Advances and Applications in Fluid Mechanics

Published Online: December 2015
http://dx.doi.org/10.17654/FM019010101
Volume 19, Number 1, 2016, Pages 101-139 **ISSN: 0973-4686**

AN ASYMPTOTIC ANALYSIS FOR GENERATION OF UNSTEADY SURFACE WAVES ON DEEP WATER BY TURBULENCE

S. G. Sajjadi

Department of Mathematics
Embry-Riddle Aeronautical University
FL, U. S. A.

Trinity College
Cambridge, U. K.

Abstract

The detailed mathematical study of the recent paper by Sajjadi et al. [21] is presented. The mathematical development considered by them, for unsteady growing monochromatic waves is also extended to Stokes waves. The present contribution also demonstrates agreement with the pioneering work of Belcher and Hunt [1] which is valid in the limit of the complex part of the wave phase speed $c_i \downarrow 0.$ It is further shown that the energy-transfer parameter and the surface shear stress for a Stokes wave revert to a monochromatic wave when the second harmonic is excluded. Furthermore, the present theory can be used to estimate the amount of energy transferred to each component of nonlinear surface waves on deep water from a turbulent shear flow blowing over it. Finally, it is demonstrated that in the presence of turbulent eddy viscosity, the Miles [13] critical-layer does not play an

Received: July 10, 2015; Accepted: July 28, 2015

2010 Mathematics Subject Classification: 76F55, 76B15, 74J15.

Keywords and phrases: wave generation, Air-Sea interactions, surface water waves, turbulence model.

Communicated by K. K. Azad

important role. Thus, it is concluded that in the limit of zero growth rate, the effect of the wave growth arises from the elevated critical-layer by finite turbulent diffusivity, so that the perturbed flow and the drag force are determined by the asymmetric and sheltering flow in the surface shear layer and its matched interaction with the upper region.

1. Introduction

It is well known that a surface wave traveling along a water surface can force a couple motion in the air and water, both propagating at the same speed, namely the eigenvalue c_r, being the real part of the wave speed c. Hence, the surface wave could force an unstable shear mode in the air, which then grows and induces growth of the water wave. In a pioneering work, Miles [13] constructed a model for generation of waves by shear flows by assuming that the critical height is sufficiently high that the turbulent stresses could be neglected. Given this assumption, he argued that the airflow perturbations are described by Rayleigh equation

$$(U - c)(\phi'' - k^2\phi) - U''\phi = 0 \tag{1.1}$$

for the non-dimensional perturbation stream function ϕ. In (1.1), $U(z)$ is the undisturbed velocity profile for the wind, blowing over the waves, and k is the wave number. Clearly, unless the wave amplitude varies with time, i.e., $c_i \neq 0$, equation (1.1) is singular at critical point z_c. By solving (1.1) above and below the air-water interface and by matching the vertical velocity and pressure at z_c, Miles [13] calculated c_i in the limit as $c_i/U_* \downarrow 0$ from the resulting eigenvalue relationship.

Miles [13] showed

$$c^2 = c_w^2 + s(\alpha + i\beta)U_1^2, \tag{1.2}$$

where $c_w = \sqrt{g/k}$ is the free surface wave speed, $s = \rho_a/\rho_w \ll 1$ and $U_1 = U_*/\kappa$, with U_* representing the wind friction velocity, ρ_a and ρ_w are air and water densities, respectively, and κ is the von Kármán's constant. He then deduced the growth of a monochromatic wave is given by the following expression:

$$\zeta_a = 2\mathscr{I}\{c\}/\mathscr{R}\{c\} = s\beta(U_1/c). \tag{1.3}$$

The very important aspect of (1.2) and (1.3) is that, for a steady wave (in which the wave amplitude *a* remains constant), *c* must have a non-zero imaginary part, i.e., such a wave will only grow if $c_i \neq 0$, This is quite evident from equations (1.2) and (1.3).

In his paper, Miles derived an integral expression for, what is commonly known as, the 'energy-transfer parameter' β,

$$\beta = -\mathscr{I}\left\{\int_{\eta_c}^{\infty} |\phi|^2 (w''/w)\, d\eta\right\}, \tag{1.4}$$

where *w* is the dimensionless wind velocity profile and the suffix *c* indicating evaluation at the critical point $\eta = \eta_c$, where the wind velocity equals the wave speed. However, in evaluation the integral in (1.4) and hence arriving at his 'well known' inviscid expression for β at $\eta = \eta_c$, for $w_c = 0$ and $c_i \neq 0$,

$$\beta = -\pi |\phi_c|^2 (w_c''/w_c'), \tag{1.5}$$

he states "The path of integration in (1.5) must be indented either over or under the singularity at $\eta = \eta_c$, where $w(\eta_c) = 0$, and on this choice depends on the sign of β…". At the bottom of the same paragraph, he concludes that "... the path must be indented under the singularity*". Note the asterisks attached to the word 'singularity' refer to a crucial footnote which holds the key to Miles' [13] critical-layer theory. In this footnote, he states "*This assumes $c = c_w$ is real. In the next approximation $\mathscr{I}\{c\} > 0$, so that the singularity lies slightly above the real axis $(\text{assuming } w_c''/w_c < 0)$, and the path of integration in (1.4) passes under the singularity without the necessity of indentation".

This important footnote is generally overlooked by many who refer to Miles' [13] critical-layer mechanism. This footnote is very significant in that (a) has a physical consequence which contradicts the results in (1.2) and (1.3) which show clearly if $c_i = 0,$ then $\beta = 0$ and hence waves cannot grow; and (b) has a mathematical consequence which indicates that equation (1.5) is valid if $c_i \neq 0$.

1.1. Physical mechanisms

We now present two alternative physical arguments that not only their results agree with each other but proves rigorously that Miles [13] critical-layer mechanism is valid only for slowly-growing waves which do not apply to growth of surface waves by strong or turbulent shear flows in open ocean.

Belcher and Hunt [1] (referred to as BH therein) considered a fully developed boundary layer over a two-dimensional monochromatic wave of small steepness *ak* propagating with small wave speed *c* and calculated the perturbations in the asymptotic limit $\mathscr{U} \equiv (U_* + c)/U \downarrow 0.$ In this limit, the critical height z_c lies within the inner surface layer, where the perturbation Reynolds shear stress varies slowly. Then, by considering the equation for the shear stress, they constructed solutions across the critical-layer and demonstrated that the shear stress perturbation plays an important role at the critical height which in turn implies Miles' [13] inviscid theory is *not* the dominating mechanism for the wave growth in this parameter range. Note that the perturbations above the inner surface layer are not directly influenced by the critical height and the region below $z_c,$ where the flow reversal occurs (see Figure 1). In fact, this is very similar to the perturbations due to a static undulation, but with the difference that the effective roughness length, that determines the shape of the unperturbed velocity profile, is modified according to

$$z_c = z_0 \exp(\kappa c/U_*).$$

BH then used the solutions for the perturbations to the boundary layer and calculated the wave growth, which is determined, in the leading order of perturbation, by the asymmetric pressure perturbation induced by the thickening of the perturbed boundary layer on the leeside of the wave crest. To the first order in $\mathscr{U}$, BH discovered that there are new effects that contribute significantly to the rate of growth: (a) the asymmetries in both the normal and shear Reynolds shear stresses associated with the leeside thickening of the boundary layer, they termed the *non-separated sheltering* (cf. Jeffreys [8]); (b) asymmetrical perturbations which are induced by the varying surface velocity associated with the fluid motion in the wave; and (c) asymmetries induced by the variation in the surface roughness along the wave. The theoretical value, predicted by their theory, for the shear stress perturbation at the crest of the wave on the wave surface as well as on the top of the inner region is in good agreement with laboratory measurements. Hence, despite the restriction that $\mathscr{U} \ll 1$, their theory describes a large portion of the experimental observations of the wave growth rate made at sea and in the laboratory.

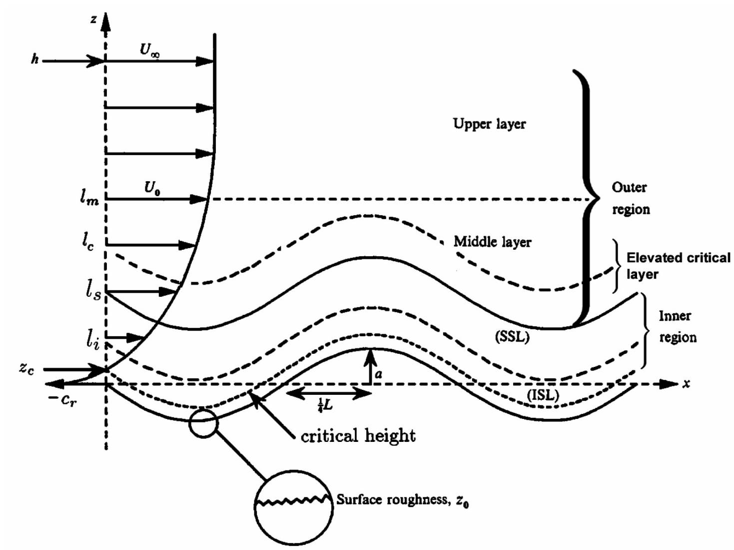

Figure 1. Schematic diagram for flow geometry and asymptotic multi-layer structure for analyzing turbulent shear flow over steady and unsteady monochromatic waves. Taken from Sajjadi et al. [21].

In a subsequent study, Belcher et al. [2] (hereafter will be referred to as

BHC) considered turbulent flow over growing waves, using triple-deck boundary layer theory originally developed by Lighthill [10] and Stewartson [22], to analyze the sheltering mechanism described above. They suggested that Miles' [13] critical-layer theory generates growth of waves but this was not demonstrated in data. In this model, they assumed the atmospheric mean flow is neutrally stable and is logarithmic and the surface wave, moving in the positive x-direction, is monochromatic whose profile is given by

$$z' = Re\{ae^{ik(x'-ct)}\},$$

where $c = c_r + ic_i$ is a complex wave speed such that c_r is the phase speed and the wave amplitude, a, grows exponentially at the rate kc_i. They considered a frame of reference moving with the wave at a speed c_r so that the mean velocity profile can be expressed as

$$U(z) = U_1 \log(z/z_0) - c_r,$$

where $U_1 = U_*/\kappa$, and that this wave speed vanishes at the critical height z_c. In this frame of reference, the surface wave is described by

$$z = Re\{ae^{ik(x-ic_i t)}\}.$$

The boundary conditions imposed at the wave surface are that the wind velocity is equal to the surface velocity of the water flow, being approximated by the surface value of the orbital velocity of an irrotational wave on deep water. The other boundary condition is that perturbations in the basic flow vanish as $kz \uparrow \infty$.

