we have that $\Lambda_{\mathfrak{a}}(M)$ is an Artinian R-module. Therefore, by [2, Corollary 2.3(ii)], we have the isomorphism

$$D(H_i^I(\Lambda_{\mathfrak{a}}(M))) \cong H_I^i(D(\Lambda_{\mathfrak{a}}(M))), \text{ for all } i \geq 0,$$

as required. □

Remark 2.7. Let M be a linearly compact R-module, with R Noetherian ring. Then

(i) $H_i^I(\mathfrak{F}_j^{\mathfrak{a},I}(M)) \cong \mathfrak{F}_j^{\mathfrak{a},I}(M)$, if $i = 0$, $j \geq 0$ and

(ii) $H_i^I(\mathfrak{F}_j^{\mathfrak{a},I}(M)) \cong 0$, if $i > 0$, $j \geq 0$.

This is a consequence of [5, Lemma 3.9]. Indeed, we have that

(i) $H_i^I(\mathfrak{F}_j^{\mathfrak{a},I}(M)) \cong \varprojlim_{n\in\mathbb{N}} H_i^I(H_j^I(M/\mathfrak{a}^n M)) \cong 0$, if $i > 0$ and

(ii) $H_i^I(\mathfrak{F}_j^{\mathfrak{a},I}(M)) \cong \varprojlim_{n\in\mathbb{N}} H_i^I(H_j^I(M/\mathfrak{a}^n M)) \cong \mathfrak{F}_j^{\mathfrak{a},I}(M)$, if $i = 0$

by [5, Lemma 3.9].

Proposition 2.8. *Let M be a linearly compact R-module, with R Noetherian ring. Then for all* $j \geq 0$,

(i) $\mathfrak{F}_i^{\mathfrak{a},I}(H_j^I(M)) \cong \varprojlim_{n\in\mathbb{N}}(H_j^I(M) \otimes_R R/\mathfrak{a}^n)$, *if* $i = 0$;

(ii) $\mathfrak{F}_i^{\mathfrak{a},I}(H_j^I(M)) \cong 0$, *if* $i > 0$.

Proof. By [5, Lemma 2.6], we have that $\{\mathrm{Tor}_j^R(R/I^t, M)\}_{t\in\mathbb{N}}$ form an inverse system of linearly compact R-modules. Note that

$$\begin{aligned}
&\mathfrak{F}_i^{\mathfrak{a},I}(H_j^I(M))\\
&\cong \varprojlim_{n\in\mathbb{N}} H_i^I(H_j^I(M)/\mathfrak{a}^n H_j^I(M))\\
&\cong \varprojlim_{n\in\mathbb{N}}(\varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/I^t, H_j^I(M) \otimes_R R/\mathfrak{a}^n))\\
&\cong \varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/I^t, \varprojlim_{s\in\mathbb{N}} \mathrm{Tor}_j^R(R/I^s, M) \otimes_R R/\mathfrak{a}^n).
\end{aligned}$$

Now, by [5, Lemma 2.7], we have that

$$\varprojlim_{s\in\mathbb{N}} \mathrm{Tor}_j^R(R/I^s, M) \otimes_R R/\mathfrak{a}^n \cong \varprojlim_{s\in\mathbb{N}}(\mathrm{Tor}_j^R(R/I^s, M) \otimes_R R/\mathfrak{a}^n)$$

and so, it follows that $\mathfrak{F}_i^{\mathfrak{a}, I}(H_j^I(M))$ is equal to

$$\varprojlim_{n\in\mathbb{N}}\varprojlim_{t\in\mathbb{N}}\operatorname{Tor}_i^R(R/I^t, \varprojlim_{s\in\mathbb{N}}(\operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n)).$$

Thus, by [5, Lemma 2.3(iii)] and [5, Lemma 2.7], we have that $\mathfrak{F}_i^{\mathfrak{a}, I}(H_j^I(M))$ is isomorphic to

$$\varprojlim_{n\in\mathbb{N}}\varprojlim_{t\in\mathbb{N}}\varprojlim_{s\in\mathbb{N}}\operatorname{Tor}_i^R(R/I^t, \operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n).$$

