Far East J. Appl. Math. **21**(1) (2005), 17-30

GENERAL-ENTROPY ANALYSIS OF NAVIER-STOKES EQUATIONS

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Abstract

New methods of analytical hydrodynamics are developed for studying the existence, smoothness, and structure of solutions of Navier-Stokes equations. The subject of this work is the laminar (regular) flows of a viscous incompressible liquid considered on general-entropy manifolds. Hydrodynamic flows support two symmetries: a conservation of the general entropy and a duality of the impulse. Several theorems and lemmas are presented for analysis of structures of regular flows. The main result is the separation theorem that reduces Navier-Stokes equations to the two sets of equations, the canonical system of equations of the acceleration potential and the equations of the vortex flux. It is shown that laminar flows contain potential and vortical components.

Introduction

At the present time, Navier-Stokes equations are attracting significant attention of scientists working in the area of analytical hydrodynamics. In the theory of Navier-Stokes equations, starting from the work by Leray [4], main efforts have been devoted to "weak" (turbulent) solutions [1, 3, 11, 12]. However, knowledge of the structure

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2000 Mathematics Subject Classification: 76D03, 76D05.

Key words and phrases: Navier-Stokes equations, general entropy, entropy manifold.

Received April 11, 2005

and properties of the solutions of Navier-Stokes equations remains rather limited. Bringing new ideas and methodologies into analytical hydrodynamics will help to advance this science.

A new concept and mathematical tools for continuum media have been developed by Panchenkov [7-10]. In this article, the general-entropy analysis is applied for studying Navier-Stokes equations. Two main problems are addressed: an investigation of the structure of laminar fields and flows of a viscous incompressible liquid and a separation of Navier-Stokes equations into two systems. At the same time, a new, effective mathematical technique based on the general entropy concept is developed.

The set of axioms of the general-entropy theory of Navier-Stokes equations identifies a viscous incompressible liquid as an object of the general-entropy conceptual model [7]:

(1) Navier-Stokes equations are the equations of the general-entropy conceptual model.

(2) States of a viscous incompressible liquid satisfy a global symmetry of the conservation of the general entropy.

(3) Continuum medium of a viscous incompressible liquid contains two types of objects: hydrodynamic fields and hydrodynamic flows.

(4) Hydrodynamic flows are located on the general-entropy manifolds.

(5) There are two types of hydrodynamic fields: vorticity fields and potential fields.

(6) There are two types of hydrodynamic flows: laminar flows and turbulent flows.

(7) Turbulence is a type of chaos.

(8) Laminar flows possess smoothness (regularity) and are realized on real geometrical objects-the general-entropy manifolds.

Main Objects

The basic geometrical object of the general-entropy theory is a phase space, which is a smooth manifold with local coordinates q and p:

$$\Lambda = \{q,\ p \,|\, \Lambda = \Lambda_q \times \Lambda_p;\ \Lambda_q \subset R^3;\ \Lambda_p \subset R_3;\ \Lambda \subset R^3 \oplus R_3\},$$

where q is the generalized coordinate, p is the impulse, R^3 is a three-dimensional real Euclidean space, and R_3 is a conjugate three-dimensional real Euclidean space. The phase space is formed by a configuration space, $\Lambda_q = \{q \,|\, \Lambda_q \subset R^3\}$, and an impulse space, $\Lambda_p = \{p \,|\, \Lambda_p \subset R_3\}$.

The general entropy has dual representation [7]

$$H_f = H_q + H_p; \quad \{q,\ p\} \in \Lambda, \tag{1}$$

where H_q is a structural entropy and H_p is an impulse entropy. In thermodynamic interpretation, the general entropy H_f contains the classical thermodynamic entropy H_T, and $\dot{H}_T = -\dot{H}_q = \dot{H}_p$ in dissipative systems [7]. States of a continuum medium in the phase space maintain the global symmetry

$$H_f = \text{const.} \tag{2}$$

This symmetry follows from the principle of the general-entropy maximum applied to the Boltzman concept of entropy [7]. The principle of the general-entropy maximum on a set of regular solutions in a Hilbert space is adequate to Hamilton principle.

A reduction of the phase space that satisfies Eq. (2) is a general-entropy manifold

$$E = \{q,\ p \,|\, E \subset \Lambda,\ H_f\}. \tag{3}$$

The general-entropy manifold has a structure of the direct product

$$E = E_q \times E_p,$$

$$E_q = \{q \,|\, E_q \subset E,\ H_q\}; \quad E_p = \{q \,|\, E_p \subset E,\ H_p\}, \tag{4}$$

where E_q is a general-entropy manifold of the configuration space and E_p is a general-entropy manifold of the impulse space.