BHC modelled the turbulent shear stress in the inner region, adjacent to the wave surface, using a mixing-length model, and in the region above this, the outer region, they invoked rapid-distortion theory to describe the turbulence. They showed that the depth of the inner region, ℓ_i, may be obtained from the following implicit relation:

$$k\ell_i = \frac{2\kappa^2}{|\ln(\ell_i/z_0) - \kappa c_r/U_*|},$$

where the variation of solution to this equation for ℓ_i as a function of c_r/U_* for $kz_0 = 10^{-4}$ is shown in Figure 2.

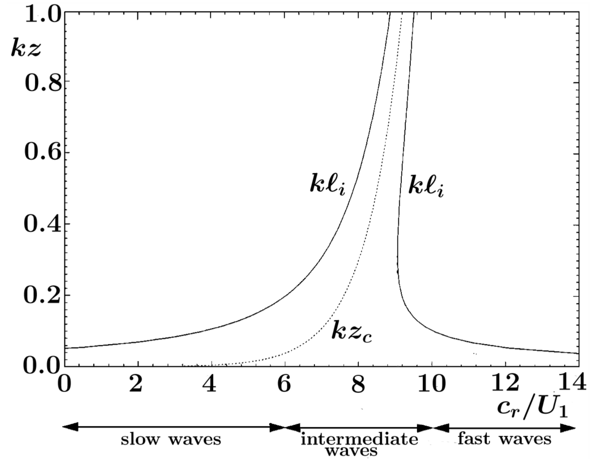

Figure 2. Variation with c_r/U_1 of solutions for the normalized inner region height, $k\ell_i$, and critical height, kz_c, when $kz_0 = 10^{-4}$. For given c_r/U_1, an inner, local equilibrium region lies between $kz = 0$ and the smallest value of $k\ell_i$, and an outer, rapid-distortion, region lies above. Solid lines: $k\ell_i$; dotted lines: kz_c.

They further assumed the perturbations to the airflow are governed by equation (1.7) where the turbulent stresses on the right-hand side are modelled by an eddy viscosity. The vertical component of the velocity perturbation $\mathscr{U} \equiv \mathscr{U}_i = (\mathscr{U}, 0, \mathscr{W})$ is expanded in the normal form

$$\mathscr{W}(x, z, t) = \hat{\mathscr{W}}(k, z)e^{i(kx - ic_i t)},$$

where the amplitude of the perturbation, $\hat{\mathscr{W}}$, satisfies the inhomogeneous Rayleigh equation

$$\frac{\partial^2 \hat{\mathscr{W}}}{\partial z^2} - \left(k^2 + \frac{U''}{U - ic_i}\right)\hat{\mathscr{W}} = \frac{i}{U - ic_i}\frac{\partial^2}{\partial z^2}\left(\nu_e \frac{\partial^2 \hat{\mathscr{W}}}{\partial z^2}\right), \tag{1.6}$$

where ν_e is an eddy viscosity.

BHC showed that in the middle layer, $\ell_i \ll z \ll \ell_m$, the advection term is negligible compared with the curvature term and thus (1.6) reduces to

$$\frac{\partial^2 \hat{\mathscr{W}}}{\partial z^2} - \frac{U''}{U - ic_i}\hat{\mathscr{W}} \sim 0 \tag{1.7}$$

whose leading order solution may be expressed as

$$\hat{\mathscr{W}} \sim \{U(z) - ic_i\}\left\{A + B\int \frac{d\zeta}{[U(\zeta) - ic_i]^2}\right\}. \tag{1.8}$$

In (1.8), A and B are constants which may be determined by matching the inner and upper layer solutions. Now, for slow waves, the critical-layer lies close to the surface in the inner region. Hence, the solution (1.8) is regular, since $U(z) > 0$ and does not vanish there. Therefore, the integrand (1.8) is regular throughout the middle layer. Note, the same argument applies to the moderate waves, but there are differences in its application.

Suppose now for a range of intermediate waves, the critical-layer lies in the outer region. Thus, we can expect, for a particular range of c_r/U_*, the critical-layer lies within $\ell_i \ll z \ll \ell_m$. Neglecting the Reynolds stresses in the vicinity of the critical-layer, $\hat{\mathscr{W}}$ satisfies (1.7) with the solution given by (1.8).

However, in this scenario, the critical-layer lies within the range of the integral in (1.8) and if $c_i = 0$, then there is a singularity in this integrand at the critical-layer, where $U(z_c) = 0$. This singularity may be resolved by inertial effects, i.e., inviscid processes that control its dynamics.

If we now suppose the wave grows so that $c_i > 0$, then the integral (1.8) is regular at $z = z_c$. If we further assume the wave grows slowly such that $c_i \ll U_c'^2/U_c''$, where the suffix c indicates evaluation at $z = z_c$, then the integral in (1.8) can be evaluated approximately. To do this, $U(z)$ is Taylor expanded in the vicinity of the critical-layer, i.e., $U(z) \sim \zeta U'_c + \frac{1}{2}\zeta^2 U''_c$, where $\zeta = z - z_c$ and thus the integral becomes

$$I = \int^{z-z_c} \frac{d\zeta}{[U(\zeta) - ic_i]^2} \sim \int \frac{d\zeta}{\left\{\zeta U'_c + \frac{1}{2}\zeta^2 U''_c - ic_i\right\}^2}$$

$$= \frac{1}{c_i U'_c} \int \frac{d\xi}{(\xi U'_c - i)^2} \left\{1 + \frac{\varepsilon}{2}\frac{\xi^2}{\xi - i}\right\}^{-2}, \tag{1.9}$$

where $\xi = \zeta U'_c / c_i$ and $\varepsilon = c_i U''_c / U'^2_c \ll 1$. Note that, if $U(z)$ has a logarithmic profile, then $\varepsilon = c_i / U_* \ll 1$ which confirms that these are slowly-growing waves. In the limit of slow-growing waves, the factor in braces can be expanded for $\varepsilon \ll 1$ to give

$$I \sim \frac{1}{c_i U'_c} \int \frac{d\xi}{(\xi U'_c - i)^2} \left\{1 - \varepsilon \frac{\xi^2}{\xi - i} + O(\varepsilon^2)\right\}$$

$$= \frac{1}{c_i U'_c} \left[\frac{1}{\xi - i} + \varepsilon \left\{\frac{1}{2(\xi - i)^2} - \frac{2i}{\xi - i} + \ln(\xi - i)\right\} + O(\varepsilon^2)\right]. \tag{1.10}$$

From (1.10), we can deduce that far from the critical height I is dominated by the logarithmic term and hence (1.10) reduces to

$$I \sim \frac{\varepsilon}{c_i U'_c} \ln(\xi - i) \text{ as } \xi \to \pm\infty$$

$$= \frac{\varepsilon}{2 c_i U'_c} \ln(\xi^2 + 1) + i\theta, \tag{1.11}$$

where θ is given by

$$\tan\theta = -\xi^{-1} = -c_i / U'_c (z - z_c). \tag{1.12}$$

For a logarithmic velocity profile $\tan\theta = \varepsilon z_c / (z - z_c)$ and hence θ varies between

$$\theta \to 0 \text{ as } (z - z_c)/\ell_c \to \infty \text{ and } \theta \to \pi \text{ as } (z - z_c)/\ell_c \to \infty. \qquad (1.13)$$

The imaginary part of the integral for $z \gg z_c$ is then

$$\mathscr{I}\{I\} \sim \varepsilon / c_i \, U_c' \theta = U_c'' / U_c'^3 \mathsf{H}(z - z_c) \text{ as } \xi \to \pm\infty, \qquad (1.14)$$

where $\mathsf{H}(z - z_c)$ is the Heaviside step function. The result given by (1.14) is remarkable since it is independent of c_i which means even for a slowly-growing wave, it leads to an out of phase component of the motion that is independent of the growth rate.

The significance of the term $iU_c''/U_c'^3$ in the solution for I is that it yields an out of phase contribution to the vertical velocity that ultimately leads to the same contribution to the wave growth from the critical-layer as found by Miles [13]. This result shows the solution found by Miles [13] is valid only when the waves grow sufficiently slowly such that

$$c_i \ll U_c' z_c \sim U_* \qquad (1.15)$$

and hence the effects of the critical-layer calculated by Miles [13] are valid *only* in the limit $c_i / U_* \downarrow 0$.

BHC further argued that in the vicinity of the critical height, the turbulent shear stress perturbation, $\Delta\tau$, can be modelled by an eddy viscosity model via

$$\Delta\tau = \nu_e \frac{\partial \Delta u}{\partial z}, \qquad (1.16)$$

where $\nu_e = \alpha U_* z$ is an eddy viscosity and $\alpha = 2\kappa$ is a parameter which reduces (1.16) to the mixing-length model used in the inner region. Thus, there is a layer around the critical height whose thickness is ℓ_r in which the stresses cannot be neglected. By balancing the shear stress term in (1.6) with the gradient term, $\partial^2 \hat{\mathscr{W}} / \partial z^2$, they then estimated

$$\ell_r \sim z_c (\alpha / k z_c)^{\frac{1}{2}}.$$

Hence, they found that the thickness of the shear-stress-layer surrounding the critical height is much smaller than the critical-layer thickness, i.e., $\ell_r \ll \ell_c$, provided

$$(\alpha/kz_c)^{\frac{1}{2}} \ll c_i/U_*.$$

The main conclusions arrived by BHC, which has motivated the present investigation, are summarized below.

(a) From various studies and experimental data, it is well known that c_i varies with governing parameters according to

$$c_i/U_* = s(U_*/c_r)\beta,$$

where $s \ll 1$ is the ratio of the air density, ρ_a, to that of water density, ρ_w, and β is the energy-transfer parameter for the wind wave interaction. Using this expression, the ratio of the thickness of an inertial critical-layer, ℓ_c to the thickness of a stress-dominated critical-layer, ℓ_r, varies according to

$$\ell_c/\ell_r \sim (kz_c/\alpha)^{\frac{1}{3}} s(U_*/c_r)\beta.$$

For growing waves in the ocean $s \sim 1/800$ and $\beta \approx O(25-30)$ then for intermediate waves when $kz_c \sim 1$ and $c_r/U_* \approx O(15-25)$, we see that $\ell_c/\ell_r \approx 1/(500\alpha^{\frac{1}{3}})$. We remark that for the mixing-length model $\alpha = 0.8$ and thus ℓ_c/ℓ_r is likely to be small for a fully developed turbulent flow. Hence, one is led to conclusion that in the intermediate regime, the critical-layer is dominated by effects of Reynolds stress and, contrary to Miles' [13] conclusion, there will not be any contribution from the critical-layer to the wave growth. We further remark that for those ocean waves that are rapidly-growing in time such that $c_i/U_* \sim 1$, the inertial effects are dominant in the critical-layer and thus the critical-layer mechanism (c.f. Miles [13]). This suggests that such 'rapidly-growing' waves might occur as a wave crest moves through a wave group. This is currently being investigated by the present authors.