Since, by [17, Theorem 2.26], inverse limits commute, we have that

$$\mathfrak{F}_i^{\mathfrak{a}, I}(H_j^I(M)) = \varprojlim_{n\in\mathbb{N}}\varprojlim_{s\in\mathbb{N}}H_i^I(\operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n).$$

Now, by [5, Lemma 3.2(i)], we have that

$$H_i^I(\operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n)$$

$$\cong \varprojlim_{k\in\mathbb{N}}H_i(\overline{x}(k), \operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n),$$

where $\overline{x} = (x_1, ..., x_r)$ is a generator system of I and $\overline{x}(k) = (x_1^k, ..., x_r^k)$. Since $\overline{x}(k)\operatorname{Tor}_j^R(R/I^s, M) = 0$, for all $k \geq s$, we get

(i) $\varprojlim_{k\in\mathbb{N}}H_i(\overline{x}(k), \operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n) \cong \operatorname{Tor}_j^R(R/I^s, M)$ $\otimes_R R/\mathfrak{a}^n$, if $i = 0$; and

(ii) $\varprojlim_{k\in\mathbb{N}}H_i(\overline{x}(k), \operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n) \cong 0$, if $i > 0$.

Therefore, $\mathfrak{F}_i^{\mathfrak{a}, I}(H_j^I(M)) \cong \varprojlim_{n\in\mathbb{N}}\varprojlim_{s\in\mathbb{N}}\operatorname{Tor}_j^R(R/I^s, M)\otimes_R R/\mathfrak{a}^n$, if $i = 0$ and $j \geq 0$, and $\mathfrak{F}_i^{\mathfrak{a}, I}(H_j^I(M)) \cong 0$, if $i > 0$ and $j \geq 0$. Thus, we conclude the proof. □

Remark 2.9. Note that for $i = 0$ and $j = 0$ in Proposition 2.8, we have that $\mathfrak{F}_0^{\mathfrak{a}, I}(H_0^I(M)) \cong \mathfrak{F}_0^{\mathfrak{a}, I}(M)$. Indeed, $\mathfrak{F}_0^{\mathfrak{a}, I}(H_0^I(M)) \cong \mathfrak{F}_0^{\mathfrak{a}, I}(\Lambda_I(M))$ and thus,

$$\begin{aligned}
\mathfrak{F}_0^{\mathfrak{a},I}(\Lambda_I(M)) &= \varprojlim_{n\in\mathbb{N}} H_0^I(\Lambda_I(M)/\mathfrak{a}^n\Lambda_I(M)) \\
&= \varprojlim_{n\in\mathbb{N}} \Lambda_I(\Lambda_I(M)\otimes_R R/\mathfrak{a}^n) \\
&= \varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}} \Lambda_I(M)\otimes_R R/\mathfrak{a}^n \otimes_R R/I^t \\
&= \varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}} \varprojlim_{k\in\mathbb{N}} M\otimes_R R/I^k \otimes_R R/I^t \otimes_R R/\mathfrak{a}^n \\
&= \varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}} M/I^tM \otimes_R R/\mathfrak{a}^n \\
&= \varprojlim_{n\in\mathbb{N}} \Lambda_I(M/\mathfrak{a}^nM) = \mathfrak{F}_0^{\mathfrak{a},I}(M),
\end{aligned}$$

as required.

Proposition 2.10. *Let M be a linearly compact R-module, where R is a Noetherian ring and has a topological structure. Then, for all* $i \geq 0$, *the inverse limit of local homology* $\mathfrak{F}_i^{\mathfrak{a},I}(M)$ *is a linearly compact R-module.*

