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FUZZY SOFT IDEALS BASED ON FUZZY SOFT SPACES

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Abstract

In this paper, we introduce the notion of fuzzy soft ideals along with certain other related concepts and obtain some of their properties.

1. Introduction

A classical algebraic system is a universal set with one or more binary operations. However, fuzzy algebraic systems are not the same. Rosenfeld

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[1] found an adequate outlet, to overcome the absence of the fuzzy universal set through the introduction of the notion of a fuzzy subgroup of a group as a generalization of the fuzzy subset of a set introduced by Zadeh in [2]. Selvachandran and Salleh introduced the algebraic and hyperalgebraic structures pertaining to vague soft sets, fuzzy soft sets and soft sets ([3-10]) using Rosenfeld's approach. Liu [11] introduced the notion of fuzzy subrings and fuzzy ideals in Rosenfeld's sense. Dib [12] introduced the concept of a fuzzy space to enable the accurate formulation of the intrinsic definition of fuzzy algebraic systems. In this paper, we apply the theory of soft sets introduced by Molodtsov [13] to establish the notion of fuzzy soft ideal and fuzzy soft ring homomorphism based on the concept of fuzzy soft spaces.

2. Preliminaries

In this section, we recall some basic definitions and results pertaining to the theory of soft sets, fuzzy sets and fuzzy spaces.

Definition 2.1 [13]**.** Let U be an initial universe set and let A be the set of parameters. Let $P(U)$ denote the power set of U. A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2.2 [14]**.** Let U be an initial universe set and E be a set of parameters. Let $F(U)$ denote the fuzzy power set of U. Let $A \subset E$. A pair (F, A) is called a *fuzzy soft set* over U, where F is a mapping given by $F : A \to F(U)$.

Definition 2.3 [14]**.** For two fuzzy soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a *fuzzy soft subset* of (G, B) if:

(i) $A \subset B$ and

(ii) $\forall \varepsilon \in A,\ F(\varepsilon)$ is a fuzzy subset of $G(\varepsilon)$.

Definition 2.4 [12]**.** A *fuzzy space*, denoted by X_L, is the set of all ordered pairs (x, L), $x \in X$, that is, $X_L = \{(x, L) : x \in X\}$, where $(x, L) = \{(x, r) : r \in L\}$ is called a *fuzzy element* of the fuzzy space X_L.

Definition 2.5 [12]**.** A *fuzzy subspace* *U* of the fuzzy space X_L is a collection of ordered pairs (y, L_y), where $y \in U_0$ for a given subset U_0 of *X* and L_y is an *M*-sublattice of *L* and is denoted by $U = \{(y, L_y) : y \in U_0\}$, where (y, L_y) is called a *fuzzy element* of the fuzzy subspace *U*. For an *L*-fuzzy subset *A* of *X*, *A* induces the following fuzzy subspace $H(A)$, called the *induced fuzzy subspace* by *A* of X_L:

$$H(A) = \{(x, [0, A(x)]) : x \in A_0\},$$

where $A_0 = \{x \in X : A(x) \neq 0\}$ is the support of *A*.

Definition 2.6 [12]**.** A *fuzzy binary operation* $\underline{F} = (F, f_{xy})$ on the fuzzy space X_L is a fuzzy function $\underline{F}$ from $X \times X$ to *X* with comembership functions $f_{xy} : L * L \to L$ that satisfies the following conditions:

(i) $f_{xy}(r, s) = 0$ if and only if $r = 0$ or $s = 0$,

(ii) f_{xy} are onto, that is, $f_{xy}(L * L) = L$ for all $(x, y) \in X \times X$.

3. Fuzzy Soft Ideals over Fuzzy Soft Spaces

In this section, the concept of fuzzy soft ideals defined over fuzzy soft spaces is established. This concept is a generalization of the notion of soft rings introduced by Acar et al. [15] and further extends the notion of fuzzy soft rings based on fuzzy soft spaces introduced by Selvachandran [16].

Definition 3.1 [16]**.** A *fuzzy soft ring*, denoted by $((R, I), F^+, F^*)$ is a fuzzy soft space $((R, I), I)$ together with two fuzzy soft binary operations, namely, F^+ and F^*, satisfying the following conditions:

(i) $((R, I), F^{+})$ is an abelian fuzzy soft group.