(b) For slow waves, $c_r/U_* \le 15,$ the critical height z_c lies within the lower part of the inner region and thus the reverse-flow region, situated below z_c and the inner layer itself plays no significant dynamical role. In such circumstances, asymmetry in the flow is generated by the frictional effect of the shear stress through the inner region, resulting to lower wind speeds on the downwind side of the wave and consequently leads to a sheltering in the lee of the wave crest. This asymmetry results to an out-of-phase pressure perturbation which subsequently yields wave growth. We emphasize that in this case the airflow perturbations are similar to those over a stationary undulation, but in the range $1 \le c_r/U_* \le 15,$ the flow is similar to that over a rough surface except now the surface roughness is now effectively $z_c,$ which increases the value of β. It is to be noted that small corrections to the velocity of $O(akc_r k\ell_i)$ due to the orbital motions at the wave surface reduce β.

For fast waves, on the other hand, the critical-layer is far above the surface, $kz_c > 1,$ and again there is no significant dynamical role for the wave growth. In this scenario, the air above the wave flows largely against the wave which induces a 'negative' asymmetry from sheltering. Furthermore, orbital motions at the water surface generate additional air flow perturbations that contribute comparable 'negative' asymmetries. This 'negative' asymmetry causes an out-of-phase pressure perturbation which makes waves to decay.

Finally, between the two regions discussed above, there is also an intermediate region in which $15 \le c_r/U_* \le 30$ and $z_c \sim \ell.$ In this region, numerical simulations show that as c_r/U_* increases from slow to the fast region, the reverse-flow below the critical height becomes stronger and produces a 'negative' asymmetric displacement of streamlines upwind of the crest. However, above the critical height, the asymmetric displacement is 'positive' downwind of the crest, similar to that for slow waves. Moreover, the critical-layer mechanism also displaces streamlines downwind of the crest. Therefore, as c_r/U_* increases across the intermediate region, the asymmetric component of the flow peaks to its maximum and then decreases to zero, with the wave growth following the same trends as that of the asymmetric component of the flow.

2. Shear Stress Model for Unsteady Wave Growth

In this section, we consider the perturbation Reynolds stresses in the flow of a turbulent wind over the surface wave

$$z = a\cos[k(x - ct)] \equiv h(x, t), \quad (ak \ll 1) \quad c = c_r + ic_i \tag{2.1}$$

through an interpolation between inner, mixing-length approximation and an outer, rapid-distortion approximation. We will show that the wind-to-wave energy-transfer predicted by this model is substantially larger than that predicted by either Miles' [13] quasi-laminar model (in which the perturbation Reynolds stresses are neglected) or Townsend's viscoelastic model (Townsend [23]) and is in very good agreement with the model proposed by Belcher and Hunt [1]. Here we will point out, Townsend's inner approximation differs from the conventional mixing-length approximation and yields a ratio of perturbation shear stress to perturbation shear that is negative for his choice of parameters.

The equations of motion for a viscous incompressible fluid may be cast in Cartesian tensor form as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j}, \tag{2.2}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.3}$$

where x_i denotes the Cartesian coordinate, u_i is a velocity component, p_{ij} is a component of a stress tensor and ρ is the fluid density. Decomposing the variables according to

$$u_i = U_i + u_i' + u_i'', \quad p_{ij} = P_{ij} + p_{ij}' + p_{ij}'', \tag{2.4}$$

where $U_i + u_i'$ and $P_{ij} + p_{ij}'$ represent a solution to (2.2), being functions of coordinates x_1 and x_3 and having mean values with respect to x_2. Note that U_i and P_{ij} represent the mean components and u_i' and p_{ij}' represent the turbulent fluctuations. In (2.4), u_i'' and p_{ij}'' represent a small perturbation with respect to the solution of (2.2)-(2.3).

Substituting (2.4) in (2.2), neglecting second-order terms in the perturbation flow, and noting the fact that the unperturbed flow satisfies (2.2), we obtain

$$\frac{\partial u_i''}{\partial t} + (U_j + u_j')\frac{\partial u_i''}{\partial x_j} + u_j''\frac{\partial(U_i + u_i')}{\partial x_j} = \frac{1}{\rho}\frac{\partial p_{ij}''}{\partial x_j}, \tag{2.5}$$

$$\frac{\partial u_i''}{\partial x_i} = 0. \tag{2.6}$$

Taking the mean values with respect to x_3, the results can be expressed in the following forms:

$$\frac{\partial \overline{u_i''}}{\partial t} + U_j\frac{\partial \overline{u_i''}}{\partial x_j} + \overline{u_j''}\frac{\partial U_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}(\overline{p_{ij}''} - \overline{r_{ij}''}), \tag{2.7}$$

$$\frac{\partial u_i''}{\partial x_i} = 0. \tag{2.8}$$

Invoking the equations of continuity for both u_i' and u_i'', the perturbation Reynolds stress may be written as

$$\overline{r_{ij}''} = \rho(\overline{u_i'u_j''} + \overline{u'_j u_i''}) \tag{2.9}$$

$$= \rho[\overline{u_i'(u_j'' - \overline{u_j''})} + \overline{u'_j(u_i'' - \overline{u_i''})}] \tag{2.10}$$

with (2.10) follows from (2.9) by virtue of $\overline{u_i'} = 0$.

If we now set $x_1 = x$, $x_3 = z$, $U_1 = U(z)$, $U_2 = U_3 = 0$, $\overline{u_1''} = u$, $\overline{u_3''} = w$, $\overline{p_{ij}''} = -\delta_{ij}\wp$ and $\overline{r_{ij}''} = \rho\overline{u_i'u_j'}$, after taking the time average, we obtain the linearized, Reynolds-averaged equations governing u and w, the x (horizontal) and z (vertical) components of the mean perturbation velocity, and the kinematic perturbation pressure p as

$$u_x + w_z = 0, \tag{2.11}$$

$$(U - c)u_x + U'w = -\wp_x + \sigma_x + \tau_z, \tag{2.12}$$

$$(U - c)w_x = \wp_z + \tau_x, \tag{2.13}$$

where the subscript x and z signify partial differentiation, $U' \equiv dU/dz$,

$$\wp \equiv \mathsf{p} + \overline{w'^2} - (\overline{w'^2})_0, \quad \sigma \equiv -(\overline{u'^2} - \overline{w'^2}) - \sigma_0, \quad \tau \equiv -\overline{u'w'} - \tau_0, \tag{2.14}$$

$(\overline{w'^2})_0$, σ_0 and τ_0 are the unperturbed values of $\overline{w'^2}$, $-(\overline{u'^2} - \overline{w'^2})$ and $-\overline{u'w'}$, and σ is Townsend's τ_n.

In this paper, we consider a turbulent shear flow blowing over the surface wave (2.1) whose mean velocity profile is given by

$$U(z) = (\tau_0^{1/2}/\kappa)\log(z/z_0), \tag{2.15}$$

where τ_0 is the kinematic shear stress in the basic flow, κ is von Kármán's constant, z is the elevation and c is the complex phase speed of the surface wave (2.1).

2.1. Energy-transport equation

As a first step toward a Reynolds stress closure, Townsend points out the transport equation for the turbulent kinetic energy $\frac{1}{2}\overline{\mathsf{q}^2}$ may be expressed in the form

$$(U - c)\partial_x\left(\frac{1}{2}\overline{\mathsf{q}^2}\right) = D + G - \varepsilon', \tag{2.16}$$

where

$$D = \rho\kappa\tau_0^{1/2}\partial_z\left[z\partial_z\left(\frac{1}{2}\overline{\mathsf{q}^2}\right)\right] \tag{2.17}$$

represents diffusion[1]

$$G = -\overline{u'^2}u_x - \overline{w'^2}w_z - \overline{u'w'}(U' + u_z + w_x) - \tau U' \tag{2.18}$$

$$= \sigma_0 u_x + \tau_0(u_z + w_x) + U'\tau \tag{2.19}$$

represents generation (Launder et al. [9]), (2.19) follows (2.18) through (2.11), (2.14) and linearization, and ε' represents dissipation (see below). Townsend neglects w_x in G, although this appears to be inconsistent with his subsequent rapid-distortion approximation (see below).

Townsend's approximation to the dissipation rate ε' involves h, the surface displacement (2.1), but this may be eliminated through a transformation to wave-following coordinates, in which (Miles [14, equation (3.4)])

$$\varepsilon' = \frac{3}{2}\tau_0 U'(e/e_0), \quad e \equiv \overline{\mathsf{q}^2} - e_0, \tag{2.20}$$

and $e_0 \equiv \overline{\mathsf{q}^2}$ in the basic flow. Substituting (2.19) and (2.20) into (2.16), neglecting diffusion (see Miles [14, Section 3]), and multiplying the result by $2a_1$, we obtain

$$(\mathscr{D} + \lambda)a_1 e - 2\lambda(\tau - a_1 e) = 2\tau_0[a_1(u_z + w_x) + a_n u_x] \equiv \mathscr{A}_i, \tag{2.21}$$

where

$$\mathscr{D} \equiv (U - c)\partial_x, \quad \lambda \equiv a_1 U', \quad a_1 \equiv \tau_0/e_0, \quad a_n = \sigma_0/e_0. \tag{2.22}$$

The relaxation rate λ is a reciprocal measure of eddy life and dominates (is dominated by) $\mathscr{D}$ in the inner (outer) domain.

---

[1]Townsend chooses $\rho = 0.3$ but states that 'the value ... is not critical'.