A solenoid manifold is obtained by imposing divergence on the general-entropy manifold:

$$\sigma = \operatorname{div} A; \quad A = \left[\frac{\partial q}{\partial t}, \frac{\partial p}{\partial t}\right],$$

$$M = \{q,\ p \mid M \subset E;\ \sigma = \operatorname{div} A\}. \tag{5}$$

The following equation supports the symmetry (2) on the solenoid manifold [7]:

$$\sigma = 0; \quad \{q,\ p\} \in M. \tag{6}$$

Introduction of an acceleration potential, Θ, leads to another reduction of the general-entropy manifold, a manifold of the acceleration potential with metric ξ,

$$\pi = \left\{q,\ p \mid \pi \subset M;\ \Theta;\ \xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}. \tag{7}$$

On this manifold, the canonical equations of the acceleration potential are held [7]

$$\frac{\partial q}{\partial t} = -\frac{\partial \Theta}{\partial p}; \quad \frac{\partial p}{\partial t} = \frac{\partial \Theta}{\partial q}; \quad \{q,\ p\} \in \pi. \tag{8}$$

A reduction of the manifold of the acceleration potential that contains the impulse potential is an important concept in our theory. This submanifold is a Hilbert field and it has a form

$$Z = \{q,\ p \mid Z \subset \pi;\ \Psi\}. \tag{9}$$

On a Hilbert field, $p = \operatorname{grad}\Psi;\ \ \Psi = \Psi(q,\ t),$ and the equation for the acceleration potential [6, 7] is

$$\frac{\partial \Psi}{\partial t} = \Theta; \quad \{q,\ p\} \subset Z. \tag{10}$$

The second important symmetry in our theory is a dual representation of the impulse

$$p = \begin{cases} p \in \Lambda_p, \\ p(q,\ t);\ \ q \in E_q;\ \ t \in [0,\ T]. \end{cases} \tag{11}$$

The first component of this duality is a free impulse, and the second component defines a bound impulse. The bound impulse is realized when the following diffeomorphism exists:

$$T_s : \Lambda_q \to \Lambda_p; \quad p(q, t) \in C^\infty(\Lambda_q^T); \quad \Lambda_q^T = \Lambda_q \times (0, T),$$

where T is the time of destruction of a laminar flow.

Theory Development

Using Helmholtz decomposition theorem [2], a laminar flow on a general-entropy manifold can be taken in the form

$$\dot{q} = u : u = p + \omega; \quad \operatorname{div} u = 0; \quad q \in E_q; \quad t \in [t, T],$$

$$\omega = \omega(q, t); \quad q \in E_q; \quad \omega = \operatorname{rot} B; \quad B \in C^\infty(\Lambda_q^T),$$

$$p = \begin{cases} p \in \Lambda_p, \\ p(q, t); \;\; q \in E_q; \;\; t \in [0, T], \end{cases}$$

$$p = p(q, t); \quad p = \operatorname{grad} \Psi; \quad \Psi = \Psi(q, t); \quad \Psi \in C^\infty(\Lambda_q^T),$$

$$T_E : E_q \to E_p. \tag{12}$$

In a laminar flow described by Eq. (12), the key role is played by the hypothesis of the vorticity independence on the impulse

$$\omega = \omega(q, t); \quad \{q, p, t\} \in \Lambda_T; \quad \Lambda_T = \Lambda \times (0, T). \tag{13}$$

The equation of hydrodynamic fields in the configuration space is

$$\operatorname{div} u = 0; \quad q \in \Lambda_q. \tag{14}$$

Let us introduce an extension of these fields into the phase space

$$\{\operatorname{div} u = 0;\ q \in \Lambda_q\} \to \{\operatorname{div} \omega = 0;\ \{q, p\} \in \Lambda;\ p \in \Lambda_p\}. \tag{15}$$

Upon this transformation, the impulse becomes free. Only the vorticity field, independent of the free impulse, will exist in the phase space. This fact confirms the hypothesis of independence (13). From Eqs. (13) and (14), it follows that a laminar flow maintains two types of hydrodynamic fields, vortical and potential:

$$\Delta B = -\Omega; \quad \Omega = \mathrm{rot}\,\omega; \quad q \in \Lambda_q,$$

$$\Delta \Psi = 0; \quad q \in \Lambda_q,$$

where B is the vorticity three-potential and Ψ is the impulse potential.

Two theorems of the necessary conditions for the existence of a laminar flow are now formulated.