Proof. Since, by [11, Properties 3.14, 3.3 and 3.5], for each positive integer *n*, we have that $M/\mathfrak{a}^nM$ is a linearly compact *R*-module it follows, by [5, Proposition 3.3], that $H_i^I(M/\mathfrak{a}^nM)$ is a linearly compact *R*-module. Therefore, the family $\{H_i^I(M/\mathfrak{a}^nM)\}_{n\in\mathbb{N}}$ form an inverse system of linearly compact *R*-modules with continuous homomorphisms. Thus, we have, by [5, Lemma 2.3(iv)], that $\mathfrak{F}_i^{\mathfrak{a},I}(M) = \varprojlim_{n\in\mathbb{N}} H_i^I(M/\mathfrak{a}^nM)$ is a linearly compact *R*-module, as required. □

Theorem 2.11. *Let R be a Noetherian ring and let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of linearly compact R-modules, and suppose that* $M' \cap \mathfrak{a}^nM$ *is equivalent to the* $\mathfrak{a}$*-adic topology on* M'. *Thus, we have a long exact sequence of inverse limits of local homology*

$$\begin{aligned}
\cdots &\to \mathfrak{F}_i^{\mathfrak{a},I}(M') \to \mathfrak{F}_i^{\mathfrak{a},I}(M) \to \mathfrak{F}_i^{\mathfrak{a},I}(M'') \to \cdots \to \mathfrak{F}_0^{\mathfrak{a},I}(M') \\
&\to \mathfrak{F}_0^{\mathfrak{a},I}(M) \to \mathfrak{F}_0^{\mathfrak{a},I}(M'') \to 0.
\end{aligned}$$

Proof. By hypothesis, we may assume that M' is a submodule of M and the short exact sequence $0 \to M' \to M \to M'' \to 0$ induces

$$0 \to M'/(M' \cap \mathfrak{a}^n M) \to M/\mathfrak{a}^n M \to M''/\mathfrak{a}^n M'' \to 0$$

which is a short exact sequence of inverse systems of linearly compact modules, by [11, Properties 3.14, 3.3 and 3.5]. By the supposition that $M' \cap \mathfrak{a}^n M$ is equivalent to the $\mathfrak{a}$-adic topology on M' we have that the previous short exact sequence it gives us a short exact sequence, by [5, Lemma 2.4],

$$0 \to \varprojlim_{n \in \mathbb{N}} M'/\mathfrak{a}^n M' \to \varprojlim_{n \in \mathbb{N}} M/\mathfrak{a}^n M \to \varprojlim_{n \in \mathbb{N}} M''/\mathfrak{a}^n M'' \to 0.$$

By [5, Lemma 2.3(iv)], the previous sequence is a short exact sequence of linearly compact modules. With this exact sequence, we have a long exact sequence of local homology modules, by [5, Corollary 3.7]. Now, from Proposition 2.3 or [5, Proposition 3.4], we obtain the result. □

Remark 2.12. Let R be a Noetherian ring and let M be a finitely generated R-module. In this case, we have that the R-module $\mathfrak{F}_i^{\mathfrak{a},I}(M)$ is zero for all $i > 0$. Indeed, as R is Noetherian and M is finitely generated, then $H_i^I(M/\mathfrak{a}^n M) = 0$, for all $i > 0$, and for all $n \geq 1$, by [4, Remark 3.2(ii)]. Thus, we get that $\mathfrak{F}_i^{\mathfrak{a},I}(M) := \varprojlim_{n \in \mathbb{N}} H_i^I(M/\mathfrak{a}^n M) = 0$, for all $i > 0$.

Corollary 2.13. *Let R be a Noetherian ring and let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated and linearly compact R-modules. Then we have an exact sequence*

$$0 \to \mathfrak{F}_0^{\mathfrak{a},I}(M') \to \mathfrak{F}_0^{\mathfrak{a},I}(M) \to \mathfrak{F}_0^{\mathfrak{a},I}(M'') \to 0.$$

Proof. By hypothesis, we may assume that M' is a submodule of M and as M is a finitely generated R-module, by Artin-Rees Lemma [13, Theorem 8.5], we have $M' \cap \mathfrak{a}^n M$ is equivalent to the $\mathfrak{a}$-adic topology on M'. Therefore, by Theorem 2.11 and Remark 2.12, we obtain the result. □

Here we want to introduce a variant of the Mayer-Vietoris sequence for inverse limit of local homology.