2.2. Rapid-distortion approximation

To determine the outer departure of τ/e and σ/e from their equilibrium values a_1 and a_n, Townsend posits the rapid-distortion approximations

$$\mathscr{D}\left\{\frac{\tau_0 + \tau}{e_0 + e}\right\} \simeq \mathscr{D}\left\{\frac{\tau - a_1 e}{e_0}\right\} \sim A_1 u_z + A_2 w_x + A_3 u_x \tag{2.23}$$

and

$$\mathscr{D}\left\{\frac{\sigma_0 + \sigma}{e_0 + e}\right\} \simeq \mathscr{D}\left\{\frac{\sigma - a_n e}{e_0}\right\} \sim B_1 u_z + B_2 w_x + B_3 u_x, \tag{2.24}$$

where $A_{1,2,3}$ and $B_{1,2,3}$ are 'the incremental rates of change for suddenly imposed additional distortions'. He then 'interpolates' between (2.23)-(2.24) and the inner domain, in which (by hypothesis) $\tau \to a_1 e$ and $\sigma \to a_n e$, by replacing $\mathscr{D}$ by $\mathscr{D} + \lambda$:

$$(\mathscr{D} + \lambda)(\tau - a_1 e) = e_0(A_1 u_z + A_2 w_x + A_3 u_x) \equiv \mathscr{A}_0 \tag{2.25}$$

and

$$(\mathscr{D} + \lambda)(\sigma - a_n e) = e_0(B_1 u_z + B_2 w_x + B_3 u_x) \equiv \mathscr{B}_0. \tag{2.26}$$

However, the elimination of $a_1 e$ between (2.21) and (2.25) in the inner domain (in which $|\mathscr{D}| \ll |\lambda|$) yields

$$\tau \to \lambda^{-1}(\mathscr{A}_i + 3\mathscr{A}_0), \tag{2.27}$$

which differs from the mixing-length approximation obtained by invoking $|\mathscr{D}| \ll \lambda$ in (2.21):

$$\tau \to a_1 e \to \lambda^{-1}\mathscr{A}_i = 2\nu_0(u_z + w_x + \mathscr{H} u_x), \tag{2.28}$$

where

$$\mathscr{H} \equiv a_n/a_1, \quad \nu_0 \equiv \tau_0/U' = \kappa\tau_0^{1/2} z. \tag{2.29}$$

Indeed, if (as typically assumed in the mixing-length approximation) $|u_x, w_x| \ll |u_z|$, (2.27) implies

$$\frac{\tau}{2\nu_0 u_z} \to 1 + \frac{3A_1}{2a_1^2}, \tag{2.30}$$

which reduces to the conventional mixing-length approximation for $A_1 = 0$ but is negative for Townsend's values of a_1 and A_1, $\frac{1}{6}$ and -0.03.

3. Solution of Boundary-value Problem

Miles-Sajjadi theory of wave generation by turbulent wind (Miles [14] and Sajjadi [16]) reduces to the solution of the Orr-Sommerfeld-like equation

$$(\nu_e \Phi'')'' = ik[(U - c)(\Phi'' - k^2\Phi) - U''\Phi] \equiv \mathscr{T}'' \tag{3.1}$$

subject to the boundary conditions

$$\Phi = ac, \quad \Phi' = a(kc - U') \quad \text{on} \quad \eta = 0, \tag{3.2}$$

$$\Phi, \quad \nu_e\Phi \to 0 \quad \text{as} \quad k\eta \to \infty, \tag{3.3}$$

where $\Phi = \Phi(\eta)$, $U = U(\eta)$ and $(\ )' \equiv d/d\eta$. In equation (3.1), ν_e is a complex eddy viscosity given by

$$\nu_e = 2U_*^2(U' + ikV)^{-1}, \quad V \equiv (U - c)/a_1, \tag{3.4}$$

where a_1 ($\simeq \kappa^2$; $\kappa = 0.4$ is von Kármán's constant) is Townsend's boundary layer constant, and a, c and k are the amplitude, speed and wave number of the surface wave. The velocity profile has the logarithmic asymptote

$$U \sim U_1 \ln(\eta/z_0), \quad (\eta \gg z_0), \tag{3.5}$$

where

$$U_1 = U_*/\kappa, \quad z_0 = \Omega U_1^2/g. \tag{3.6}$$

U_* is the wind friction velocity and Ω is Charnock's constant. We seek the impedance

$$\alpha + i\beta = (\mathscr{R}_0 + i\mathscr{T}_0)/(kaU_1^2), \tag{3.7}$$

where

$$\mathscr{R}_0 = kac^2 + (ik)^{-1}(\nu_e\Phi'')' \quad \text{and} \quad \mathscr{T}_0 = \nu_e\Phi'' \quad \text{on} \quad \eta = 0 \tag{3.8}$$

are complex amplitudes of the wind-induced perturbation pressure and shear stress action on the wave.

3.1. Reduction to second-order differential equations

It is convenient to reduce the fourth-order differential equation (3.1) to the pair of second-order equations

$$\mathscr{T}'' = (ik/\nu_e)(U - c)\mathscr{T} - ik[U'' + k^2(U - c)]\Phi, \tag{3.9}$$

$$\Phi'' = \mathscr{T}/\nu_e \tag{3.10}$$

for which the respective boundary conditions are

$$\mathscr{T} = \mathscr{T}_0, \quad \mathscr{T}' = ik(\mathscr{R}_0 - kac^2) \quad \text{on} \quad \eta = 0 \tag{3.11}$$

and (3.2). Note that, $\mathscr{T}_0$ and $\mathscr{R}_0$ are implicitly determined by the null conditions (3.3).

We render the formulation dimensionless by referring to η to k^{-1}, c, U and V to U_1, Φ to aU_1, $\mathscr{P}$ and $\mathscr{T}$ to kaU_1^2, thereby reducing (3.4), (3.9)-(3.11) and (3.2) to

$$\mathscr{T}'' = \frac{1}{2}iV(U' + iV)\mathscr{T} - i(U'' + U - c)\Phi, \tag{3.12}$$

$$\Phi'' = (2\kappa^2)^{-1}(U' + iV)\mathscr{T}, \tag{3.13}$$

$$\mathscr{T} = \mathscr{T}_0, \quad \mathscr{T}' = i(\mathscr{R}_0 - c^2), \quad \Phi = c, \quad \Phi' = c - U' \quad \text{on} \quad \eta = 0. \tag{3.14}$$

3.2. Inner expansion for surface layer

A constant-stress interpolation between the logarithmic profile (3.5) and a viscous sublayer of vanishing thickness is given by (Rotta [15])

$$\frac{U}{U_1} = \log(\zeta + \sqrt{\zeta^2 + 1}) - \frac{\zeta}{1 + \sqrt{\zeta^2 + 1}} \equiv \hat{U}(\zeta), \quad \zeta = \frac{1}{2} e \frac{\eta}{z_0} \tag{3.15}$$

in which

$$\frac{dU}{d\zeta} = \frac{U_1}{1 + \sqrt{1 + \zeta^2}}.$$

Note that $U'' = O(U_1/z_0^2)$ for $\zeta = O(1)$ and therefore, in contrast to the conventional Orr-Sommerfeld problem, is not negligible near the boundary, although it does vanish at $\zeta = 0$.

The inner and outer length scales are z_0 (or, more conveniently, $\hat{z}_0 \equiv 2z_0/e$) and k^{-1}, and the introduction of the inner variable ζ and the small parameter

$$\varepsilon \equiv k\hat{z}_0 = (2\Omega/e)c^{-2} \ll 1 \tag{3.16}$$

in (3.12-3.13) and (3.14) leads to the inner expansions

$$\Phi = c + \left(\varepsilon c - \frac{1}{2}\right)\zeta + (\varepsilon/2\kappa^2)\left[\mathscr{T}_0 \int_0^\zeta U d\zeta + i\mathscr{G}(\zeta)\right] + O(\varepsilon^2), \tag{3.17}$$

where

$$\mathscr{G}(\zeta) = \int_0^\zeta d\zeta_1 \int_0^{\zeta_1} U'(\zeta_2) d\zeta_2 \int_0^{\zeta_2} \left[\frac{1}{2}(c - U + \zeta U') - cU'\right] d\zeta_3, \tag{3.18}$$

and

$$\mathscr{T} = \mathscr{T}_0 + i\left[\frac{1}{2}(c + U)\zeta - cU - \int_0^\zeta U d\zeta\right] + O(\varepsilon). \tag{3.19}$$

Letting $\zeta \to \infty$ in (3.17) and (3.18) and invoking $\zeta = \eta/\varepsilon$, we obtain

$$\Phi \sim c + \left(c - \frac{1}{2}\varepsilon^{-1}\right)\eta + \frac{1}{2}\kappa^{-1}\left[\mathscr{T}_0\eta(U-1) + \frac{1}{4}i\varepsilon^{-1}\eta^2\left(c - U + \frac{7}{2}\right)\right]. \quad (3.20)$$

Hence, the problem is reduced in determining $\mathscr{T}$ and $\mathscr{T}_0$. The form given by (3.19) is not very convenient for this task. A better approach is to determine $\mathscr{T}$ in the inner via shear-stress-layer approximation.

In this region, the complex amplitude of the wind-induced perturbation shear stress may be expressed as (Sajjadi [16])

$$\mathscr{T} = \nu_e[\mathscr{U}(\Phi/\mathscr{U})'' - 2U'(\Phi/\mathscr{U})'], \quad \mathscr{U} \equiv U - c. \quad (3.21)$$

Rearranging this equation, we obtain

$$\frac{\mathscr{U}\mathscr{T}}{\nu_e} = \mathscr{U}\Phi'' - U''\Phi. \quad (3.22)$$

Under the assumption that $k\eta \ll 1$ in this region, we may neglect $k^2\mathscr{U}\Phi$ in (3.1) and upon combining the result with (3.22), we arrive at the shear-stress-layer approximation

$$\mathscr{T}'' - (ik\mathscr{U}/\nu_e)\mathscr{T} = 0; \quad (k\eta \ll 1) \quad (3.23)$$

whose asymptotic solution for $k\eta \downarrow 0$, which may be expressed in terms of modified Bessel function of the first kind (Sajjadi [20]), is

$$\mathscr{T}(\eta) = -2\{1 + 4\eta K_0'(\eta)\} + O(\varepsilon)$$

(cf. Appendix A of BH)[2], whence for growing waves we obtain

$$\tau = -2e^{kc_it}\mathscr{T}(\eta)e^{ik(x-c_rt)}.$$

The real part of the complex amplitude of shear stress as a function of non-dimensional height is shown in Figure 3. This figure clear that the shear

[2]The asymptotic solution given here reduces to that given by BH in the limit as $c_i \downarrow 0$.

stress becomes negative after descending from a maxima and then rising again. The latter part is not depicted in this figure.

However, for the purpose of calculating the energy-transfer parameter, from wind to wave β, we multiply (3.23) by $\mathscr{T}$ and integrate by parts over $0 < \eta < \infty$ to obtain

$$(\mathscr{T}\mathscr{T}')_0 = -\int_0^{\infty}[\mathscr{T}'^2 + (ik\mathscr{U}/\nu_e)\mathscr{T}^2]d\eta. \tag{3.24}$$

The integral (3.24) is stationary with respect to first-order variations in $\mathscr{T}$ about the true solution (3.23).