(ii) $((R, I), F^{*})$ is a fuzzy soft semigroup.

(iii) F^{*} is distributive over F^{+}. That is, for any three fuzzy soft elements, $(\langle x, \mu_{F(e)}(x)\rangle, I), (\langle y, \mu_{F(e)}(y)\rangle, I), (\langle z, \mu_{F(e)}(z)\rangle, I),$ the left and right distributive properties are satisfied.

Definition 3.2 [16]**.** If $(\langle x, \mu_{F(e)}(x)\rangle, I)$ and $(\langle y, \mu_{F(e)}(y)\rangle, I)$ are two nonzero elements of a fuzzy soft ring $((R, I), F^{+}, F^{*})$ such that $(\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I) = (0_R, I),$ then $(\langle x, \mu_{F(e)}(x)\rangle, I)$ and $(\langle y, \mu_{F(e)}(y)\rangle, I)$ are *divisors of zero* or *zero divisors*. In particular, $(\langle x, \mu_{F(e)}(x)\rangle, I)$ is called the *left divisor of zero* and $(\langle y, \mu_{F(e)}(y)\rangle, I)$ is called the *right divisor of zero*. In a commutative fuzzy soft ring, every left divisor of zero is also a right divisor of zero and conversely. Thus, there is no distinction between the left and right divisors of zero in a commutative fuzzy soft ring.

Definition 3.3. A *fuzzy soft integral domain D* is a commutative fuzzy soft ring with unity that contains no fuzzy soft zero divisors.

Definition 3.4. Let $((R, I), F^{+}, F^{*})$ be a fuzzy soft ring with unity. $((R, I), F^{+}, F^{*})$ is called a *fuzzy soft division ring* if for each nonzero element $(\langle x, \mu_{F(e)}(x)\rangle, I) \in ((R, I), F^{+}, F^{*}),$ there exists a fuzzy soft element $(\langle y, \mu_{F(e)}(y)\rangle, I) \in ((R, I), F^{+}, F^{*})$ such that

$$\begin{aligned} &(\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I) \\ &= (\langle y, \mu_{F(e)}(y)\rangle, I)F^{*}(\langle x, \mu_{F(e)}(x)\rangle, I) \\ &= (1_R, I), \end{aligned}$$

where $(1_R, I)$ is the unity element of $((R, I), F^{+}, F^{*}).$ If such a

$(\langle y, \mu_{F(e)}(y)\rangle, I)$ exists, then we say that $(\langle y, \mu_{F(e)}(y)\rangle, I)$ is a *multiplicative inverse* of $(\langle x, \mu_{F(e)}(x)\rangle, I)$.

Definition 3.5 [16]**.** A *fuzzy soft field* $((R, I), F^{+}, F^{*})$ is a commutative fuzzy soft ring with unity in which every nonzero fuzzy soft element $(\langle x, \mu_{F(e)}(x)\rangle, I)$ in $((R, I), F^{+}, F^{*})$ is a unit.

Definition 3.6 [16]**.** A *fuzzy soft field* $((R, I), F^{+}, F^{*})$ is a fuzzy soft set, (R, I) of elements together with two operations, F^{+} and F^{*}, satisfying the following axioms:

(i) (R, I) is a commutative fuzzy soft group under the additive binary operation, F^{+}.

(ii) If $(0_R, I)$ (which is the identity for the additive binary operation F^{+}) is excluded, then (R, I) is a commutative fuzzy soft group under the multiplicative binary operation F^{*}.

(iii) The multiplicative binary operation F^{*} is distributive over the additive binary operation F^{+}, i.e., for all $(\langle x, \mu_{F(e)}(x)\rangle, I)$, $(\langle y, \mu_{F(e)}(y)\rangle, I)$ and $(\langle z, \mu_{F(e)}(z)\rangle, I) \in ((R, I), F^{+}, F^{*})$, the following holds true:

$$(\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}((\langle y, \mu_{F(e)}(y)\rangle, I)F^{+}(\langle z, \mu_{F(e)}(z)\rangle, I))$$

$$= ((\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I))$$

$$\times F^{+}((\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle z, \mu_{F(e)}(z)\rangle, I)).$$

