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# CARISTI AND BANACH FIXED POINT THEOREM ON PARTIAL METRIC SPACE

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**Abstract**

This paper shows Caristi and Banach fixed point theorems type in the partial metric space. Both will be proven by using Ekeland's variational principle in partial metric space which also introduced in this article.

## 1. Introduction

Caristi fixed point theorem was generalized by several authors. For example, Bae [1] generalized Caristi's theorem to prove the fixed point theorem for weakly contractive set-valued mapping as well as Banach fixed point theorem in the other way.

In recent years, many works on domain theory have been made in order to equip semantics domain with a notion of distance, see [2-3, 6-9]. In particular, Matthews [8] introduced the notion of a partial metric space as a part of the study of denotational semantic of data flow network, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

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In this paper, we present the Caristi and Banach fixed point theorem type in partial metric spaces. We would also introduce Ekeland variational principle on partial metric spaces and its applications to fixed point.

## 2. Preliminaries

First, we start with some preliminaries on partial metric spaces. For more details, we refer to reader to [8].

**Definition 2.1.** Let $X$ be nonempty set. The mapping $p : X \times X \to \mathbb{R}^+$ is said *partial metric* on $X$ if satisfies

(P1) $p(x, x) \leq p(x, y)$ for all $x, y \in X$;

(P2) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$;

(P3) $p(x, y) = (y, x)$ for all $x, y \in X$;

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ for all $x, y, z \in X$.

The pair $(x, p)$ is called a *partial metric space*. Note that the self-distance of any point need not be zero. A partial metric is a metric on $X$ if $p(x, x) = 0$ for any $x \in X$.

**Example 2.2.** Let $\mathbb{R}$ be a real number set and the distance function $p$ : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$p(x, y) = \frac{1}{2}\{| x - y | + | y |\}, \quad \forall x, y \in \mathbb{R}.$$

Then $p$ is a partial metric on $R$.

**Lemma 2.3.** *Let* $(X, p)$ *be a partial metric space, and the function* $d_p$ : $X \times X \to [0, \infty)$ *be defined by*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad \forall x, y \in X.$$

*Then* $d_p$ *is a metric*.

**Proof.** (i) Clear for all $x, y \in X$, then $d_p(x, y) \geq 0.$

(ii) From (P2), we have

$$\begin{aligned} x = y &\Leftrightarrow p(x, x) = p(y, y) = p(x, y) \\ &\Leftrightarrow d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \\ &\Leftrightarrow d_p(x, y) = 2p(x, y) - 2p(x, y) \\ &\Leftrightarrow d_p(x, y) = 0. \end{aligned}$$

(iii) Clear for all $x, y \in X$, $d_p(x, y) = d_p(y, x).$

(iv) For all $x, y, z \in X$ and from (P4), we obtain

$$\begin{aligned} d_p(x, z) &= 2p(x, z) - p(x, x) - p(z, z) \\ &\leq 2(p(x, y) + p(y, z) - p(y, y)) - p(x, x) - p(z, z) \\ &= (2p(x, y) - p(x, x) - p(y, y)) + (2p(y, z) - p(y, y) - p(z, z)) \\ &= d_p(x, y) + d_p(y, z). \end{aligned}$$ □

Lemma 2.3 describes that metric is a special case of partial metric. Therefore, the partial metric is a generalization of metric.

**Definition 2.4.** Let $(X, p)$ be a partial metric space, a point $x_0 \in X$ and $\varepsilon > 0.$ The open ball for a partial metric *p* is set of the form

$$B_\varepsilon(x_0) = \{x \in X \mid p(x_0, x) < \varepsilon\}.$$

Since $p(x_0, x_0) > 0,$ the open ball are sets of the form

$$B_{\varepsilon + p(x_0, x_0)}(x_0) = \{x \in X \mid p(x_0, x) < \varepsilon + p(x_0, x_0)\}.$$

Contrary to the metric space case, some open balls may be empty. If $\varepsilon > p(x_0, x_0),$ then $B_\varepsilon(x_0) = B_{\varepsilon - p(x_0, x_0)}(x_0).$ If $0 < \varepsilon \leq p(x_0, x_0),$ then we obtain

$$B_\varepsilon(x_0) = \{x \in X \mid p(x_0, x) < \varepsilon \leq p(x_0, x_0)\} = \emptyset.$$

This mean the open ball $B_{p(x_0, x_0)}(x_0)$ be empty set. However, may be point $x_0 \notin B_{p(x_0, x_0)}(x_0)$.