Theorem 2.14. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R. Let M be a linearly compact R-module, and suppose that the $(\mathfrak{a} \bigcap \mathfrak{b})$-adic filtration on M is equivalent to the filtration $\{(\mathfrak{a}^n \bigcap \mathfrak{b}^n)M\}_{n\in\mathbb{N}}$. Thus, there is the long exact sequence of inverse limits of local homology*

$$\cdots \to \mathfrak{F}_i^{\mathfrak{a}\bigcap\mathfrak{b},\,\mathfrak{m}}(M) \to \mathfrak{F}_i^{\mathfrak{a},\,\mathfrak{m}}(M) \oplus \mathfrak{F}_i^{\mathfrak{b},\,\mathfrak{m}}(M) \to \mathfrak{F}_i^{(\mathfrak{a},\,\mathfrak{b}),\,\mathfrak{m}}(M) \to \cdots$$

for all $i \geq 0$.

Proof. Let $n \in \mathbb{N}$ be a natural. Then there is the following natural exact sequence

$$0 \to M/(\mathfrak{a}^n M \bigcap \mathfrak{b}^n M) \to M/\mathfrak{a}^n M \oplus M/\mathfrak{b}^n M \to M/(\mathfrak{a}^n, \mathfrak{b}^n)M \to 0$$

which is a short exact sequence of linearly compact modules by [11, Properties 3.14, 3.3 and 3.5]. By the hypothesis, the $(\mathfrak{a} \bigcap \mathfrak{b})$-adic filtration on M is equivalent to the filtration $\{(\mathfrak{a}^n \bigcap \mathfrak{b}^n)M\}_{n\in\mathbb{N}}$, and noting that the $(\mathfrak{a}, \mathfrak{b})$-adic filtration is equivalent to the filtration $\{(\mathfrak{a}^n, \mathfrak{b}^n)M\}_{n\in\mathbb{N}}$ it follows that the previous exact sequence it gives us a short exact sequence of inverse systems of linearly compact modules, and by [5, Lemma 2.4], we have a short exact sequence of inverse limits

$$0 \to \varprojlim_{n\in\mathbb{N}} M/(\mathfrak{a} \bigcap \mathfrak{b})^n M \to \varprojlim_{n\in\mathbb{N}} (M/\mathfrak{a}^n M \oplus M/\mathfrak{b}^n M)$$

$$\to \varprojlim_{n\in\mathbb{N}} M/(\mathfrak{a}, \mathfrak{b})^n M \to 0.$$

By [5, Lemma 2.3(iv)], the previous sequence is a short exact sequence of linearly compact modules. With this exact sequence, we have a long exact sequence of local homology modules, by [5, Corollary 3.7]. Now, from Proposition 2.3 or [5, Proposition 3.4], we obtain the result. □

Corollary 2.15. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and let* M *be a finitely generated and linearly compact* R*-module. Thus, we have an exact sequence*

$$0 \to \mathfrak{F}_0^{\mathfrak{a}\cap\mathfrak{b},\,\mathfrak{m}}(M) \to \mathfrak{F}_0^{\mathfrak{a},\,\mathfrak{m}}(M) \oplus \mathfrak{F}_0^{\mathfrak{b},\,\mathfrak{m}}(M) \to \mathfrak{F}_0^{(\mathfrak{a},\,\mathfrak{b}),\,\mathfrak{m}}(M) \to 0.$$

Proof. As M is a finitely generated R-module, we have that $(\mathfrak{a} \cap \mathfrak{b})$-adic filtration on M is equivalent to the filtration given by $\{(\mathfrak{a}^n \cap \mathfrak{b}^n)M\}_{n\in\mathbb{N}}$, by Artin-Rees Lemma [13, Theorem 8.5]. Therefore, by Theorem 2.14 and Remark 2.12, it follows the result. □

Proposition 2.16. *Let* M *be a linearly compact* R*-module with* R *be a Noetherian ring and suppose that* $I^tM \cap \mathfrak{a}^nM$ *is equivalent to the* $\mathfrak{a}$*-adic topology on* I^tM*. Then we have that*