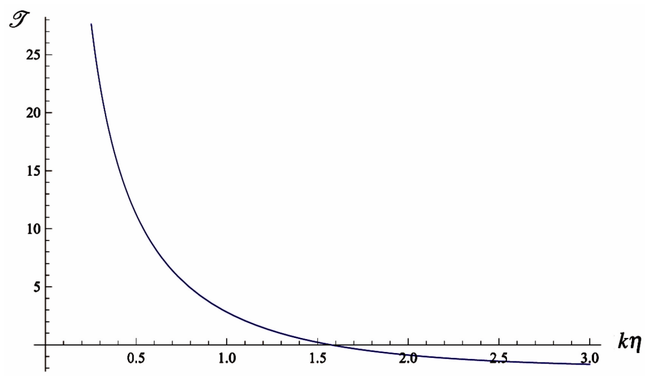

Figure 3. The real part of the complex amplitude of $\mathscr{T}$ as a function of non-dimensional height $k\eta$.

Substituting the trial function

$$\mathscr{T} = \mathscr{T}_0 e^{-k\eta/\delta}, \tag{3.25}$$

where δ is a (complex) free parameter to be determined, and ν_e is given by (3.4), with $a_1 = \kappa^2$, into (3.24) and invoking the condition $\partial(\mathscr{T}_0'/\mathscr{T}_0)/\partial\delta = 0$, we obtain

$$\frac{\mathscr{T}_0'}{\mathscr{T}_0} = -\frac{ik}{4\kappa^2}\left[(1+\delta_*)\left(L_\delta^2 + \frac{1}{6}\pi^2\right) + 2\delta_*^{-1} - \hat{c}^2\right], \tag{3.26}$$

$$\left(L_\delta^2 + 2L_\delta + \frac{1}{6}\pi^2\right)\delta_*^2 + 2L_\delta\delta_* - 2 = 0, \quad L_\delta \equiv L_0 + \ln\delta, \quad \delta_* \equiv i\kappa^{-2}\delta, \tag{3.27a-c}$$

where $L_0 \equiv \Lambda^{-1} = -\gamma - \ln 2\mathsf{k}$, $\mathsf{k} = \Omega\hat{c}^{-2}e^{\hat{c}}$, $\hat{c} = c/U_1$, $\gamma = 0.5772$ is the Euler's number and Ω is the Charnock's constant.

Solving (3.27a) as a quadratic in $\delta_* L_\delta$ and letting $\delta_* \to 0$, we get

$$\frac{i\delta L_\delta}{\kappa^2} = \sqrt{3} - 1 + O(\delta). \tag{3.28}$$

4. Outer Approximation Above Inertial Critical-layer

In contrast to the inner region, the solution in the outer region is very straightforward. In the outer domain $k\eta \gg 1$, U may be approximated by (3.5), and Φ admits the Green-Liouville approximation (Olver [11, chap. 6])

$$\Phi \sim \mathscr{F}(\mathscr{Z})\exp\left[\int_0^\eta \mathscr{W}(\mathscr{Z})d\eta\right], \quad \mathscr{Z}(\eta) \sim (1/\kappa^2)\ln(\eta/\eta_c). \tag{4.1}$$

Substituting (4.1) into the dimensionless counterpart of (3.1), we obtain

$$\mathscr{W}^4 + \frac{1}{2}\mathscr{Z}^2(\mathscr{W}^2 - 1) = 0 \tag{4.2}$$

and

$$\frac{\mathscr{W}(\mathscr{W}^2 + \mathscr{Z}^2)}{\mathscr{F}}\frac{d\mathscr{F}}{d\mathscr{Z}} + \left(6\mathscr{W}^2 + \frac{1}{2}\mathscr{Z}^2\right)\frac{d\mathscr{W}}{d\mathscr{Z}} + \frac{(i\kappa^2\mathscr{W} - 2)\mathscr{W}^3}{\mathscr{Z}} = 0 \tag{4.3}$$

from which it follows that

$$\mathscr{W}^2 = \frac{1}{4}[-\mathscr{Z}^2 \pm (\mathscr{Z}^4 + 8\mathscr{Z}^2)^{1/2}], \tag{4.4}$$

$$\frac{d\mathscr{W}}{d\mathscr{Z}} = \frac{-\mathscr{Z}(\mathscr{W}^2 - 1)}{\mathscr{W}(4\mathscr{W}^2 + \mathscr{Z}^2)} = \frac{2\mathscr{W}^3}{\mathscr{Z}(4\mathscr{W}^2 + \mathscr{Z}^2)} = \frac{\pm 2\mathscr{W}^3}{\mathscr{Z}(\mathscr{Z}^4 + 8\mathscr{Z}^2)^{1/2}} \tag{4.5}$$

and

$$\frac{d\ln\mathscr{F}}{d\mathscr{Z}} = -\frac{\mathscr{W}^2(4\mathscr{W}^2 - \mathscr{Z}^2)}{\mathscr{Z}(4\mathscr{W}^2 + \mathscr{Z}^2)^2} - \frac{i\kappa^2\mathscr{W}^3}{\mathscr{Z}(4\mathscr{W}^2 + \mathscr{Z}^2)}. \tag{4.6}$$

We restrict further consideration to the asymptotic limit $V \to \infty$, for which the admissible roots of (4.4) may be approximated by

$$\mathscr{W} \sim -1, \quad -i\mathscr{Z}/\sqrt{2} \tag{4.7}$$

(the roots $+1$ and $+i\mathscr{Z}/\sqrt{2}$ are ruled out by the null condition at $\mathscr{Z} = \infty$). The corresponding approximations to (4.6) are

$$\frac{d\ln\mathscr{F}}{d\mathscr{Z}} \sim \frac{1 + i\kappa^2}{\mathscr{Z}^3}, \quad -\frac{3}{2}\mathscr{Z}^{-1} - 2^{-3/2}\kappa^2, \tag{4.8}$$

the integration of which leads, through (4.1), to

$$\Phi \sim C_1 e^{-\eta} + C_2\mathscr{Z}^{-3/2}\exp\left[-\frac{i\eta(\mathscr{Z} - \kappa^{-2})}{\sqrt{2}} - \frac{\kappa^2\mathscr{Z}}{2\sqrt{2}}\right], \text{ for } \mathscr{Z} \to \infty, \tag{4.9}$$

where C_1 and C_2 are constants. (The term $-\frac{1}{2}\mathscr{Z}^{-2}$, derived from the real part of (4.8), has been neglected in the exponent in (4.9) since it is dominated by the error implicit in the approximation (4.7).) The corresponding approximation to $\mathscr{T}$, obtained through substitution of (4.9) into (3.13) is

$$\mathscr{T} \sim 2i\kappa^2\left[C_1\mathscr{Z}^{-1}e^{-\eta} - \frac{1}{2}C_2\mathscr{Z}^{-1/2}\exp\left\{-\frac{i\eta(\mathscr{Z} - \kappa^{-2})}{\sqrt{2}} - \frac{\kappa^2\mathscr{Z}}{2\sqrt{2}}\right\}\right]. \tag{4.10}$$

Note that, using the standard exponential substitution of the Liouville-Green method for the asymptotic solution of (3.1), we obtain the following expansion of the phase function:

$$\begin{aligned}\Phi \sim \hat{\eta}^{\gamma_1}[\log(\hat{\eta}/z_0)]^{\gamma_2} \exp[\gamma_3/\log(\hat{\eta}/z_0)]\cdots\exp\{i[\vartheta_1\hat{\eta}\log(\eta/z_0)\\ + \vartheta_2\hat{\eta} + \vartheta_3 Ei(1, -\log(\hat{\eta}/z_0)) + \vartheta_4\hat{\eta}/\log(\hat{\eta}/z_0) + \cdots]\},\end{aligned} \tag{4.11}$$

where ϑ_k and γ_k are real, $\hat{\eta} \approx k\eta$ and Ei is the exponential integral. Formally there is an infinite number of terms with coefficients ϑ_k that precede the determination of the γ_k. However, as only the first of these, namely ϑ_1, enters into the formula for γ_1, the first two into γ_2 (although the ϑ contributions happen to cancel), the first four into γ_3, and so on. The result is depicted in Figure 4 where we see that an initial exponential decay follows by an algebraic tail. It should be noted that this is agreement with numerical simulation of Ierley and Miles [6], and the expression (4.11) is originally found by them. However, the results of Figure 4 are drawn from the analytical expression given by equation (4.11).

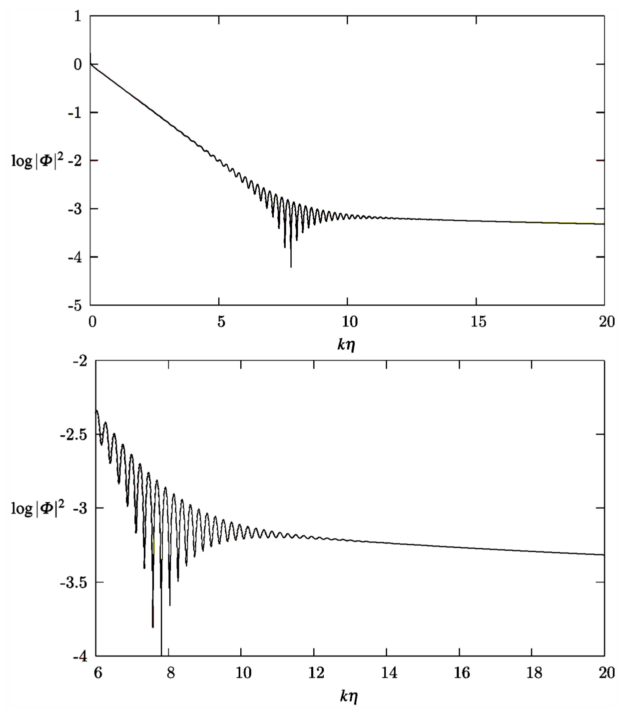

Figure 4. The variation of the log of modulus square for perturbation stream function Φ against the non-dimensional distance $k\eta$ showing the initial exponential decay follows by an algebraic tail. The lower figure is a close of the upper figure in the range $6 \leq k\eta \leq 20$.