**Definition 2.5.** A sequence $\langle x_n \rangle$ in a partial metric space $(X, p)$ converges to $x_0 \in X$ if, for any $\varepsilon > 0$ such that $x_0 \in B_\varepsilon(x_0)$ there exists $N \in \mathbb{N}$ so that for any $n \geq N$, $x_n \in B_e(x_0)$. We write $\lim_{n\to\infty} x_n = x_0$.

**Lemma 2.6.** *A sequence* $\langle x_n \rangle$ *in a partial metric space* $(X, p)$. *Then* $\langle x_n \rangle$ *converges to point* $x_0 \in X$ *if and only if* $\lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0)$.

**Proof.** By Definition 2.5, for any $\varepsilon > 0$, $p(x_n, x_0) < \varepsilon$ for any $n \geq N$. Since $B_\varepsilon(x_0) \neq \varnothing$ of course $p(x_0, x_0) \leq \varepsilon$ this implies $p(x_n, x_0) - p(x_0, x_0) < \varepsilon$, for any $n \geq N$ so that $\lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0)$.

Conversely suppose that $p(x_0, x_0) = \lim_{n\to\infty} p(x_n, x_0)$. If $x_0 \in B_\varepsilon(x_0)$, then there exists $n \in \mathbb{N}$ such that for any $n \geq N$, $p(x_n, x_0) < \varepsilon$. This mean $x_n \in B_\varepsilon(x_0)$ for any $n \geq N$. By Definition 2.5, $\langle x_n \rangle$ converges to point $x_0 \in X$. If $x_0 \in B_\varepsilon(a)$ with $a \in X$, that is $p(a, x_0) < \varepsilon$, then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $p(x_n, x_0) - p(x_0, x_0) < \varepsilon - p(x_0, a)$ so that for any $n \geq N$ we obtain

$$\begin{aligned} p(x_n, a) &\leq p(x_n, x_0) + p(x_0, a) - p(x_0, x_0) \\ &< (\varepsilon - p(x_0, a)) + p(x_0, a) = \varepsilon. \end{aligned}$$

This means for any $n \geq N$, $x_n \in B_\varepsilon(a)$. □

**Definition 2.7.** A sequence $\langle x_n \rangle$ in a partial metric space $(X, p)$ is called *properly convergence* to $x \in X$ if $\langle x_n \rangle$ converges to $x$ and

$$\lim_{n\to\infty} p(x_n, x_n) = p(x, x).$$

In other words, a sequence $\langle x_n \rangle$ properly converges to $x \in X$ if $\lim_{n\to\infty} p(x_n, x)$ and $\lim_{n\to\infty} p(x_n, x_n)$ exist and

$$\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

Notice that every convergent sequence in a metric space converging in partial metric spaces.

**Definition 2.8.** A sequence $\langle x_n \rangle$ in a partial metric space $(X, p)$ is called a *Cauchy sequence* if $\lim_{m, n\to\infty} p(x_n, x_m)$ exists and is finite.

In other words, $\langle x_n \rangle$ is Cauchy if the number sequence $p(x_n, x_m)$ converges to some $\lambda \in \mathbb{R}$ as $n$ and $m$ approach to infinity, that is, if $\lim_{n, m\to\infty} p(x_n, x_m) = \lambda < \infty$. This means for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $| p(x_n, x_m) - \lambda | < \varepsilon$. If $(X, p)$ is a metric space, then $\lambda = 0$.

**Lemma 2.9.** *Let $\langle x_n \rangle$ be a sequence in $(X, p)$. If $\langle x_n \rangle$ properly converges to x, then $\langle x_n \rangle$ is Cauchy sequence.*

**Proof.** By Definition 2.7,

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n).$$

By (P1) and (P4), we have

$$\begin{aligned} p(x, x) &= \lim_{m, n\to\infty} p(x_n, x_n) \leq \lim_{n, m\to\infty} p(x_n, x_m) \\ &\leq \lim_{n, m\to\infty} p(x_n, x) + \lim_{n, m\to\infty} p(x_m, x) - p(x, x) \\ &= p(x, x). \end{aligned}$$

Hence $\lim_{n, m\to\infty} p(x_n, x_m) = p(x, x)$. This means, there exists $\lambda \in \mathbb{R}^+$ such that $\lambda = p(x, x)$ and $\lim_{n, m\to\infty} p(x_n, x_m) = \lambda$. The sequence $\langle x_n \rangle$ is Cauchy proved. □

**Theorem 2.10.** *A sequence $\langle x_n \rangle$ in a partial metric space $(X, p)$ is a Cauchy, if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all*