(i) $\mathfrak{F}_i^{\mathfrak{a},\,I}\left(\bigcap_{t>0} I^tM\right) \cong 0,$ *if* $i = 0;$

(ii) $\mathfrak{F}_i^{\mathfrak{a},\,I}\left(\bigcap_{t>0} I^tM\right) \cong \mathfrak{F}_i^{\mathfrak{a},\,I}(M),$ *if* $i > 0.$

Proof. From the short exact sequence of linearly compact R-modules

$$0 \to I^tM \to M \to M/I^tM \to 0,$$

for all $t > 0$ we derive by [5, Lemma 2.4] and by [5, Lemma 2.3(iv)] a short exact sequence of linearly compact R-modules

$$0 \to \bigcap_{t>0} I^tM \to M \to \Lambda_I(M) \to 0.$$

Hence, by Theorem 2.11, we get a long exact sequence

$$\cdots \to \mathfrak{F}_{i+1}^{\mathfrak{a},\,I}(\Lambda_I(M)) \to \mathfrak{F}_i^{\mathfrak{a},\,I}\left(\bigcap_{t>0} I^tM\right) \to \mathfrak{F}_i^{\mathfrak{a},\,I}(M) \to \mathfrak{F}_i^{\mathfrak{a},\,I}(\Lambda_I(M)) \to$$

$$\cdots \to \mathfrak{F}_1^{\mathfrak{a},\,I}(\Lambda_I(M)) \to \mathfrak{F}_0^{\mathfrak{a},\,I}\left(\bigcap_{t>0} I^tM\right) \to \mathfrak{F}_0^{\mathfrak{a},\,I}(M) \to \mathfrak{F}_0^{\mathfrak{a},\,I}(\Lambda_I(M)) \to 0.$$

So, the statement it follows from Proposition 2.8 and Remark 2.9. □

Remark 2.17. If M is I-separated, it means that $\bigcap_{t>0} I^t M = 0,$ then

(i) $\mathfrak{F}_0^{\mathfrak{a}, I}(M) \cong \mathfrak{F}_0^{\mathfrak{a}, I}(\Lambda_I(M));$

(ii) $\mathfrak{F}_i^{\mathfrak{a}, I}(M) = 0,$ if $i > 0.$

3. Some Results of Artinianness

Here we want to discuss about Artinianness of inverse limit of local homology.

Proposition 3.1. *Let R be an Artinian ring and let M be an Artinian R-module. Then we have that*

$$\mathfrak{F}_i^{\mathfrak{a}, I}(M) \cong \operatorname{Tor}_i^R(R/I^r, \varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M),$$

for some integer $r \geq 1$.

Proof. Since R is an Artinian ring, there is a positive integer r such that $I^t = I^r,$ for all $t \geq r.$ As

$$\mathfrak{F}_i^{\mathfrak{a}, I}(M) = \varprojlim_{n\in\mathbb{N}} \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^R(R/I^t, M/\mathfrak{a}^n M),$$

it follows that $\mathfrak{F}_i^{\mathfrak{a}, I}(M) = \varprojlim_{n\in\mathbb{N}} \operatorname{Tor}_i^R(R/I^r, M/\mathfrak{a}^n M).$ As we have that the family $\{M/\mathfrak{a}^n M\}_{n\in\mathbb{N}}$ is an inverse system of Artinian R-modules and R/I^r is a finitely generated R-module, by [4, Lemma 4.3], we have that

$$\varprojlim_{n\in\mathbb{N}} \operatorname{Tor}_i^R(R/I^r, M/\mathfrak{a}^n M) \cong \operatorname{Tor}_i^R(R/I^r, \varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M), \text{ for all } i \geq 0.$$

Therefore, $\mathfrak{F}_i^{\mathfrak{a}, I}(M) \cong \operatorname{Tor}_i^R(R/I^r, \varprojlim_{n\in\mathbb{N}} M/\mathfrak{a}^n M),$ for some integer $r \geq 1,$ as required. □