Since the governing equation is fourth order, we find ϑ by solving for the roots of the fourth degree polynomial

$$\vartheta_1^2[(2\kappa^4 + A_1)\vartheta_1^2 + 1] = 0.$$

Two roots are zero and the other two constitute an imaginary pair. The double zero leads to a particularly simple result at next order: $\vartheta_2^2 - 1 = 0$, thus one pair of solutions is approximately $\exp(\pm\eta)$. The pure imaginary root pair exhibits weak algebraic growth (or decay), as reflected in γ_1. A suitable boundary condition for (3.1) is to match the decaying exponential and the decaying algebraic solutions. For this purpose, we employ the following expressions:

$$\vartheta_1 = -(A_1 + 2\kappa^4)^{-1/2},$$

$$\vartheta_2 = (1 + \kappa^3)(A_1 + 2\kappa^4)^{-1/2} - \frac{A_3 + B_1}{2(2\kappa^4 + A_1)} - \frac{b(4\kappa^4 + A_1)}{2(2\kappa^4 + A_1)},$$

$$\gamma_1 = -\frac{1}{2}\kappa^2(A_1 + 2\kappa^4)^{-1/2}; \quad \gamma_2 = -\frac{3}{2}$$

after selecting $\tilde{a}_1 = \kappa^2$.

5. Effect of the Inertial Critical-layer

5.1. Comparison with BHC

In the case of slow waves, where $c \ll 1$, the perturbation shear stress in (3.1) can be neglected in the outer region and thus Φ will satisfy the Rayleigh equation

$$(U - c)(\Phi'' - k^2\Phi) - U''\Phi = 0, \tag{5.1}$$

where now we shall assume $c = c_r$ and $c_i = 0$. The corresponding expressions for unsteady waves will be given in the next subsection.

As was shown by Sajjadi [16], the leading order solution to (5.1) is

$$\Phi = (U - c)e^{-k\eta}\left[\mathsf{A} + \Phi_c U_c' e^{k\eta_c}\int_0^{\infty}\left\{\frac{1}{(U-c)^2} - 1\right\}d\eta\right]. \tag{5.2}$$

The integral (5.2) is regular at the critical height and hence, by indenting the path of integration in (5.2) under the singularity $\eta = \eta_c$, we obtain

$$\Phi = (U - c)e^{-k\eta}\left[\mathsf{A} + \Phi_c U_c' e^{k\eta_c}\left(\,⨍_0^{\infty}\left\{\frac{1}{(U-c)^2} - 1\right\}d\eta - I\,\right)\right], \tag{5.3}$$

where A is a constant which can be determined from the boundary conditions and

$$I = \lim_{\varpi\to 0}\int_{\eta_c-\varpi}^{\eta_c+\varpi}\left\{\frac{1}{(U-c)^2} - 1\right\}d\eta. \tag{5.4}$$

Expanding $U(\eta)$ in a Taylor expansion in the vicinity of the critical point, setting $\eta = \eta_c\varpi e^{i\theta}$, where $\varpi \equiv c_i/U_* \ll 1$, and

$$\tan\theta = -c_i/U_c'(\eta - \eta_c), \tag{5.5}$$

then (5.4) becomes

$$\begin{aligned} I &= \frac{1}{U_c'^2}\left\{\lim_{\varpi\to 0}\int_{\eta_c-\varpi}^{\eta_c+\varpi}\frac{d\eta}{(\eta-\eta_c)^2} + i\pi\frac{U_c''}{U_c'}\right\} \\ &= \frac{i\pi U_c''}{U_c'^3} \end{aligned} \tag{5.6}$$

which agrees with the result obtained by Belcher et al. [2].

As also pointed out by Belcher et al. [2], a logarithmic mean velocity profile (5.5) yields $\tan\theta = \varpi\eta_c/(\eta - \eta_c)$. Hence, θ varies between 0 and π as $(\eta - \eta_c)/\ell_c$ tends to $\pm\infty$, respectively. Note that, the transition between these limiting values occurs across the layer of thickness $\ell_c = \varpi\eta_c$. Note also, the significance of the term $iU_c''/U_c'^3$ in the solution for I is that it leads to an out of phase contribution to the wave induced vertical velocity that gives rise to the same contribution to the wave growth from Miles [13] critical-layer mechanism.

The result of the present analysis confirms the earlier finding of Belcher et al. [2] in that Miles [13] solution is *only* valid when the waves grow significantly slowly such that

$$c_i \ll U_c' \eta_c \sim U_*. \tag{5.7}$$

As in Belcher et al. [2], our analysis also shows that when inertial effects control the behaviour around the critical-layer, there is a smooth behaviour around the critical-layer of thickness

$$\ell_c \sim c_i / U_c' \sim \eta_c c_i / U_*. \tag{5.8}$$

Hence, this proves the effects of critical-layer, as calculated by Miles [13], are *only* valid in the limit $c_i / U_* \downarrow 0$.

5.2. Steady monochromatic waves

For comparison with the waves that are unsteady, we calculate the energy-transfer parameter due to critical-layer, β_c, for steady monochromatic waves. Thus, we let $\Phi = -\mathscr{U}\mathscr{M}$. Thus, (3.1) becomes

$$[\nu_e(\mathscr{U}\mathscr{M}'' + 2U'\mathscr{M}' + U''\mathscr{M})]'' = ik[(\mathscr{U}^2\mathscr{M}')' - k^2\mathscr{U}^2\mathscr{M}]. \tag{5.9}$$

In quasi-laminar limit, the left-hand side of (5.9) is negligible and thus (5.9) reduces to

$$(\mathscr{U}^2\mathscr{M}')' - k^2\mathscr{U}^2\mathscr{M} = 0. \tag{5.10}$$

Multiplying (5.10) by $\mathscr{M}$, integrating by parts over $0 < \eta < \infty$, and invoking the inner limits $\mathscr{M} \to a$ and $\mathscr{U}^2\mathscr{M}' \to \mathscr{R}_0$ and a null condition at $\eta = \infty$, we obtain

$$a\mathscr{R}_0 = -\int_0^\infty \mathscr{U}^2(\mathscr{M}'^2 + k^2\mathscr{M}^2)d\eta. \tag{5.11}$$

Using the simplest admissible trial function for the variational integral (5.11), i.e.,

$$\mathscr{M} = ae^{-k\eta/\varsigma}, \tag{5.12}$$

where ς is a free parameter. Substituting (5.12) into (5.11) together with the approximation $\mathscr{U} \approx U_1 \ln(\eta/\eta_c)$, we get

$$\hat{\mathscr{P}}_0 \equiv \mathscr{P}/kaU_1^2 = -k(\varsigma^{-2} + 1)\int_0^\infty e^{-2k\eta/\varsigma} \ln^2(\eta/\eta_c)\,d\eta$$

$$= -\xi_c(\varsigma^{-2} + 1)\int_0^\infty e^{-st} \ln^2 t\,dt$$

$$= -\frac{\varsigma + \varsigma^{-1}}{2}\left\{\frac{\pi^2}{6} + \ln^2\left(\frac{2\gamma\xi_c}{\varsigma}\right)\right\}, \tag{5.13}$$

where $\xi_c \equiv k\eta_c$. It then follows from the variational condition $\partial\hat{\mathscr{P}}_0/\partial\varsigma = 0$ that

$$\varsigma^2 = \frac{L_\varsigma^2 - 2L_\varsigma + \pi^2/6}{L_\varsigma^2 + 2L_\varsigma + \pi^2/6}, \tag{5.14}$$

where $L_\varsigma \equiv 2\gamma\xi_c/\varsigma$ and $\varsigma = O(1)$.

The corresponding, quasi-laminar approximation to the energy-transfer parameter may be calculated from (5.3), which implies $\Phi_c = \mathscr{P}_c/U_c' \approx \mathscr{P}_0/U_c'$, and (5.6), which yields

$$\beta_c = \pi\xi_c|\,\Phi_c/U_1a\,|^2 = \pi\xi_c^3|\,\hat{\mathscr{P}}_0\,|^2 = \frac{1}{4}\pi(\varsigma + \varsigma^{-1})^2\left(L_\varsigma + \frac{1}{6}\pi^2\right)^2 \tag{5.15}$$

$$= \pi\xi_c^3L_0^4\left[1 - \left(4 - \frac{1}{3}\pi^2\right)\Lambda^2 + O(\Lambda^3)\right], \tag{5.16}$$

where $\Lambda = L_0^{-1} = -\gamma - \ln(2k)$, $L_0 = L_\delta - \ln\delta$ and $i\delta L_\delta/\kappa^2 = (\sqrt{3} - 1) + O(\delta)$.

To obtain the corresponding expression for the component of the energy-transfer parameter, β_T, due to turbulence, we multiply (3.1) by $-\mathscr{M}$, integrating over $0 < z < \infty$, invoking the conditions

$$\mathscr{M} = a, \quad \mathscr{M}' = ka, \quad \mathscr{T}' = ik[\mathscr{R}_0 - kac^2]$$

on $z = 0$ and the null condition for $z \to 0$, we obtain

$$\int_0^\infty \mathscr{M}\mathscr{T}'' dz = ka[\mathscr{T}_0 - i\mathscr{R}_0] + i(kac_r)^2 + \int_0^\infty \mathscr{M}''\mathscr{T} dz$$

$$= i(kac_r)^2 + ik\int_0^\infty \mathscr{V}^2(\mathscr{M}'^2 + k^2\mathscr{M}^2)dz,$$

with $c = c_r + ic_i$. Then, in the limit as $s \equiv \rho_a/\rho_w$, where ρ_a and ρ_w are densities of the air and water, respectively, we obtain from

$$\alpha + i\beta \equiv (c^2 - c_w^2)/sU_1^2 = (\mathscr{R}_0 + i\mathscr{T}_0)/kaU_1^2 \equiv (\hat{\mathscr{R}}_0 + i\hat{\mathscr{T}}_0), \tag{5.17}$$

where c is the complex wave speed, $c_w = \sqrt{g/k}$ is the speed of water waves in the absence of the airflow above it and the suffix zero denotes evaluation at $z = 0$,

$$\alpha_T + i\beta_T = (kaU_1)^{-2}\int_0^\infty \{i\nu_e[\mathscr{V}\mathscr{M}''^2 + 2U'\mathscr{M}/\mathscr{M}'' + U''\mathscr{M}\mathscr{M}'']$$

$$- k\mathscr{V}^2(\mathscr{M}'^2 + k^2\mathscr{M}^2)\}dz. \tag{5.18}$$

The above integral can be evaluated asymptotically[3] whose imaginary part yields

$$\beta_T = 5\kappa^2 L_0^{-1}. \tag{5.19}$$

Therefore, in summary, the energy-transfer parameter from wind to surface waves for steady monochromatic waves may be given by the following formulae:

[3]The detail evaluations may be obtained found in the appendix of the paper by Sajjadi [19].

$$\beta = \beta_T + \beta_c, \quad \beta_T = \frac{5\kappa^2}{\Lambda}, \quad \beta_c = \frac{5}{2}\pi \hat{W} L_0^4 \left[1 - \left(4 - \frac{\pi^2}{3}\right)\varepsilon^2\right],$$

$$\Lambda = L_0^{-1}, \quad L_0 = -\gamma - \log \hat{W}, \quad \hat{W} = kz_0 e^{c_r/U_\lambda}(U_\lambda / c_r)^2, \quad U_\lambda = 2U_*,$$

where $\kappa = 0.4$ is von Kármán's constant and $\gamma = 0.5772$ is an Euler's constant.