*$n, m \geq N$ we have*

$$p(x_n, x_m) - p(x_m, x_m) < \varepsilon.$$

**Proof.** Since $\langle x_n \rangle$ is Cauchy, there exists $\lambda \in \mathbb{R}^+$ such that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and for all $n, m \geq N$ we have

$$| p(x_n, x_m) - \lambda | < \frac{\varepsilon}{2}.$$

Let $n = m \geq N$. Then $| p(x_m, x_m) - \lambda | < \frac{\varepsilon}{2}$. Therefore

$$| p(x_n, x_m) - p(x_m, x_m) | \leq | p(x_n, x_m) - \lambda | + | p(x_m, x_m) - \lambda | < \varepsilon.$$

By (P1), we obtain $p(x_n, x_m) - p(x_m, x_m) < \varepsilon$. Conversely it is obvious. □

**Definition 2.11.** A partial metric space is complete if every Cauchy sequence properly converges.

**Definition 2.12.** Let $(X, p)$ be a partial metric space and $A$ be a nonempty subset of $X$. The *diameter* of $A$, denoted by $D(A)$, is given by

$$D(A) = \sup\{p(x, y) - p(x, x) : x, y \in A\}.$$

**Theorem 2.13.** *Let $(X, p)$ be a complete partial metric space and $F_n$ be a decreasing sequence (that is, $F_n \supset F_{n+1}$) of nonempty closed subsets of $X$ such that $D(F_n) \to 0$ as $n \to \infty$. Then the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.*

**Proof.** The first, construct a sequence $\langle x_n \rangle$ in $X$ by selecting a point $x_n \in F_n$ for each $n \in \mathbb{N}$. Since $F_n \supset F_{n+1}$ for all $n$, we have $x_n \in F_n \subset F_m$ for all $n > m$.

Let $\varepsilon > 0$ be given. Since $D(F_n) \to 0$, there exists $N \in \mathbb{N}$ such that $D(F_n) < \varepsilon$ for each $n \geq N$. Since $F_m, F_n \subseteq F_N$ for each $n, m \geq N$. Therefore, $x_n, x_m \in F_N$ for each $n, m \geq N$ and thus, we have

$$p(x_n, x_m) - p(x_m, x_m) \leq D(F_n) < \varepsilon.$$

By Theorem 2.10, $\langle x_n \rangle$ is Cauchy sequence. Since $X$ is complete, there exists $x^* \in X$ such that $p(x_n, x^*) - p(x^*, x^*) < \varepsilon$ for each $n \geq N$.

Let $n = N$ be fixed. Then the subsequence $\{x_n, x_{n+1}, ...\}$ of the sequence $\langle x_n \rangle$ is contained in $F_n$ and still converges to $x^*$. $F_n$ is closed in complete partial metric space $(X, p)$, it is complete and so $x^* \in F_n$ for each $n \in \mathbb{N}$. Hence $x^* \in \bigcap_{n=1}^{\infty} F_n$, that is $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Finally, we show that $x^*$ is unique in $\bigcap_{n=1}^{\infty} F_n$. If $y \in \bigcap_{n=1}^{\infty} F_n$, then $x^*, y \in F_n$ for each $n \in \mathbb{N}$. Therefore, $0 \leq p(x^*, y) - p(y, y) \leq D(F_n) \to 0$ as $n \to \infty$ and $0 \leq p(x^*, y) - p(x^*, x^*) \leq D(F_n) \to 0$ as $n \to \infty$. Thus $p(x^*, y) = p(y, y) = p(x^*, x^*)$. By (P2), $x^* = y$. □

**Definition 2.14** [8]**.** Let $(X, p)$ be a partial metric space. The mapping $f : X \to X$ is called a *contraction* on $X$ if there exists $k \in (0, 1)$ such that for every $x, y \in X$ we have

$$p(f(x), f(y)) - p(f(y), f(y)) \leq k(p(x, y) - p(y, y)). \tag{2.1}$$

**Theorem 2.15** [8]**.** *Each contraction in a complete partial metric space has a unique fixed point.*

**Proof.** Suppose $f : X \to X$ is contraction in a complete partial metric space. Let $x_{n+1} = f(x_n)$ for $n \geq 0$. We will first show that $\langle x_n \rangle$ is a Cauchy sequence. Since $f$ is contraction, we obtain

$$p(f(x_0), f(x_1)) - p(f(x_1), f(x_1)) \leq k(p(x_0, x_1) - p(x_1, x_1))$$

$$p(f(x_1), f(x_2)) - p(f(x_2), f(x_2)) \leq k^2(p(x_0, x_1) - p(x_1, x_1))$$

$$\vdots$$

$$p(f(x_n), f(x_{n+1})) - p(f(x_{n+1}), f(x_{n+1})) \leq k^{n+1}(p(x_0, x_1) - p(x_1, x_1)).$$

