**Far East Journal of Mathematical Sciences (FJMS)**

Published Online: December 2015
http://dx.doi.org/10.17654/MS099010133
Volume 99, Number 1, 2016, Pages 133-152 **ISSN: 0972-0871**

# SOME CONVERGENCE RESULTS FOR MODIFIED ABBAS AND NAZIR ITERATION PROCESS ON A NONLINEAR DOMAIN

**Duangkamon Kitkuan**

Department of Mathematics
Faculty of Science and Technology
Rambhai Barni Rajabhat University
Chanthaburi 22000, Thailand
e-mail: or_duangkamon@hotmail.com

## Abstract

In this paper, we establish some fixed point theorems by using a modified Abbas and Nazir iteration process for total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces, a nonlinear domain. The results presented here extend and improve some well-known results in the current literature.

## 1. Introduction

Fixed point theory is a branch of nonlinear analysis which has attracted much attention in recent times due to its possible applications. Approximating fixed points of nonlinear mappings using different iteration processes on different domains has remained at the heart of fixed point theory. Nonexpansive mappings constitute one of the most important classes of nonlinear mappings which have remained a crucial part of such studies.

Received: August 15, 2015; Revised: September 7, 2015; Accepted: September 21, 2015
2010 Mathematics Subject Classification: 47H09, 47H10.
Keywords and phrases: nonlinear domain, approximating fixed point, hyperbolic spaces.
Communicated by Simeon Reich; Editor: JP Journal of Fixed Point Theory and Applications: Published by Pushpa Publishing House.

The Picard iteration process $\{x_n\}$ is defined by

$$x_{n+1} = Tx_n, \quad n \geq 1. \tag{1.1}$$

In 1953, Mann defined the Mann iteration process [16] as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence of positive numbers in $(0, 1)$.

In 2007, Agarwal et al. defined the $S$-iteration process [2] as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $(0, 1)$.

In 2013, Abbas and Nazir defined the iteration process [1] as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Ty_n + \alpha_n Tz_n, \\ y_n &= (1 - \beta_n)Tx_n + \beta_n Tz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $(0, 1)$ and converge faster than all Picard, Mann and $S$-iterations for nonexpansive mappings.

Khan [24] proved the strong and $\Delta$-convergence theorems of the Abbas and Nazir iteration process on a nonlinear domain in a hyperbolic space for a nonexpansive mapping.

The purpose of this paper is to study the iterative scheme defined as follows. Let $C$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T : C \to C$ be a uniformly $L$-Lipschitzian and asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$x_{n+1} = W(T^n y_n, T^n z_n, \alpha_n),$$

$$y_n = W(T^n x_n, T^n z_n, \beta_n),$$

$$z_n = W(x_n, T^n x_n, \gamma_n). \tag{1.5}$$

In this paper, we prove the strong and $\Delta$-convergence of a three-step iteration process for totally asymptotically nonexpansive maps on a nonlinear domain in hyperbolic spaces.

## 2. Preliminaries

Let $(X, d)$ be a metric space and $C$ be its nonempty subset. Let $T : C \to C$ be a mapping. A point $x \in C$ is called a *fixed point* of $T$ if $Tx = x$. We will also denote by $F(T)$ the set of fixed points of $T$, that is, $F(T) = \{x \in C : Tx = x\}$. $T$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C. \tag{2.1}$$

A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 0$ such that

$$d(T^n x, T^n y) \leq (1 + k_n) d(x, y), \quad \forall n \geq 1, \quad x, y \in C. \tag{2.2}$$

A mapping $T : C \to C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y), \quad \forall n \geq 1, \quad x, y \in C. \tag{2.3}$$

A mapping $T : C \to C$ is said to be *totally asymptotically nonexpansive* if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \mu_n \zeta(d(x, y)) + \xi_n, \quad \forall n \geq 1, \quad x, y \in C. \tag{2.4}$$

It follows from the above definitions that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1, \ \forall n \geq 1$ and that each asymptotically nonexpansive mapping is a totally asymptotically nonexpansive mapping with $\mu_n = k_n - 1, \ \xi_n = 0, \ \forall n \geq 1, \ \zeta(t) = t, \ \forall t \geq 0.$ Moreover, each asymptotically nonexpansive mapping is a uniformly $L$-Lipschitzian mapping with $L = \sup_{n \geq 1}\{k_n\}$. However, the converse of these statements is not true, in general.

**Example 2.1.** Let $X = [0, 3], \ d(x, y) = |x - y|, \ \zeta(t) = t, \ \forall t \geq 0,$ $\mu_n = 1, \ \xi_n = 0, \ \forall n \geq 1$ and define $T$ by

$$T(x) = \begin{cases} 0, & \text{if } x \neq 3, \\ 1, & \text{if } x = 3. \end{cases}$$

By taking $x = 3$ and $y = 2.5,$ we have

$$d(T^n(3), T^n(2.5)) \leq d(3, 2.5) + \zeta(d(3, 2.5)) = 1.$$

But

$$d(T(3), T(2.5)) = 1 \not\leq 0.5 = d(3, 2.5).$$

Therefore, $T$ is a totally asymptotically nonexpansive mapping, but $T$ is not nonexpansive mapping.