Corollary 3.2. *Let R be an Artinian ring and let M be an Artinian R-module. If M is complete with respect to the $\mathfrak{a}$-adic topology, then $\mathfrak{F}_i^{\mathfrak{a}, I}(M)$ is an Artinian R-module, for all $i \geq 0$.*

Proof. In fact, by Proposition 3.1, we have that

$$\mathfrak{F}_i^{\mathfrak{a}, I}(M) \cong \mathrm{Tor}_i^R(R/I^r, \varprojlim_{n \in \mathbb{N}} M/\mathfrak{a}^n M),$$

for some integer $r \geq 1$. Since $\varprojlim_{n \in \mathbb{N}} M/\mathfrak{a}^n M \cong M$, we have that

$$\mathfrak{F}_i^{\mathfrak{a}, I}(M) \cong \mathrm{Tor}_i^R(R/I^r, M), \text{ for all } i \geq 0.$$

As R/I^r is a finitely generated R-module and M is an Artinian R-module, it follows that $\mathrm{Tor}_i^R(R/I^r, M)$ is an Artinian R-module, by [16, Proposition 2.13] and [15, Proposition 2.5(i)]. Therefore, $\mathfrak{F}_i^{\mathfrak{a}, I}(M)$ is an Artinian R-module for all $i \geq 0$, as required. □

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and let* M *be an Artinian* R*-module. Then, for a positive integer* $s \geq 1$, *the following statements are equivalent*:

(i) $\mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M)$ *is an Artinian* R*-module, for all* $i < s$;

(ii) $\mathfrak{m} \subseteq \mathrm{Rad}(\mathrm{Ann}_R(\mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M)))$, *for all* $i < s$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $i < s$. Since $\mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M)$ is Artinian for all $i < s$, we have that there exists a positive integer n such that $\mathfrak{m}^t \mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M) = \mathfrak{m}^n \mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M)$, for all $t \geq n$. Thus, $\mathfrak{m}^n \mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M) = \bigcap_{t > 0} \mathfrak{m}^t \mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M) = 0$, since $\mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M)$ is $\mathfrak{m}$-separated. Therefore,

$$\mathfrak{m} \subseteq \mathrm{Rad}(\mathrm{Ann}_R(\mathfrak{F}_i^{\mathfrak{a}, \mathfrak{m}}(M))), \text{ for all } i < s.$$

(ii) $\Rightarrow$ (i) Since M is an Artinian R-module, we have that $\{M/\mathfrak{a}^n M\}_{n \in \mathbb{N}}$ is an inverse system of Artinian R-modules. Therefore, $\Lambda_{\mathfrak{a}}(M)$ is an Artinian R-module, by [11, Properties 2.4, 3.6 and 3.3] and [10, Properties 27.2 and

27.3]. By Proposition 2.3, it follows that $\mathfrak{F}_i^{\mathfrak{a},\mathfrak{m}}(M) = H_i^{\mathfrak{m}}(\Lambda_{\mathfrak{a}}(M))$, for all $i \geq 0$. By the implication (ii) $\Rightarrow$ (i) of [2, Theorem 3.4], we have that $H_i^{\mathfrak{m}}(\Lambda_{\mathfrak{a}}(M))$ is an Artinian R-module for all $i < s$. This finishes the proof. □

Remark 3.4. We note that the implication (i) implies (ii) in the proof of Theorem 3.3 we need not assume that *M* is an Artinian *R*-module.

4. Vanishing and Non-vanishing Results

Recall that a module *M* is simple if it is non-zero and does not admit a proper non-zero submodule. Simplicity of a module *M* is equivalent to say that $Rm = M$, for every *m* non-zero in *M*. The Soc (*M*) the socle of *M* is the sum of all simple submodules of *M*, i.e., the submodule

$$\mathrm{Soc}(M) = \sum \{N \mid N \text{ is simple submodule of } M\}.$$

We recall also that a module *M* is said to be *semisimple* if it satisfies any of the equivalent conditions:

(i) It is a sum of simple submodules.