For all $n,\ m \in \mathbb{N}$, we obtain

$$
\begin{aligned}
& p(f(x_n),\ f(x_{n+m})) - p(f(x_{n+m}),\ f(x_{n+m})) \\
\le\ & p(f(x_n), f(x_{n+m-1})) \\
& + p(f(x_{n+m-1}),\ f(x_{x+m})) - p(f(x_{n+m-1}),\ f(x_{n+m-1})) \\
& - p(f(x_{n+m}) - f(x_{n+m})) \\
\le\ & k^{n+m-1}(p(x_0,\ x_1) - p(x_1,\ x_1)) - p(f(x_{n+m-1}),\ f(x_{n+m})) \\
& - p(f(x_{n+3}),\ f(x_{n+m})) \\
\le\ & k^{n+m-1}(p(x_0,\ x_1) - p(x_1,\ x_1)) + p(f(x_{n+m-1}),\ f(x_{n+m-2})) \\
& + p(f(x_{n+m-2}),\ f(x_{n+m})) - p(f(x_{n+m-2}),\ f(x_{n+m-2})) \\
& - p(f(x_{n+m}) - f(x_{n+m})) \\
\le\ & (k^{n+m-1} + k^{n+m-2} + \cdots + k^n)(p(x_0,\ x_1) - p(x_1,\ x_1)) \\
=\ & \frac{k^n}{1-k}(p(x_0,\ x_1) - p(x_1,\ x_1)).
\end{aligned}
$$

By Theorem 2.10, $\langle x_n \rangle$ to be a Cauchy sequence. Since $(X,\ p)$ is a complete partial metric space, $\langle x_n \rangle$ properly converges to $x^* \in X$ say.

We now show that $x^*$ is a fixed point of *f*. For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \ge N$,

$$p(x_n,\ x^*) - p(x_n,\ x_n) < \frac{\varepsilon}{1+k}$$

and

$$p(x_n,\ x^*) - p(x^*,\ x^*) < \frac{\varepsilon}{1+k}.$$

Thus for $n \geq N$,

$$
\begin{aligned}
& p(f(x^*), x^*) - p(x^*, x^*) \\
\leq\ & p(f(x^*), f(x_n)) \\
& + p(f(x_n), x^*) - p(f(x_n), f(x_n)) - p(x^*, x^*) \\
\leq\ & (p(f(x_n), x^*) - p(f(x_n, f(x_n))) + (p(f(x^*), f(x_n)) \\
& - p(x^*, x^*)) \\
\leq\ & k(p(x_n, x^*) - p(x_n, x_n) + k(p(x^*, x_n) - p(x^*, x^*))) \\
<\ & k\left(\frac{\varepsilon}{1+k} + \frac{\varepsilon}{1+k}\right) \\
<\ & \varepsilon.
\end{aligned}
$$

Thus, as $\varepsilon > 0$ arbitrary, then

$$p(f(x^*), x^*) = p(x^*, x^*). \tag{2.2}$$

Similarly, for $n \geq N$,

$$
\begin{aligned}
& p(f(x^*), x^*) - p(f(x^*), f(x^*)) \\
\leq\ & p(f(x^*), f(x_n)) + p(f(x_n), x^*) \\
& - p(f(x_n), f(x_n)) - p(f(x^*), f(x^*)) \\
\leq\ & (p(f(x_n), x^*) - p(f(x_n), f(x_n)) + (p(f(x^*), f(x_n)) \\
& - p(f(x^*), f(x^*)) \\
\leq\ & k(p(x_n, x^*) - p(x_n, x_n) + k(p(x^*, x_n) - p(x^*, x^*))
\end{aligned}
$$

$$< k\left(\frac{\varepsilon}{1+k} + \frac{\varepsilon}{1+k}\right)$$

$$< \varepsilon.$$

Thus, as $\varepsilon > 0$ arbitrary, then

$$p(f(x^*), x^*) = p(f(x^*), f(x^*)). \qquad (2.3)$$

Thus, from (2.2), (2.3) and by (P2), $f(x^*) = x^*$ and so $f$ has been shown to have a fixed point.

It just remains to show that $x^*$ is unique.

Suppose $y^* \in X$ and $y^* = f(y^*)$. Then

$$p(x^*, y^*) - p(y^*, y^*) = p(f(x^*), f(y^*)) - p(f(y^*), p(y^*))$$

$$\leq k(p(x^*, y^*) - p(y^*, y^*)).$$

It follows $p(x^*, y^*) - p(y^*, y^*) = 0$ as $0 \leq k < 1$.

Similarly, suppose $x^* \in X$ and $x^* = f(x^*)$. Then

$$p(x^*, y^*) - p(x^*, x^*) = p(f(x^*), f(y^*)) - p(f(x^*), p(x^*))$$

$$\leq k(p(x^*, y^*) - p(x^*, x^*)).$$

It follows $p(x^*, y^*) - p(x^*, x^*) = 0$ as $0 \leq k < 1$.