Definition 3.7. A *fuzzy soft* (*ring*, *division ring*, *field*) *integral domain* $((R, I), F^{+}, F^{*})$, where $\boldsymbol{F}^{+} = (F^{+}, f^{+}_{xy})$ and $\boldsymbol{F}^{*} = (F^{*}, f^{*}_{xy})$ is said to be *uniform* if $\boldsymbol{F}^{+}$ and $\boldsymbol{F}^{*}$ are uniform fuzzy binary operations, i.e., $f^{+}_{\underline{xy}}(r, s) =$

$f^{+}(r, s)$ and $\underline{f}^{*}_{xy}(r, s) = f^{*}(r, s)$ for all

$$(\langle x, \mu_{F(e)}(x)\rangle, I), (\langle y, \mu_{F(e)}(y)\rangle, I) \in (R, I).$$

Theorem 3.8. *Every fuzzy soft field* $((R, I), F^{+}, F^{*})$ *is a fuzzy soft integral domain.*

Proof. Let $(\langle x, \mu_{F(e)}(x)\rangle, I), (\langle y, \mu_{F(e)}(y)\rangle, I) \in ((R, I), F^{+}, F^{*})$ and suppose that $(\langle x, \mu_{F(e)}(x)\rangle, I) \neq 0.$ Hence, if

$$(\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I) = (0_R, I),$$

then we have

$$(\langle x^{-1}, \mu_{F(e)}(x^{-1})\rangle, I)F^{*}[(\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I)]$$

$$= (\langle x^{-1}, \mu_{F(e)}(x^{-1})\rangle, I)F^{*}(0_R, I)$$

$$= (0_R, I).$$

But then,

$$(0_R, I) = (\langle x^{-1}, \mu_{F(e)}(x^{-1})\rangle, I)F^{*}[(\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I)]$$

$$= [(\langle x^{-1}, \mu_{F(e)}(x^{-1})\rangle, I)F^{*}(\langle x, \mu_{F(e)}(x)\rangle, I)]F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I)$$

$$= (1_R, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I)$$

$$= (\langle y, \mu_{F(e)}(y)\rangle, I).$$

We have shown that $(\langle x, \mu_{F(e)}(x)\rangle, I)F^{*}(\langle y, \mu_{F(e)}(y)\rangle, I) = (0_R, I)$ with $(\langle x, \mu_{F(e)}(x)\rangle, I) \neq (0_R, I)$ implies that $(\langle y, \mu_{F(e)}(y)\rangle, I) = (0_R, I)$ in $((R, I), F^{+}, F^{*})$. Hence, there are no zero divisors in $((R, I), F^{+}, F^{*})$. Therefore, $((R, I), F^{+}, F^{*})$ is a commutative fuzzy soft ring with unity. □

Theorem 3.9. *Consider the set given below*:

$$(\langle x, \mu_{F(e)}(x)\rangle, I), (\langle y, \mu_{F(e)}(y)\rangle, I), (\langle z, \mu_{F(e)}(z)\rangle, I) \in ((R, I), F^+, F^*),$$

where $((R, I), F^+, F^*)$ *is a fuzzy soft integral domain. If* $(\langle x, \mu_{F(e)}(x)\rangle, I) \neq (0_R, I)$ *and*

$$(\langle x, \mu_{F(e)}(x)\rangle, I)F^*(\langle y, \mu_{F(e)}(y)\rangle, I)$$

$$= (\langle x, \mu_{F(e)}(x)\rangle, I)F^*(\langle z, \mu_{F(e)}(z)\rangle, I)$$

and

$$(\langle y, \mu_{F(e)}(y)\rangle, I)F^*(\langle x, \mu_{F(e)}(x)\rangle, I)$$

$$= (\langle z, \mu_{F(e)}(z)\rangle, I)F^*(\langle x, \mu_{F(e)}(x)\rangle, I),$$

then $(\langle y, \mu_{F(e)}(y)\rangle, I) = (\langle z, \mu_{F(e)}(z)\rangle, I)$.

Proof. The proof is straightforward. □

Next, we proceed to introduce the main concept in this paper, namely the notion of fuzzy soft ideals based on fuzzy soft spaces.