First Theorem. *On the general-entropy manifold*

$$E = \{q, p \mid E \subset \Lambda;\ E = E_q \times E_p;\ H_f\}$$

of the phase space

$$\Lambda = \{q, p \mid \Lambda = \Lambda_q \times \Lambda_p;\ q \in \Lambda_q;\ p \in \Lambda_p;\ \Lambda \subset R^3 \oplus R_3\}$$

the necessary conditions for the existence of the laminar flow

$$\dot{q} = u : u = p + \omega; \quad \omega = \mathrm{rot}\,B; \quad \mathrm{div}\,u = 0; \quad \{q, p\} \in E; \quad t \in [0, T]$$

that satisfies the independence provision

$$\omega = \omega(q, t); \quad q \in E_q$$

are the following:

(1) *The existence of the vorticity three-potential*

$$B \in C^\infty(\Lambda_q^T) : \Lambda_q^T = \Lambda_q \times (0, T)$$

that satisfies Poisson equation

$$\Delta B = -\Omega; \quad \Omega = \mathrm{rot}\,\omega; \quad q \in \Lambda_q.$$

(2) *The realization of the free impulse* $p \in \Lambda_p$ *in the form of a function*

$$p \in C^\infty([0, T]) \text{ on } [0, T].$$

Second Theorem. *On the general-entropy manifold of the configuration space*

$$E_q = \{q \mid E_q \subset \Lambda_q;\ H_q\}, \quad \Lambda_q = \{q \mid \Lambda_q \subset R^3\}$$

the necessary conditions for the existence of the laminar flow

$$\dot{q} = u : u = p + \omega;\ \omega = \text{rot}\, B;\ p = \text{grad}\Psi;\ \text{div}\, u = 0;\ q \in E_q;\ t \in [0, T]$$

that satisfies the independence provision

$$\omega = \omega(q, t); \quad q \in E_q$$

are the following:

(1) *The existence of the vorticity three-potential*

$$B \in C^{\infty}(\Lambda_q^T) : \Lambda_q^T = \Lambda_q \times (0, T)$$

that satisfies Poisson equation

$$\Delta B = -\Omega; \quad \Omega = \text{rot}\, \omega; \quad q \in \Lambda_q.$$

(2) *The existence of the impulse potential* $\Psi \in C^{\infty}(\Lambda_q^T)$ *that satisfies Laplace equation*

$$\Delta \Psi = 0; \quad q \in \Lambda_q.$$

The following theorem contains the main result.

Separation Theorem. *If on the solenoid manifold*

$$M = \{q, p \,|\, M \subset E;\ \text{div}\, A\},$$

$$M = M_q \times M_p; \quad M_q = \{q \,|\, M_q \subset E_q\},$$

$$M_p = \{p \,|\, M_p \subset E_p\} : A = \left[\frac{\partial q}{\partial t}, \frac{\partial p}{\partial t}\right]; \quad \text{div}\, A = 0$$

of the phase space

$$\Lambda = \{q, p \,|\, \Lambda = \Lambda_q \times \Lambda_p;\ q \in \Lambda_q;\ p \in \Lambda_p;\ \Lambda \subset R^3 \oplus R_3\}$$

the laminar flow exists

$$\dot{q} = u : u = p + \omega; \quad \omega = \text{rot}\, B; \quad \text{div}\, u = 0$$

$$\{B, p\} \in C^{\infty}(\Lambda_q^T) : \Lambda_q^T = \Lambda_q \times (0, T)$$

that is described by Navier-Stokes equations

$$\frac{Du}{Dt} = -\text{grad} \prod - \nu\, \text{rot}\, \Omega + f; \quad \{q, p\} \in M; \quad t \in (0, T)$$

and

(1) *the external force has a form*

$$f = \operatorname{grad} \Phi_f + \operatorname{rot} \Omega_f; \quad f \in C^\infty(M_q \times (0, T)),$$

(2) *the condition of independence is fulfilled*

$$[p \times \Omega] = \operatorname{grad} \Phi; \quad \Phi \in C^\infty(M_q \times (0, T)),$$

then Navier-Stokes equations can be separated into the two sets of equations:

(1) *The canonical set of equations for the acceleration potential*:

$$\frac{\partial q}{\partial t} = -\frac{\partial \Theta}{\partial p}; \quad \frac{\partial p}{\partial t} = \frac{\partial \Theta}{\partial q}; \quad \{q,\ p\} \in \pi,$$

$$\Theta = -\frac{1}{2}\| u \|^2_{R^3} - \prod\nolimits_p; \quad \prod\nolimits_p = \prod - \Phi - \Phi_f,$$

$$[p \times \Omega] = \operatorname{grad} \Phi; \quad \Phi \in C^\infty(M_q \times (0, T)),$$

$$\Delta\Theta = 0; \quad q \in \Lambda_q; \quad \Theta \in C^\infty(\Lambda_q^T). \tag{16}$$