According to axiom (P2), $p(x^*, y^*) = p(x^*, x^*) = p(y^*, y^*)$, thus $y^* = x^*$ is unique. □

## 3. Main Results

In this section, main result will be shown as a fixed point theorem of both Caristi [4] and Banach [8] on partial metric space. In addition, it shall be shown the prove Caristi fixed point theorem with two methods, that is, without and use Ekeland's variational principle. Similarly, for the Banach fixed point theorem.

We start with the following lemma needed to prove our main result.

**Lemma 3.1.** *Let* $(X, p)$ *be a partial metric space and the function* $\varphi : X \to [0, \infty)$ *be lower semicontinuous. For any* $x, y \in X$ *we define relation* "$\preceq_p$" *on X by*

$$x \preceq_p y \Leftrightarrow p(x, y) - p(x, x) \le \varphi(x) - \varphi(y). \tag{3.1}$$

*Then the relation* "$\preceq_p$" *is partial ordered on X.*

**Proof.** (i) It is clear that $p(x, y) - p(x, x) = 0 = \varphi(x) - \varphi(x)$ so that $x \preceq_p x$ is reflexive.

(ii) If $x \preceq_p y$, then $p(x, y) - p(x, x) \le \varphi(x) - \varphi(y)$ and if $y \preceq_p x$, then $p(y, x) - p(y, y) \le \varphi(y) - \varphi(x)$. This implies $2p(x, y) - p(x, x) - p(y, y) = 0$. Of course $p(x, y) = p(x, x) = p(y, y)$. By (P2), we obtain $x = y$.

(iii) If $x \preceq_p y$, then $p(x, y) - p(x, x) \le \varphi(x) - \varphi(y)$ and if $y \preceq_p z$, then $p(y, z) - p(y, y) \le \varphi(y) - \varphi(x)$. This implies $2p(x, z) - p(x, x) \le p(x, y) + p(y, z) - p(y, y) - p(x, x) \le \varphi(x) - \varphi(z)$ and hence $x \preceq_p z$. □

**Lemma 3.2** (Zorn's lemma)**.** *Let X be nonempty partially ordered set in which every totally set has an upper bound. Then X has at least one maximal element.*

The following is Caristi fixed point theorem type on partial metric space.

**Theorem 3.3.** *Let* $(X, p)$ *be a complete partial metric space and* $f : X \to X$ *be a mapping on X. Suppose there exists a lower semicontinuous function* $\varphi : X \to [0, \infty)$ *such that*

$$p(x, f(x)) - p(f(x), f(x)) \le \varphi(x) - \varphi(f(x)) \tag{3.2}$$

*for all* $x \in X$. *Then f has a fixed point.*

**Proof.** For any $x, y \in X$, we define the relation "$\preceq_p$" on $X$ by

$$x \preceq_p y \Leftrightarrow p(x, y) - p(y, y) \leq \varphi(x) - \varphi(y). \tag{3.3}$$

By Lemma 3.1, $(X \preceq_p)$ is a partial ordered. Let $x_0 \in X$ be an arbitrary but fixed element of $X$. Then by Zorn's lemma, we obtain totally ordered subset $M$ of $X$ containing $x_0$.

Let $M = \{x_\alpha\}_{\alpha \in I}$, where $I$ is totally ordered and

$$x_\alpha \preceq_p x_\beta \Leftrightarrow \alpha \preceq_p \beta \tag{3.4}$$

for all $\alpha, \beta \in I$.

Since $\{\varphi(x_\alpha)\}$ is decreasing net in $\mathbb{R}^+$, there exists $r \geq 0$ such that $\varphi(x_\alpha) \to r$ as $\alpha$ increases.

Let $\varepsilon > 0$ be given. Then there exists $\alpha_0 \in I$ such that for $\alpha \succeq_p \alpha_0$ we have

$$r \leq \varphi(x_\alpha) \leq \varphi(x_{\alpha_0}) \leq r + \varepsilon. \tag{3.5}$$

Let $\beta \succeq_p \alpha \succeq_p \alpha_0$. Then by (4) we obtain

$$p(x_\alpha, x_\beta) - p(x_\beta, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta) \leq r + \varepsilon - r = t \tag{3.6}$$

which implies that $\{x_\alpha\}$ is a Cauchy net in $X$ by Theorem 2.10. Since $X$ is complete, there exists $x \in X$ such that $x_\alpha \to x$ as $\alpha$ increases. From the lower semicontinuity of $\varphi$ we deduce that $\varphi(x_\alpha) \leq r$. If $\beta \succeq_p \alpha$, then $p(x_\alpha, x_\beta) - p(x_\beta, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta)$.