Definition 3.10. Let $((R, I), F^+, F^*)$ be a fuzzy soft ring. A fuzzy soft subring $((S, I), F^+, F^*)$ of $((R, I), F^+, F^*)$ having the property

$$(\langle x, \mu_{F(e)}(x)\rangle, I)F^*(\langle y, \mu_{F(e)}(y)\rangle, I) \in ((S, I), F^+, F^*)$$

for all $(\langle x, \mu_{F(e)}(x)\rangle, I) \in ((R, I), F^+, F^*)$ and

$$(\langle y, \mu_{F(e)}(y)\rangle, I) \in ((S, I), F^+, F^*)$$

is called the *left fuzzy soft ideal* in $((R, I), F^+, F^*)$. $((S, I), F^+, F^*)$ is called a *right fuzzy soft ideal* in $((R, I), F^+, F^*)$ if it satisfies the property

$$(\langle y, \mu_{F(e)}(y)\rangle, I)F^*(\langle x, \mu_{F(e)}(x)\rangle, I) \in ((S, I), F^+, F^*)$$

for all $(\langle x, \mu_{F(e)}(x)\rangle, I) \in ((R, I), F^+, F^*)$ and

$$(\langle y, \mu_{F(e)}(y)\rangle, I) \in ((S, I), F^+, F^*).$$

Also, $((S, I), F^+, F^*)$ is called a *fuzzy soft ideal* if it is both the right and left fuzzy soft ideals of $((R, I), F^+, F^*)$.

$\{(0_R, I)\}$ and $((R, I), F^+, F^*)$ are both the left and right fuzzy soft ideals in $((R, I), F^+, F^*)$.

$\{(0_R, I)\}$ and $((R, I), F^+, F^*)$ are called *improper* fuzzy soft ideals in $((R, I), F^+, F^*)$.

All the other fuzzy soft ideals in $((R, I), F^+, F^*)$ are called *proper* left or right fuzzy soft ideals.

Theorem 3.11. *If* $((R, I), F^+, F^*)$ *is a fuzzy soft ring with unity and* $((S, I), F^+, F^*)$ *is a fuzzy soft ideal in* $((R, I), F^+, F^*)$ *containing a unit, then* $((S, I), F^+, F^*) = ((R, I), F^+, F^*)$.

Proof. Let $((S, I), F^+, F^*)$ be a fuzzy soft ideal in $((R, I), F^+, F^*)$ and suppose that $(\langle x, \mu_{F(e)}(x)\rangle, I) \in ((S, I), F^+, F^*)$ for some unit $(\langle x, \mu_{F(e)}(x)\rangle, I)$ in $((R, I), F^+, F^*)$. Then the condition

$$(\langle r, \mu_{F(e)}(r)\rangle, I)F^*((S, I), F^+, F^*) \subseteq ((S, I), F^+, F^*)$$

for all $(\langle r, \mu_{F(e)}(r)\rangle, I)$ in $((R, I), F^+, F^*)$ implies that if we take

$$(\langle r, \mu_{F(e)}(r)\rangle, I) = (\langle x^{-1}, \mu_{F(e)}(x^{-1})\rangle, I)$$

and $(\langle x, \mu_{F(e)}(x)\rangle, I) \in ((S, I), F^+, F^*)$ such that

$$(1_R, I) = (\langle x^{-1}, \mu_{F(e)}(x^{-1})\rangle, I)F^*(\langle x, \mu_{F(e)}(x)\rangle, I)$$

is in $((S, I), F^+, F^*)$.

But then, $(\langle r, \mu_{F(e)}(r)\rangle, I)F^*((S, I), F^+, F^*) \subseteq ((S, I), F^+, F^*)$ for all $(\langle r, \mu_{F(e)}(r)\rangle, I)$ in $((R, I), F^+, F^*)$ implies that

$$(\langle r, \mu_{F(e)}(r)\rangle, I)F^*(1_R, I) = (\langle r, \mu_{F(e)}(r)\rangle, I)$$

is in $((S, I), F^+, F^*)$ for all $(\langle r, \mu_{F(e)}(r)\rangle, I) \in ((R, I), F^+, F^*)$.

Hence, $((S, I), F^+, F^*) = ((R, I), F^+, F^*)$. □

Corollary 3.12. *A fuzzy soft field contains no proper fuzzy soft ideals*.

4. Conclusion

In this paper, we further developed the theory of fuzzy soft rings based on fuzzy soft spaces through the introduction of the notion of fuzzy soft integral domain, fuzzy soft division rings, fuzzy soft fields and fuzzy soft ideals based on fuzzy soft spaces in Dib's sense. The properties and structural characteristics of these concepts are also studied and investigated.

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