(2) *The vortex flux equations*:

$$\frac{\partial \omega}{\partial t} = [\omega \times \Omega] + \nu\Delta\omega + \operatorname{rot} \Omega_f,$$

$$\| \Omega \|^2_{R^3} = (\omega \,|\, \operatorname{rot} \Omega)_{R^3}; \quad \Omega = \operatorname{rot} \omega; \quad q \in M; \quad \omega \in C^\infty(M_q \times (0, T)). \tag{17}$$

Proof. The full time derivative in Navier-Stokes equation is

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \angle_\xi u,$$

where Lie derivative is defined as

$$\angle_\xi u = (u \,|\, \operatorname{grad})_{R^3} u,$$

where

$$(u \,|\, \operatorname{grad})_{R^3} u = \frac{1}{2} \operatorname{grad} \| u \|^2_{R^3} - [u \times \Omega]. \tag{18}$$

For a laminar flow the following representation of the vector product is valid:

$$[u \times \Omega] = \text{grad}\,\Phi + \text{rot}\,\chi.$$

Invoking the hypothesis of the vortex independence on the impulse, Eq. (18) can be separated into the two equations:

$$\omega = \omega(q,\, t) : [p \times \Omega] = \text{grad}\,\Phi, \tag{19}$$

$$[\omega \times \Omega] = \text{rot}\,\chi. \tag{20}$$

Accounting for Eq. (19), Lie derivative becomes

$$\angle_{\xi} u = \left(\frac{1}{2}\,\text{grad} \| \, u \, \|^2_{R^3} - \text{grad}\,\Phi\right) + (-[\omega \times \Omega]). \tag{21}$$

This structure allows us to obtain the equation for the impulse from Navier-Stokes equations

$$\frac{\partial p}{\partial t} = -\frac{1}{2}\,\text{grad} \| \, u \, \|^2_{R^3} - \text{grad} \prod\nolimits_p,$$

$$\prod\nolimits_p = \prod - \Phi - \Phi_f. \tag{22}$$

Let us also consider the equation of a laminar flow

$$\frac{\partial q}{\partial t} = u; \quad u = p + \omega. \tag{23}$$

Since a manifold of the acceleration potential is located on a solenoid manifold, the system of Eqs. (22) and (23) can be interpreted as the canonical equations of the acceleration potential. In this interpretation, the acceleration potential satisfies the following equations:

$$\frac{\partial \Theta}{\partial q} = -\frac{\partial}{\partial q}\left[\frac{1}{2} \| \, u \, \|^2_{R^3} + \prod\nolimits_p\right],$$

$$\frac{\partial \Theta}{\partial p} = -u; \quad u = p + \omega.$$

The solution of this system recovers the acceleration potential

$$\Theta = -\frac{1}{2} \| \, u \, \|^2_{R^3} - \prod\nolimits_p; \quad \{q,\, p\} \in \pi.$$

This acceleration potential forms the canonical system of equations of the theorem.

Now, let us take into account

$$\operatorname{div} p = 0; \quad q \in \Lambda_q .$$

Then the equation for the field of the acceleration potential is obtained from the second equation of the canonical system:

$$\Delta\Theta = 0; \quad q \in \Lambda_q; \quad \Theta \in C^\infty(\Lambda_q^T).$$

Combining rotor terms in Navier-Stokes equations, the equation for the vortex flux is

$$\frac{\partial\omega}{\partial t} = [\omega \times \Omega] + \nu\Delta\omega + \operatorname{rot}\Omega_f . \tag{24}$$

From Eq. (20) it follows

$$\operatorname{div}[\omega \times \Omega] = 0,$$

which together with a continuity equation gives the equation of the theorem

$$\| \Omega \|_{R^3}^2 = (\omega | \operatorname{rot}\Omega)_{R^3} .$$

The theorem is proved.

Several notes to the theorem:

1. The vorticity Eq. (24) can be transformed into Helmholtz equation

$$\frac{\partial\Omega}{\partial t} = (\Omega | \operatorname{grad})\omega - (\omega | \operatorname{grad})\Omega + \nu\Delta\Omega + \operatorname{rot}\operatorname{rot}\Omega_f . \tag{25}$$

2. The most complete and systematic study of vortical flows of a viscous incompressible liquid on the base of Eq. (24) has been carried out in [5].

3. It is important that the canonical system of the acceleration potential is given on a manifold of the acceleration potential, while the vorticity equations are given on a solenoid manifold of the configuration space.