Letting $\beta$ as increases, we obtain

$$p(x_\alpha, x) - p(x, x) \leq \varphi(x_\alpha) - r \leq \varphi(x_\alpha) - \varphi(x) \tag{3.7}$$

which is given $x_\alpha \preceq_p x$ for $\alpha \in I$. In particular, $x_0 \preceq_p x$. Since $M$ is maximal, $x \in M$. Moreover, the condition (3) implies that

$$x_\alpha \preceq_p x \preceq_p f(x) \text{ for all } \alpha \in I. \tag{3.8}$$

Again by maximality, $f(x) \in M$. Since $x \in M$, $f(x) \preceq_p x$ and hence $f(x) = x$. □

The mapping $f$ satisfying (3.2) is called *Caristi's map*. Again we write self map $f : X \to X$ contraction on a partial metric space, as follows:

$$p(f(x), f(y)) - p(f(x), f(x)) \leq kk[p(x, y) - p(y, y)]$$

for all $x, y \in X$ and for some $k \in (0, 1)$.

For $y = f(x)$ it will be deduced as follows:

$$p(f(x), f^2(x)) - p(f(x), f(x)) \leq k[p(x, f(x)) - p(f(x), f(x))]$$

thus

$$[p(x, f(x)) - p(x, x)] - k[p(x, f(x)) - p(x, x)]$$

$$\leq [p(x, f(x)) - p(x, x)] - [p(f(x), f^2(x)) - p(f(x), f(x))].$$

Hence

$$(1-k)[p(x, f(x)) - p(x, x)]$$

$$\leq [p(x, f(x)) - p(x, x)] - [p(f(x), f^2(x)) - p(f(x), f(x))]$$

or

$$p(x, f(x)) - p(x, x)$$

$$\leq \frac{1}{1-k}[p(x, f(x)) - p(x, x)] - \frac{1}{1-k}[p(f(x), f^2(x)) - p(f(x), f(x))].$$

If the function $\varphi : X \to [0, \infty)$ is defined by

$$\varphi(x) = \frac{1}{1-k}[p(x, f(x)) - p(x, x)],$$

then we obtain

$$p(x,\ f(x)) - p(x,\ x) \leq \varphi(x) - \varphi(f(x)).$$

It appears that $f$ is a Caristi's mapping on partial metric space. Thus, the contraction mapping is a special case of Caristi's mapping.

The following will be given the Ekeland's variational principle on partial metric spaces.

**Theorem 3.4.** *Let $(X,\ p)$ be a complete partial metric space and $\varphi: X \to [0,\ \infty)$ be a lower semicontinuous function. Let $\varepsilon > 0$ and $\bar{x} \in X$ be given such that*

$$\varphi(\bar{x}) \leq \inf_{x \in X} \varphi(x) + \varepsilon.$$

*Then for a given $\delta > 0$ there exists $x^* \in X$ such that*

(a) $\varphi(x^*) \leq \varphi(\bar{x})$,

(b) $p(\bar{x},\ x^*) \leq \delta + p(x^*,\ x^*)$,

(c) $\varphi(x^*) \leq \varphi(x) + \frac{\varepsilon}{\delta}(p(x,\ x^*) - p(x^*,\ x^*))$ *for all* $x \in X \setminus \{x^*\}$.

**Proof.** For $\delta > 0$, we set $p_\delta(x,\ y) - p_\delta(y,\ y) = \frac{1}{\delta}(p(x,\ y) - p(y,\ y))$. Then $p_\delta$ is equivalent to $p$ and $(X,\ p_\delta)$ is complete. Let us define a partial ordering $\preceq$ on $X$ by

$$x \preceq y \Leftrightarrow \varphi(x) \leq \varphi(y) - \varepsilon(p_\delta(x,\ y) - p_\delta(y,\ y)). \tag{3.9}$$

It is easy to see that this ordering is (i) reflexive, that is, for all $x \in X$, $x \preceq x$; (ii) antisymmetric, that is, for all $x,\ y \in X$, $x \preceq y$ and $y \preceq x$ imply $x = y$; (iii) transitive, that is, for all $x,\ y,\ z \in X$, $x \preceq y$ and $y \preceq z$ imply $x \preceq z$.