5.3. Unsteady waves

The generalization of the results just obtained above follows a similar development, but with the exception that $c_i \neq 0$. Here we shall present results for Stokes waves being a sum of two harmonics $(n = 1, 2)$. Hence, we consider the surface wave expressed as

$$z = Re\left\{ae^{ik(x-ct)} + \frac{1}{2}ka^2 e^{2ki(x-ct)}\right\}, \quad c = c_r + ic_i.$$

Note that, results for monochromatic waves follow immediately from what will be developed by setting $n = 1$ and ignoring the second harmonic.

Therefore, we begin by using the expression (5.11), but now we take

$$\mathscr{U} = U - c_r - ic_i. \tag{5.20}$$

Thus, the expression (5.11) now reads

$$a\mathscr{P}_{n0} = -\int_0^\infty [(U - c_r)^2 - 2ic_i(U - c_r) - c_i^2](\mathscr{M}_n'^2 + k_n^2 \mathscr{M}_n)\, d\eta, \tag{5.21}$$

where $k_1 \equiv k$ and $k_2 \equiv 2k$.

Once again using a similar admissible trial function as that given by (5.12), namely

$$\mathscr{M}_n = ae^{k_n \eta/\varsigma_n} \quad (n = 1, 2)$$

the variational integral (5.21) becomes

$$a\mathscr{P}_{n0} = a^2k_n^2(\varsigma_n^{-2}+1)\left\{\int_0^\infty U_1^2 \ln^2(\eta/\eta_c)e^{-2k_n\eta/\varsigma_n}d\eta\right.$$

$$\left. + 2ic_i\int_0^\infty U_1 \ln(\eta/\eta_c)e^{-2k_n\eta/\varsigma_n}d\eta + c_i^2\int_0^\infty e^{-2k_n\eta/\varsigma_n}d\eta\right\}.$$

Evaluating the integrals, we obtain

$$\hat{\mathscr{P}}_{n0} \equiv \mathscr{P}_{n0}/k_na_nU_1^2$$

$$= \frac{1}{2}(\varsigma_n+\varsigma_n^{-1})\left\{\left[\frac{\pi^2}{6}+\log^2\left(\frac{2k_n\gamma\eta_c}{\varsigma_n}\right)\right]-2i\hat{c}_i\log\left(\frac{2k_n\gamma\eta_c}{\varsigma_n}\right)+\hat{c}_i^2\right\}, \quad (5.22)$$

where $\hat{c}_i = c_i/U_1$.

As before, applying the variational condition $\partial\hat{\mathscr{P}}_{n0}/\partial\varsigma_n = 0$ yields

$$L_{n\varsigma}^2 - 2(1+i\hat{c}_i)L_{n\varsigma} + \left(\hat{c}_i^2 + 2i\hat{c}_i + \frac{\pi^2}{6}\right)$$

$$= \varsigma_n^2\left[L_{n\varsigma}^2 + 2(1+i\hat{c}_i)L_{n\varsigma} + \left(\hat{c}_i^2 - 2i\hat{c}_i + \frac{\pi^2}{6}\right)\right]$$

whence

$$\varsigma_n^2 = \frac{L_{n\varsigma}^2 - 2\mathfrak{C}_iL_{n\varsigma} + \left(\hat{c}_i^2 + 2i\hat{c}_i + \frac{\pi^2}{6}\right)}{L_{n\varsigma}^2 + 2\mathfrak{C}_i^*L_{n\varsigma} + \left(\hat{c}_i^2 + 2i\hat{c}_i + \frac{\pi^2}{6}\right)}, \quad (5.23)$$

where $\mathfrak{C}_i = 1+\hat{c}_i$, the superscript * denotes the complex conjugate, and $L_{n\varsigma} \equiv 2\gamma k_n\eta_c/\varsigma_n$, where $L_{n\varsigma} = -\gamma - \log(2k_n\eta_c)$.

The expression (5.23) may be approximated to $O(\Lambda_n^2)$, to give

$$\varsigma_n = \frac{1-\mathfrak{C}_i\Lambda_n}{1+\mathfrak{C}_i^*\Lambda_n} + O(\Lambda_n^2),$$

where $\Lambda_n \equiv L_{n0}^{-1}$, and therefore we obtain the following expression for the energy-transfer parameter to the two $(n = 1, 2)$ harmonics of the wave:

$$\beta_{nc} = \pi\left(\frac{\varsigma_n L_{n\varsigma}}{2\gamma}\right)^3 L_{n0}^4\left[1 - \left(4 - \frac{\pi^2}{3} + 10\hat{c}_i^2\right)\Lambda_n^2 + O(\Lambda_n^3)\right]. \qquad (5.24)$$

Note that, for a steady wave $(c_i = 0)$ and a monochromatic wave ($n = 1$ and ignoring the second harmonic of the Stoke wave), the expression (5.24) reduces to (5.16). We remark that for a steady Stokes wave, we only need to assume $c_i = 0$.

In a similar manner, for an unsteady wave, we adopt the complex amplitude of surface shear stress is given by (3.24) but with $\mathscr{U}$ given by (5.20) and the following modification for the expression for the eddy viscosity, given by (3.4), namely

$$\nu_{ne} = 2U_*^2\{U_1[\eta^{-1} + (ik_n/\kappa^2)\ln(\eta/\eta_c)] + k_n c_i/\kappa^2\}^{-1}. \qquad (5.25)$$

Thus, upon substituting (5.25), and using an equivalent trial function to that given by (3.25), namely

$$\mathscr{T}_n = \mathscr{T}_{n0}e^{-k_n\eta/\delta_n}, \qquad (5.26)$$

the 'unsteady' version of the integral (3.24) becomes

$$(\mathscr{T}_n\mathscr{T}_n')_0 = -k_n\int_0^\infty [k_n/\delta_n^2 + U_1^2(i\mathscr{U}'^2 - 2k_n\mathscr{U}^2)]\mathscr{T}_{n0}e^{-2k_n\eta/\delta_n}d\eta. \quad (5.27)$$

Invoking the condition $\partial(\mathscr{T}_n/\mathscr{T}_n)_0/\partial\delta_n = 0$, after substituting $\mathscr{U}$ and (5.26), and evaluating the integral (5.27), we obtain

$$(\mathscr{T}_n/\mathscr{T}_n)_0 = \frac{1}{2i\delta_n} - \frac{(c_r + ic_i)^2}{4U_*^2} + \frac{\delta_n}{4\kappa^2}\left(\frac{1}{\delta_n} - \frac{ik_n}{4\kappa^2}\right)\left(\frac{\pi^2}{6} + L_{\delta_n}^2\right)$$

$$= \frac{1}{4\kappa^2}\left[\frac{2}{\delta_n} - \frac{(c_r + ic_i)^2}{U_1^2} + (1 + \hat{\delta}_n)\left(\frac{\pi^2}{6} + L_{\delta_n}^2\right)\right], \qquad (5.28)$$

where $\hat{\delta}_n = i\delta_n/\kappa^2$,

$$\left(L_{\delta_n}^2 + 2L_{\delta_n} + \frac{\pi^2}{6}\right)\hat{\delta}_n^2 + L_{\delta_n}\hat{\delta}_n - 2 = 0 \tag{5.29}$$

and $L_{\delta_n} = L_{n0} + \ln\delta_n$.

Solving equation (5.29) for $\hat{\delta}_n$, we find that

$$\hat{\delta}_n = (\sqrt{3}-1)(L_{n0} + \ln\delta_n)^{-1}. \tag{5.30}$$

Note that, δ_n may be complex and $|\delta_n| \ll 1$ (but strictly not equal to zero), we may use the expansion for $\ln\delta_n$, given by

$$\ln\delta_n = 2\sum_{m=1}^{\infty}\frac{1}{2m-1}\left(\frac{\delta_n - 1}{\delta_n + 1}\right)^{2m-1} \doteqdot 2\,\frac{\delta_n - 1}{\delta_n + 1} + O\left[\left(\frac{\delta_n - 1}{\delta_n + 1}\right)^3\right],$$

$$0 < |\delta_n| \le 2,$$

we may cast (5.30) as

$$\hat{\delta}_n = (\sqrt{3}-1)\left[L_{n0} + 2\left(\frac{\delta_n - 1}{\delta_n + 1}\right)\right]^{-1}. \tag{5.31}$$

Hence, the asymptotic evaluation of integrals in (5.27) yields to the following expression (for further details, see the appendix of Sajjadi [19])

$$\beta_n = 2\kappa^2[1 + (\sqrt{3}+1)(1 - \mathscr{E}_{\delta_n})L_{|\delta_n|} - (\sqrt{3}-1)(L_{|\delta_n|} + 4\ln 2)$$

$$- 4(\mathscr{E}_{\delta_n} - \mathscr{E}_{2\delta_n})] - 4\delta_{ni}L_{n0}^2, \tag{5.32}$$

where $\delta_{ni} = \mathscr{T}\{\delta_n\}$,

$$\mathscr{E}_p = \mathscr{E}\left(\frac{2a_1}{ipL_p}\right) \approx 1 - \frac{ipL_p}{2a_1}, \quad L_p = -\gamma - \ln(2\mathsf{k}\mathsf{p}) = L_0 + \ln p$$

and

$$\mathscr{E}(\Theta_n) = \Theta_n e^{\Theta_n}\int_0^\infty t^{-1}e^{-t}dt = \Theta_n e^{\Theta_n}\mathsf{E}_1(\Theta_n).$$

From these expressions, we obtain (see Sajjadi [19])

$$\beta_{nc} = \frac{1}{2}(\Lambda_n + \Lambda_n^{-1})\left\{L_{\varsigma n}^2 + \frac{\pi^2}{6}\right\}, \quad \beta_{nT} = \frac{4\kappa^2}{\Lambda_n}$$

which are in reasonable agreement with (5.19) and (5.24).