(ii) It is a direct sum of simple submodules.

So the socle of *M* is the largest submodule of *M* generated by simple modules, or equivalently, it is the largest semisimple submodule of *M*.

Proposition 4.1. *Let R be a Noetherian ring and let M be a semidiscrete and linearly compact R-module, and suppose that* $\mathrm{Soc}(M/\mathfrak{a}M) = 0$. *Then we have* $\mathfrak{F}_i^{\mathfrak{a},I}(M) = 0$, *for all* $i > 0$.

Proof. We have that $\mathrm{Soc}(M/\mathfrak{a}^n M) \subseteq \mathrm{Soc}(M/\mathfrak{a}M) = 0$, for all $n \geq 1$. By [5, Lemma 4.2], we have $H_i^I(M/\mathfrak{a}^n M) = 0$, for all $n \geq 1$ and for all $i > 0$. Therefore, $\mathfrak{F}_i^{\mathfrak{a},I}(M) = \varprojlim_{n\in\mathbb{N}} H_i^I(M/\mathfrak{a}^n M) = 0$, for all $i > 0$, as required. □

Lemma 4.2. *Let M be a semidiscrete linearly compact R-module, where the ring R is Noetherian. Then,* $IM = M \,(\mathrm{mod}\, \mathfrak{a}^n M)$, *for all* $n \geq 1$, *if and only if* $\mathfrak{F}_0^{\mathfrak{a}, I}(M) = 0$.

Proof. Suppose that $IM = M \,(\mathrm{mod}\, \mathfrak{a}^n M)$, for all $n \geq 1$; by [4, Corollary 2.5], we have that the *I*-adic completion $\Lambda_I(M/\mathfrak{a}^n M)$ of $M/\mathfrak{a}^n M$ is null, for all $n \geq 1$. On the other hand,

$$\Lambda_I(M/\mathfrak{a}^n M) = \varprojlim_{t \in \mathbb{N}}(R/I^t \otimes_R M/\mathfrak{a}^n M) = \varprojlim_{t \in \mathbb{N}} \mathrm{Tor}_0^R(R/I^t, M/\mathfrak{a}^n M).$$

Therefore, $H_0^I(M/\mathfrak{a}^n M) = 0$, for all $n \geq 1$. Thus, $\mathfrak{F}_0^{\mathfrak{a}, I}(M) = 0$.

Now, suppose that $IM \neq M \,(\mathrm{mod}\, \mathfrak{a}^n M)$, for some $n \geq 1$. Thus, by [4, Corollary 2.5], we have that $H_0^I(M/\mathfrak{a}^n M) \neq 0$. Then the short exact sequence

$$0 \to \mathfrak{a}^n M/\mathfrak{a}^{n+1} M \to M/\mathfrak{a}^{n+1} M \to M/\mathfrak{a}^n M \to 0$$

induces an epimorphism $\Lambda_I(M/\mathfrak{a}^{n+1} M) \to \Lambda_I(M/\mathfrak{a}^n M) \to 0$, of non-zero *R*-modules for all $n \in \mathbb{N}$. Hence, the inverse limit $\varprojlim_{n \in \mathbb{N}} \Lambda_I(M/\mathfrak{a}^n M)$ is not zero, according to [18, Remark 4.6]. It follows that $\mathfrak{F}_0^{\mathfrak{a}, I}(M) \neq 0$, as required. □

A sequence of elements $x_1, \ldots, x_r$ in *R* is said to be an *M-coregular sequence* [16, Definition 3.1] if $(0 :_M (x_1, \ldots, x_r)) \neq \{0\}$ and moreover we have that $(0 :_M (x_1, \ldots, x_{i-1})) \xrightarrow{x_i} (0 :_M (x_1, \ldots, x_{i-1}))$ is surjective, for all $i = 1, \ldots, r$. We denote by $\mathrm{width}_I(M)$ the supremum of the lengths of all maximal *M*-coregular sequences in the ideal *I*. If such sequences do not exist, then we write $\mathrm{width}_I(M) = \infty$. We remember that a sequence of elements $x_1, \ldots, x_s$ in *I* is a maximal *M*-coregular sequence if for all $x_{s+1} \in I$, the sequence $x_1, \ldots, x_s, x_{s+1}$ is not M-coregular. We have by [5, Remark 4.6(i)] and [5, Lemma 4.7] that $\operatorname{width}_I(M) < \infty,$ when M is a semidiscrete linearly compact R-module.