We define a sequence $\langle E_n \rangle$ of subsets of $X$ as follows: We start $x_1 = \bar{x}$

and define

$$E_1 = \{x \in X : x \preceq x_1\};\ x_2 \in E_1 \text{ such that } \varphi(x_2) \leq \inf_{x \in E_1} \varphi(x) + \frac{\varepsilon}{2},$$

$$E_2 = \{x \in X : x \preceq x_2\};\ x_3 \in E_2 \text{ such that } \varphi(x_3) \leq \inf_{x \in E_2} \varphi(x) + \frac{\varepsilon}{2^2},$$

and inductively

$$E_n = \{x \in X : x \preceq x_n\};\ x_{n+1} \in E_n \text{ such that } \varphi(x_{n+1}) \leq \inf_{x \in E_n} \varphi(x) + \frac{\varepsilon}{2^n}.$$

Clearly, $E_1 \supset E_2 \supset E_3 \cdots$. Let $u_m \in E_n$ with $u_m \to u \in X$. Then $u_m \preceq x_n$ and so $\varphi(u_m) \leq \varphi(x_n) - \varepsilon(p_\delta(u_m, x_n) - p_\delta(x_n, x_n))$. Therefore

$$\begin{aligned}\varphi(u) &\leq \lim_{m\to\infty}\inf \varphi(u_m)\\ &\leq \varphi(x_n) - \varepsilon \lim_{m\to\infty}\inf(p_\delta(u_m, x_n) - p_\delta(x_n, x_n))\\ &\leq \varphi(x_n) - \varepsilon(p_\delta(u, x_n) - p_\delta(x_n, x_n)).\end{aligned}$$

Thus $u \in E_n$. We conclude that each $E_n$ is closed.

Take any point $x \in E_n$, one on hand $x \preceq x_n$, implies that

$$\varphi(x) \leq \varphi(x_n) - \varepsilon(p_\delta(x, x_n) - p_\delta(x_n, x_n)). \tag{3.10}$$

We observe that $x$ also belongs to $E_{n-1} \supset E_n$. So it is one of the points which entered in the competition when we picked $x_n$. Therefore,

$$\varphi(x_n) \leq \inf_{y \in E_{n-1}} \varphi(y) + \frac{\varepsilon}{2^{n-1}} \leq \varphi(x) + \frac{\varepsilon}{2^{n-1}}. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\varphi(x) + \varepsilon(p_\delta(x, x_n) - p_\delta(x_n, x_n)) \leq \varphi(x) + \frac{\varepsilon}{2^{n-1}}.$$

It follows that

$$p_\delta(x, x_n) - p_\delta(x_n, x_n) \le 2^{-n+1}$$

for all $x \in E_n$.

Which resulted $D(E_n) \le 2^{-n}$ and hence $D(E_n) \to 0$ $(n \to \infty)$.

Since $(X, p_\delta)$ is complete and $\langle E_n \rangle$ is a decreasing sequence of closed sets, by Theorem 2.13, we infer that

$$\bigcap_{n=1}^{\infty} E_n = \{x^*\}.$$

Since $x^* \in E_1$, we have

$$x^* \preceq x_1 = \bar{x} \Leftrightarrow \varphi(x^*) \le \varphi(\bar{x}) - \varepsilon(p_\delta(x^*, \bar{x}) - p_\delta(\bar{x}, \bar{x})) \le \varphi(\bar{x}).$$

Hence, (a) is proved.

Now, we write

$$\begin{aligned} p_\delta(\bar{x}, x_n) - p_\delta(x_n, x_n) = {} & p_\delta(x_1, x_n) - p_\delta(x_n, x_n) \\ & \le \sum_{i=1}^{n-1} [p_\delta(x_i, x_{i+1}) - p_\delta(x_i, x_i)] \\ & \le \sum_{i=1}^{n-1} 2^{-i+1} \end{aligned}$$

and taking limit as $n \to \infty$, we obtain

$$\frac{1}{\delta}(p(\bar{x}, x^*) - p(x^*, x^*)) = p_\delta(\bar{x}, x^*) - p_\delta(x^*, x^*) \le 1$$

and so $(p(\bar{x}, x^*) \le \delta + p(x^*, x^*))$. This proves (b).

Finally, let $x \ne x^*$, of course $x \notin \bigcap_{n=1}^{\infty} E_n$, so $x \npreceq x^*$, which means that

$$\varphi(x) > \varphi(x^*) - \varepsilon[p_\delta(x, x^*) - p_\delta(x^*, x^*)]$$

$$= \varphi(x^*) - \frac{\varepsilon}{\delta}[p(x, x^*) - p(x^*, x^*)]$$

and hence (c) is proved. □

We now present, so called the weak formulation of Ekeland's variational principle.