6. Results and Conclusions

In Figure 5, we show comparison of the energy-transfer rate, β, between the present result for a monochromatic unsteady (growing) wave, both analytically and numerically, and those calculated by Miles [13] and Janssen [7] for the steady wave counterpart. Miles and Janssen both assume that the drag C_D, and thence β, is dominated by the limiting inviscid wave growth mechanism, thus their formulation is independent of c_i. In contrast, the present calculation is for a viscous unsteady (growing) wave, where $c_i/U_* = 0.01$ and $kz_0 = 10^{-4}$.

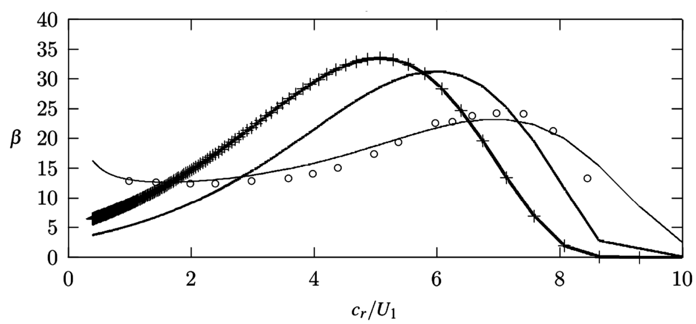

Figure 5. Total energy-transfer parameter, β, using different models for critical-layer and sheltering mechanisms for unsteady waves $(\text{where } c_i \ll U_*)$ as a function of the wave age c_r/U_1. +++++, Miles [13] calculation $(c_i = 0,\ \nu_e = 0)$ from his formula:

$$\beta = \pi\eta_c\left\{\frac{1}{6}\pi^2 + \log^2(\gamma\eta_c) + 2\sum_{n=1}^{\infty}\frac{(-1)^n\eta_c^n}{n!\,n^2}\right\}^2$$

where $\eta_c = kz_c$ is the critical height $\eta_c = \Omega(U_1/c_r)^2 e^{c_r/U_1}$ and $\Omega = gz_0/U_1^2$ is the Charnock's [3] constant. Thick solid line, parameterization of Miles [13] formula, for $c_i = 0$, $\nu_e = 0$, given by Janssen [7]: $\beta = 1.2\kappa^{-2}\eta_c \log^4 \eta_c$, where $\eta_c = \min\{1, kz_0 e^{[\kappa/(U_*/c+0.011)]}\}$. Thin solid line, present formulation: $(\beta_T + \beta_c)$ for $c_i \neq 0$, $\nu_e \neq 0$. $\circ$, numerical simulation using Launder et al. [9] Reynolds-stress closure model for $c_i \neq 0$, $\nu_e \neq 0$. Note that, β for Miles and Janssen is equivalent to β_c in our notation. Taken from Sajjadi et al. [21].

We emphasize that the various models, such as those by Belcher and Hunt [1], Mastenbroek et al. [12], etc., all generally agree with our numerical simulations performed using Launder et al. [9] Reynolds stress closure scheme for the energy-transfer parameter, β, shown in Figure 5. This shows consistency between these models and the unimportance of very small c_i for which viscous processes are significant.

Figure 6 shows comparison of β_c as a function of wave age c_r/U_1, calculated according to (Conte and Miles [4]) for numerical solution of inviscid Orr-Sommerfeld equation, against the numerical solution of equation (1.6) for $c_i/U_* = 0.01$, $kz_0 = 10^{-4}$ and $\nu_e \neq 0$. We remark that increasing c_i/U_* from 0.01 to 0.1 (not shown here) makes no significant difference in the magnitude of β_c. We conclude therefore for a finite value of ν_e the right-hand side of equation (1.6) is dominant and therefore the magnitude of β_c, calculated from the solution of (1.6), is practically zero over a wide range of the wave age, in particular for a 'young' wave, where $c_r/U_1 < 2$. We thus conclude that the critical-layer mechanism plays an insignificant role for $c_r/U_1 < 9$, and very little effect for $9 \leq c_r/U_1 \leq 10.5$.

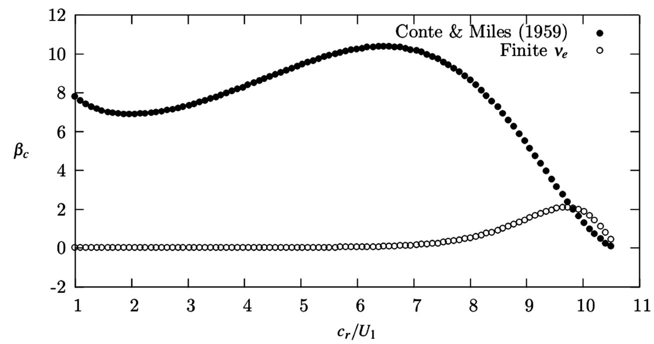

Figure 6. Component of energy-transfer parameter, β_c, by different models of critical-layer mechanisms for unsteady waves $(\text{where } c_i \ll U_*)$ as a function of the wave age c_r/U_1. •, numerical solution of inviscid Orr-Sommerfeld equation by Conte and Miles [4] for $c_i = 0$ and $\nu_e = 0$ using the singular critical-layer approach; ∘ numerical solution of equation (1) for $c_i \neq 0$ and $\nu_e \neq 0$. Taken from Sajjadi et al. [21].

We remark that these parameterizations have been incorporated and tested in a spectral wave model, Wave Watch and Wind Wave, which show superior results when compared with field data (see Fitzpatrick et al. [5] and Sajjadi et al. [18]).

In conclusion, we adopted an asymptotic multi-deck solution for turbulent shear flows over unsteady surface waves, in the limits of low turbulent stresses and small wave amplitude. The structure of the flow is defined, using an eddy viscosity turbulence model, in terms of asymptotically-matched thin-layers, namely the surface layer and a critical-layer. Solutions for both inner and outer regions are constructed through an interpolation between an inner, mixing-length and an outer, rapid-distortion approximations. The results particularly demonstrate the physical importance of the singular flow features and physical implications of the elevated critical-layer in the limit of the unsteadiness tending to zero. These agree with the variational mathematical solution of Miles [13] for small but finite growth rate. However, the results obtained here, are not consistent physically or mathematically with his analysis in the limit of growth rate tending to zero. In the present study, it is shown that in the limit of zero growth rate, the effect of the elevated critical-layer is eliminated by finite turbulent diffusivity, so that the perturbed flow and the drag force determined by the asymmetric and sheltering flow in the surface shear layer and its matched interaction with the upper region, as physically demonstrated by Sajjadi et al. [21]. The results for an unsteady monochromatic wave are also extended to those growing Stokes waves. Thus, estimation can be made as to what percentage of total energy-transfer from wind goes to each harmonic of a Stokes wave.

References

[1] S. E. Belcher and J. C. R. Hunt, Turbulent shear flow over slowly moving waves, J. Fluid Mech. 251 (1993), 109.

[2] S. E. Belcher, J. C. R. Hunt and J. E. Cohen, Turbulent flow over growing waves, Proceedings of IMA Conference on Wind over Waves, S. G. Sajjadi, N. H. Thomas and J. C. R. Hunt, eds., 1930, Oxford University Press, 1999.

[3] H. Charnock, Wind stress on a water surface, Q. J. R. Meteorol. Soc. 81 (1955), 639.

[4] S. D. Conte and J. W. Miles, On the numerical integration of the Orr-Sommerfeld equation, J. Soc. Indust. Appl. Math. 7 (1959), 361.

[5] P. Fitzpatrick, G. Mostovoi, Y. Li, M. Bettencourt and S. G. Sajjadi, Coupling of COAMPS and WAVEWATCH with Improved Wave Physics, DoD High Performance Computing Modernization Program Programming Environment and Training Report No. 0005, 2002.

[6] G. Ierley and J. W. Miles, On Townsend's rapid-distortion model of the turbulent-wind-wave problem, J. Fluid. Mech. 435 (2001), 175.

[7] P. A. E. M. Janssen, Quasi-linear theory of wind-wave generation applied to wave forecasting, J. Phys. Oceanog. 21 (1991), 1631.

[8] H. Jeffreys, On the formation of water waves by wind, Proc. R. Soc. London Ser. A 107 (1925), 189.

[9] B. E. Launder, G. J. Reece and W. Rodi, Progress in the development of a Reynolds-stress turbulence closure, J. Fluid Mech. 68 (1975), 537.

[10] M. J. Lighthill, The fundamental solution for small steady three-dimensional parallel shear flow, J. Fluid. Mech. 14 (1957), 385.

[11] F. W. J. Olver, Asymptotics and Special Functions, Academic Press, 1974.

[12] C. Mastenbroek, V. K. Makin, M. H. Garat and J. P. Giovanangeli, Experimental evidence of the rapid distortion of turbulence in air flow over waves, J. Fluid Mech. 318 (1996), 273-302.

[13] J. W. Miles, On the generation of surface waves by shear flows, J. Fluid Mech. 3 (1957), 185.

[14] J. W. Miles, Surface-wave generation: a viscoelastic model, J. Fluid Mech. 322 (1996), 131.

[15] J. Rotta, Das in Wandnähe gültige Geschwindigkeitsgesetz turbulenter Strömungen, Archive of Applied Mechanics 18(4) (1950), 277-280.

[16] S. G. Sajjadi, Shearing flows over Stokes waves, Department of Mathematics, Internal Report, Coventry Polytechnic, U.K., 1988.

[17] S. G. Sajjadi, On the growth of a fully nonlinear Stokes wave by turbulent shear flow. Part 2: Rapid distortion theory, Math. Eng. Ind. 6 (1998), 247.

[18] S. G. Sajjadi, M. Bettencourt, P. J. Fitzpatrick, G. Mostovoi and Y. Li, Sensitivity of a coupled tropical cyclone/ocean wave simulation to different energy transfer schemes, Proceeding of 25th Conference on Hurricanes and Tropical Meteorology, 15D.5, 2002.

[19] S. G. Sajjadi, Interaction of turbulence due to tropical cyclones with surface waves, Adv. Appl. Fluid Mech. 1(2) (2007), 101-145.

[20] S. G. Sajjadi, On asymptotic solution of $\mathscr{T}'' - (ik\mathscr{U}/\nu_k)\mathscr{T} = 0$ for large k, Adv. Appl. Fluid Mech. 7(1) (2010), 31-43.

[21] S. G. Sajjadi, J. C. R. Hunt and F. Drullion, Asymptotic multi-layer analysis of wind over unsteady monochromatic surface waves, J. Eng. Math. 84 (2014), 73.

[22] K. Stewartson, Boundary layers on flat plates and related bodies, Adv. Appl. Mech. 14 (1974), 145.

[23] A. A. Townsend, Flow in a deep turbulent boundary layer over a surface distorted by water waves, J. Fluid Mech. 55 (1972), 719.