A mapping $T$ from a subset $C$ of a metric space $(X, d)$ into itself is semi-compact if every bounded sequence $\{x_n\} \subset C$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to *satisfy* condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$ (see [21]).

Hyperbolic spaces in general and uniformly convex hyperbolic spaces in particular originate in [19]. An important example of such spaces is the Hilbert ball with the hyperbolic metric, which is studied in detail in the book by Goebel and Reich [25]. See also the recent paper by Kopecka and Reich [26].

A hyperbolic space [13] is a triple $(X, d, W)$, where $(X, d)$ is a metric space and a mapping $W : X^2 \times [0, 1] \to X$ satisfying

$(W1)$ $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha)d(u, y)$,

$(W2)$ $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,

$(W3)$ $W(x, y, \alpha) = W(y, x, (1-\alpha))$,

$(W4)$ $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$

for all $x, y, z \in X$ and $\alpha, \beta \in [0, 1]$.

The class of hyperbolic spaces in the sense of Kohlenbach [13] contains all normed linear spaces and convex subsets thereof as well as Hadamard manifolds and CAT(0) spaces in the sense of Gromov (see [5]).

A hyperbolic space $(X, d, W)$ is uniformly convex [22], if for all $r > 0$ and $\varepsilon \in [0, 2)$, there exists $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r \tag{2.5}$$

whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called *modulus of uniform convexity* of *X*. We call $\eta$ to be *monotone* if it decreases with *r* (for a fixed ε).

**Lemma 2.2** (See [11]). *Let* $(X, d, W)$ *be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity* $\eta$. *Let* $x \in X$ *and* $\{\alpha_n\}$ *be a sequence in* $[a, b]$ *for some* $b, c \in (0, 1)$. *If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in X such that* $\limsup_{x\to\infty} d(x_n, x) \leq r$, $\limsup_{x\to\infty} d(y_n, x) \leq r$ *and* $\lim_{x\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ *for some* $r \geq 0$, *then* $\lim_{x\to\infty} d(x_n, y_n) = 0$.

The concept of $\Delta$-convergence in a metric space was introduced by Lim [15]. $\Delta$-convergence in CAT(0) spaces has been investigated initially by Dhompongsa and Panyanak [8]. See also Khan and Abbas [12] and Khan et al. [11].

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $X$. For $x \in X$, we define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{x\to\infty} d(x, x_n)$ for all $x \in C$. The asymptotic radius of $\{x_n\}$ with respect to $C \subset X$ is defined as $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$.

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to $C \subset X$ is defined as

$$A_C(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in C\}.$$

**Lemma 2.3** (Leustean [14])**.** *Let $(X, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

A sequence $\{x_n\}$ in $X$ is said to $\Delta$-*convergence* $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta$-limit of $\{x_n\}$ as $\Delta - \lim_n x_n = x$.

**Lemma 2.4** (See [11])**.** *Let C be a nonempty closed convex subset of uniformly convex hyperbolic space and let $\{x_n\}$ be a bounded sequence in C such that $A_C(\{x_n\}) = \{y\}$ and $r_C(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{n\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{n\to\infty} y_m = y$.*

**Lemma 2.5** (See [6])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} = (1 + b_n)a_n + c_n, \quad \forall n \geq 1.$$

*If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

## 3. Main Results

In this section, we prove our main theorems.

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T : C \to C$ be a uniformly $L$-Lipschitzian and totally asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.5) *and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$,

(ii) *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b]$,

(iii) *there exists a constant* $M > 0$ *such that* $\zeta(r) \leq Mr, \ \forall r > 0$.

*Then* $\lim_{n\to\infty} d(x_n, u)$ *and* $\lim_{n\to\infty} d(x_n, F(T))$ *exist for all* $u \in F(T)$.

**Proof.** Let $u \in F(T)$. By (1.5), $T$ be a totally asymptotically nonexpansive mapping and condition (iii), we have

$$\begin{aligned} d(z_n, u) &= d(W(x_n, T^n x_n, \gamma_n), u) \\ &\leq (1 - \gamma_n) d(x_n, u) + \gamma_n d(T^n x_n, u) \\ &\leq (1 - \gamma_n) d(x_n, u) + \gamma_n [d(x_n, u) + \mu_n \zeta(d(x_n, u)) + \xi_n] \\ &= d(x_n, u) + \gamma_n \mu_n \zeta(d(x_n, u)) + \gamma_n \xi_n \\ &\leq (1 + \gamma_n \mu_n M) d(x_n, u) + \gamma_n \xi_n \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d(y_n, u) &= d(W(T^n x_n, T^n z_n, \beta_n), u) \\ &\leq (1 - \beta_n) d(T^n x_n, u) + \beta_n d(T^n z_n, u) \end{aligned}$$