**Corollary 3.5.** *Let $(X, p)$ be a complete partial metric space and $\varphi : X \to [0, \infty)$ be a lower semicontinuous function. Then for any given $\varepsilon > 0$ there exists $x^* \in X$ such that*

$$\varphi(x^*) \leq \inf_{x \in X} \varphi(x) + \varepsilon$$

*and*

$$\varphi(x^*) < \varphi(x) + \varepsilon[p(x, x^*) - p(x^*, x^*)]$$

*for all $x \in X \setminus \{x^*\}$.*

**Definition 3.6.** Let $(X, p)$ be a partial metric space. The function $f : X \to X$ is called *continuous* at the point $x_0$ if for any sequence $\langle x_n \rangle$ in $X$ converges to $x_0$, then a sequence $\langle f(x_n) \rangle$ converges to $f(x_0)$.

**Lemma 3.7.** *Let $(X, p)$ be a partial metric space and the function $f : X \to X$. Then for each $x \in X$, the function $\varphi_x : X \to [0, \infty)$ defined by*

$$\varphi_x(y) = p(x, f(y)).$$

*If f is contraction, then the function $\varphi_x$ is continuous on X.*

**Proof.** Assume that a sequence $y_n$ converges to $y$ in $X$. Then $\lim_{n\to\infty} p(y_n, y) - p(y, y) = 0$. For each $x \in X$ and $k \in (0, 1)$ and by (P4), we have

$$\begin{aligned}|\varphi_x(y_n)-\varphi_x(y)| &= |p(x, f(y_n))-p(x, f(y))| \\ &\le |p(f(y), f(y_n))-p(f(y), f(y))| \\ &= p(f(y), f(y_n))-p(f(y), f(y)) \\ &< k(p(y_n, y)-p(y, y)).\end{aligned}$$

This yields $\lim_{n\to\infty}\varphi_x(y_n)=\varphi_x(y)$ because $\lim_{n\to\infty} p(y_n, y)-p(y, y) = 0.$ □

Now, we will present the proof of fixed point theorem using Ekeland's variational principle.

As first applications of Ekeland's variational principle, we prove Caristi's fixed point theorem version.

**Theorem 3.8.** *Let $(X, p)$ be a complete partial metric space and $f: X \to X$ be a mapping on X. Suppose there exists a lower semicontinuous function $\varphi: X \to [0, \infty)$ such that*

$$p(x, f(x))-p(f(x), f(x)) \le \varphi(x)-\varphi(f(x)) \tag{3.12}$$

*for all $x \in X$. Then f has a fixed point.*

**Proof.** By using Corollary 3.5 with $\varepsilon = 1$, we obtain $x^* \in X$ such that

$$\varphi(x^*) < \varphi(x)+[p(x, x^*)-p(x, x)] \tag{3.13}$$

for all $x \in X\setminus\{x^*\}$.

We assume for all $y = f(x^*) \in X$ such that $y \ne x^*$. Then from (3.12) and (3.13), we have

$$p(x^*, y)-p(y, y) \le \varphi(x^*)-\varphi(y)$$

and

$$\varphi(x^*) < \varphi(y)+[p(y, x^*)-p(y, y)]$$

which cannot hold simultaneously. Hence, $x^* = f(x^*)$. □

As second applications of Ekeland's variational principle, we prove the well-known Banach contraction theorem.

**Theorem 3.9.** *Let $(X, p)$ be a complete partial metric space and $f : X \to X$ be a contraction mapping. Then f has a unique fixed point in X.*

**Proof.** Consider the function $\varphi : X \to [0, \infty)$ defined by

$$\varphi(x) = p(x, f(x)),$$

for all $x \in X$.

By Lemma 3.7, φ is a continuous on *X*. Choose $\varepsilon > 0$ such that $0 < \varepsilon < 1 - k$, where $k \in (0, 1)$. By Corollary 3.5, there exists $x^* \in X$ such that

$$\varphi(x^*) < \varphi(x) + \varepsilon[p(x, x^*) - p(x, x)]$$

for all $x \in X$.

Putting $x = f(x^*)$, we have

$$\begin{aligned}
&p(x^*, f(x^*)) \\
&\leq p(x, f(x)) + \varepsilon[p(x, x^*) - p(x, x)] \\
&= (f(x^*), f(f(x^*))) + \varepsilon[p(f(x^*), x^*) - p(f(x^*), f(x^*))] \\
&\leq k[p(x^*, f(x^*)) - p(f(x^*), f(x^*))] + \varepsilon[p(f(x^*), x^*) - p(f(x^*), f(x^*))] \\
&= (k + \varepsilon)[p(x^*, f(x^*)) - p(f(x^*), f(x^*))] \\
&\leq (k + \varepsilon)p(x^*, f(x^*)).
\end{aligned}$$

If $x^* \neq f(x^*)$, then we obtain $1 \leq (k + \varepsilon)$, which contradicts to our assumption that $1 > (k + \varepsilon)$. Therefore, we have $x^* = f(x^*)$. The uniqueness of $x^*$ can be proved as in Theorem 2.15. □

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