4. In the hypothesis of independence, the vortex flux equations do not depend on the impulse, and the vortex enters the equation for the impulse flux as an external flux.

Several facts following from the separation theorem are summarized in the next lemma.

Lemma. *On a Hilbert field*

$$Z = \{q,\ p \,|\, Z \subset \pi;\ \Psi\}.$$

(1) *The bound impulse is*

$$p = \operatorname{grad} \Psi; \quad \Psi \subset C^{\infty}(\Lambda_q^T).$$

(2) *The potential laminar flow describes the equation of the acceleration potential*

$$\frac{\partial \Psi}{\partial t} = \Theta,$$

$$\Theta = -\frac{1}{2}\| u \|_{R^3}^2 - \prod\nolimits_p, \quad \prod\nolimits_p = \prod - \Phi - \Phi_f,$$

$$\Delta\Theta = 0; \quad q \in \Lambda_q; \quad \Theta \in C^{\infty}(\Lambda_q^T).$$

(3) *The potential of the independence equations*

$$[p \times \Omega] = \operatorname{grad} \Phi; \quad \Phi \in C^{\infty}(M_q \times (0,\ T))$$

is orthogonal to the impulse potential

$$(\operatorname{grad} \Psi \,|\, \operatorname{grad} \Phi)_{R^3} = 0.$$

Let us now look at the symmetry of the dual representation of the impulse (Eq. 11). If a solenoid manifold contains a free impulse, Hamiltonian $H = H(q,\ p,\ t)$; $p \in \Lambda_p$, and the sidelong metric, then another subset can be constructed - a characteristic surface:

$$\Sigma = \left\{q,\ p \,|\, \Sigma \subset M;\ p \in \Lambda_p;\ H;\ \xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

The phase flux on the characteristic surface is determined by the following theorem.

Theorem. *Upon the impulse release*

$$\{p = p(q,\ t);\ q \in \Lambda_q\} \to \{p \in \Lambda_p\}$$

the acceleration potential transforms into Hamiltonian

$$\{\Theta,\ q \in \Lambda_q\} \to \{H,\ \{q,\ p\} \in \Lambda\},$$

and on the characteristic surface

$$\Sigma = \left\{q,\ p \,|\, \Sigma \subset M;\ p \in \Lambda_p;\ H;\ \xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

the potential laminar flow is described by Hamilton canonical equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p};\quad \frac{dp}{dt} = -\frac{\partial H}{\partial q};\quad \{q,\ p\} \in \Sigma,$$

$$H = \frac{1}{2}\| u \|^2_{R^3} + \Pi_p;\quad u = p + \omega;\quad H \in C^\infty(\Lambda_q^T),$$

$$\Pi_p = \Pi_p(q,\ t) : \Pi_p = \Pi - \Phi - \Phi_f;\quad \Pi_p \in C^\infty(\Lambda_q^T). \tag{26}$$

Proof. For a free impulse

$$\frac{\partial p}{\partial t} = \frac{dp}{dt}$$

and accounting for the property

$$\frac{\partial q}{\partial t} = \frac{dq}{dt},$$

vector A on the characteristic surface will be

$$A = \left[\frac{dq}{dt},\ \frac{dp}{dt}\right].$$

On a manifold with the sidelong metric, this vector nullifies the divergence invariant

$$\operatorname{div} A = \sigma : \sigma = 0$$

by the gradient condition

$$A = \left[\frac{dq}{dt},\ \frac{dp}{dt}\right];\quad A = \operatorname{grad} H;\quad H = \left\| \begin{matrix} \dfrac{\partial H}{\partial p} \\ -\dfrac{\partial H}{\partial q} \end{matrix} \right\|;\quad \{q,\ p\} \in Z. \tag{27}$$

Equation (27) is another form of Hamiltonian canonical equations.

Theorem is proved.

There is a connection between the acceleration potential and Hamiltonian

$$H = -\Theta|_{p\in\Omega_p}. \tag{28}$$

However, one cannot always identify Hamiltonian with the acceleration potential taken with an opposite sign. These concepts are the base elements of two different methods for describing laminar flows, and Eq. (28) is the equation relating these methods.

Concluding Remarks

The most important result of the general-entropy analysis is the separation theorem that contains a reduction of Navier-Stokes to the two sets of equations: the canonical system of the acceleration potential and the system of the vortex flux. These equations create a foundation for new, effective mathematical methods for analytical hydrodynamics, including problems of the existence, smoothness, and structure of solutions of Navier-Stokes equations. This work also contains an important methodological fact: laminar flows have vortical and potential components.

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