Theorem 4.3. *Let R be a Noetherian ring and let M be a linearly compact, semidiscrete R-module and $(0 :_{\Lambda_{\mathfrak{a}}(M)} I) \neq 0.$ Then, all $\Lambda_{\mathfrak{a}}(M)$-coregular sequences maximal at I have the same length. Moreover,*

$$\operatorname{width}_I(\Lambda_{\mathfrak{a}}(M)) = \inf\{i \mid \mathfrak{F}_i^{\mathfrak{a}, I}(M) \neq 0\}.$$

Proof. By Proposition 2.3, we have that $\mathfrak{F}_i^{\mathfrak{a}, I}(M) = H_i^I(\varprojlim_{n \in \mathbb{N}} M/\mathfrak{a}^n M)$ $= H_i^I(\Lambda_{\mathfrak{a}}(M)).$ Now, by the [5, Theorem 4.11], it follows that

$$\operatorname{width}_I(\Lambda_{\mathfrak{a}}(M)) = \inf\{i \mid H_i^I(\Lambda_{\mathfrak{a}}(M)) \neq 0\}$$

and so it follows the result. □

Corollary 4.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a semidiscrete linearly compact R-module. Thus, we have that*:

(i) *If M is complete and $(0 :_{\Lambda_{\mathfrak{m}}(M)} I) \neq 0,$ then*

$$\operatorname{width}_I(\Lambda_{\mathfrak{m}}(M)) = \inf\{i \mid H_i^I(M) \neq 0\}.$$

(ii) *If $\mathfrak{a} = 0$ and $(0 :_M I) \neq 0,$ then*

$$\operatorname{width}_I(M) = \inf\{i \mid H_i^I(M) \neq 0\}.$$

Proof. (i) Indeed, as we have $\Lambda_{\mathfrak{m}}(M) \cong M$ it follows, by Theorem 4.3, that we have

$$\operatorname{width}_I(M) = \operatorname{width}_I(\Lambda_{\mathfrak{m}}(M)) = \inf\{i \mid H_i^I(M) \neq 0\},$$

as required.

(ii) It is clear. □

Remark 4.5. Note that if the ideal $\mathfrak{a} = 0$ in Theorem 4.3, then we return to the usual result for the $\text{width}_I(M)$ which is given by [5, Theorem 4.11].

Proposition 4.6. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and let M be a non-zero semidiscrete linearly compact R-module. We have that if* $\mathfrak{F}_i^{\mathfrak{a},\mathfrak{m}}(M) = 0$, *for all* $i \geq 0$, *then* $\mathfrak{m}\Lambda_{\mathfrak{m}}(M) = \Lambda_{\mathfrak{m}}(M) \, (\bmod \, \mathfrak{a}^n \Lambda_{\mathfrak{m}}(M))$, *for all* $n \geq 1$.

Proof. Suppose that $\mathfrak{F}_i^{\mathfrak{a},\mathfrak{m}}(M) = 0$, for all $i \geq 0$. By Remark 2.9, we have that $\mathfrak{F}_0^{\mathfrak{a},\mathfrak{m}}(M) = \mathfrak{F}_0^{\mathfrak{a},\mathfrak{m}}(\Lambda_{\mathfrak{m}}(M))$. Thus, by Lemma 4.2, we obtain that

$$\mathfrak{m}\Lambda_{\mathfrak{m}}(M) = \Lambda_{\mathfrak{m}}(M)(\bmod \, \mathfrak{a}^n \Lambda_{\mathfrak{m}}(M)),$$

for all $n \geq 1$, as required.

Acknowledgment

The authors are deeply grateful to the referee for carefully reading the manuscript and the helpful suggestions.

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