$$\leq (1-\beta_n)[d(x_n, u) + \mu_n\zeta(d(x_n, u)) + \xi_n]$$

$$+ \beta_n[d(z_n, u) + \mu_n\zeta(d(z_n, u)) + \xi_n]$$

$$\leq (1-\beta_n)[(1+\mu_n M)d(x_n, u) + \xi_n]$$

$$+ \beta_n[(1+\mu_n M)d(z_n, u) + \xi_n]. \tag{3.2}$$

Substituting (3.1) into (3.2), we have

$$d(y_n, u) \leq (1-\beta_n)[(1+\mu_n M)d(x_n, u) + \xi_n]$$

$$+ \beta_n[(1+\mu_n M)\{(1+\gamma_n\mu_n M)d(x_n, u) + \gamma_n\xi_n\} + \xi_n]$$

$$\leq (1 + \mu_n M(1 + \beta_n\gamma_n(1+\mu_n M))d(x_n, u))$$

$$+ (1 + \beta_n\gamma_n(1+\mu_n M))\xi_n. \tag{3.3}$$

Thus,

$$d(x_{n+1}, u) = d(W(T^n y_n, T^n z_n, \alpha_n), u)$$

$$\leq (1-\alpha_n)d(T^n y_n, u) + \alpha_n d(Tz_n, u)$$

$$\leq (1-\alpha_n)[d(y_n, u) + \mu_n\zeta(d(y_n, u)) + \xi_n]$$

$$+ \alpha_n[d(z_n, u) + \mu_n\zeta(d(z_n, u)) + \xi_n]$$

$$\leq (1-\alpha_n)[(1+\mu_n M)d(y_n, u) + \xi_n]$$

$$+ \alpha_n[(1+\mu_n M)d(z_n, u) + \xi_n]. \tag{3.4}$$

Combining (3.1), (3.3) and (3.4), we have

$$d(x_{n+1}, u) \leq (1+\omega_n)d(x_n, u) + \psi_n, \quad \forall n \geq 1 \text{ and } u \in F(T) \tag{3.5}$$

and

$$d(x_{n+1}, F(T)) \leq (1+\omega_n)d(x_n, F(T)) + \psi_n, \quad \forall n \geq 1, \tag{3.6}$$

where

$$\omega_n = \mu_n M(2 + \alpha_n\gamma_n + \beta_n\gamma_n + \mu_n M(1 + \alpha_n\gamma_n + 2\beta_n\gamma_n + \beta_n\gamma_n\mu_n M))$$

and

$$\psi_n = \mu_n M(2 + \beta_n \gamma_n + \mu_n M(1 + \alpha_n \gamma_n + 2\beta_n \gamma_n + \beta_n \gamma_n \mu_n M)).$$

Furthermore, using condition (i), we have

$$\sum_{n=1}^{\infty} \mu_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \xi_n < \infty. \tag{3.7}$$

Consequently, a combination of (3.5), (3.6), (3.7) and by Lemma 2.5 shows that $\lim_{n \to \infty} d(x_n, u)$ and $\lim_{n \to \infty} d(x_n, F(T))$ exist for all $u \in F(T)$. The proof is completed. □

**Theorem 3.2.** *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity* $\eta$. *Let* $T : C \to C$ *be a uniformly L-Lipschitzian and totally asymptotically nonexpansive mapping with* $F(T) \neq \emptyset$. *Suppose that* $\{x_n\}$ *is defined by the iteration process* (1.5) *and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$,

(ii) *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b]$,

(iii) *there exists a constant* $M > 0$ *such that* $\zeta(r) \leq Mr, \ \forall r > 0$.

*Then* $\lim_{n \to \infty} d(x_n, Tx_n) = 0$.

**Proof.** By Theorem 3.1, $\lim_{n \to \infty} d(x_n, u)$ exists for all $u \in F(T)$. Assume that

$$\lim_{n \to \infty} d(x_n, u) = r \geq 0. \tag{3.8}$$

Now, taking $\limsup$ on both sides in inequality (3.3), we get

$$\begin{aligned}\limsup_{n\to\infty} d(y_n, u) &\le \limsup_{n\to\infty}(1+\mu_n M(1+\beta_n\gamma_n(1+\mu_n M))d(x_n, u)) \\ &\quad + (1+\beta_n\gamma_n(1+\mu_n M))\xi_n \\ &\le \lim_{n\to\infty} d(x_n, u) \\ &= r. \qquad (3.9)\end{aligned}$$

Noting

$$\begin{aligned}d(T^n y_n, u) &= d(T^n y_n, T^n u) \\ &\le d(y_n, u) + \mu_n\zeta(d(y_n, u)) + \xi_n \\ &\le (1+\mu_n M)d(y_n, u) + \xi_n, \quad \forall n \ge 1 \qquad (3.10)\end{aligned}$$

and

$$\begin{aligned}d(T^n z_n, u) &= d(T^n z_n, T^n u) \\ &\le d(z_n, u) + \mu_n\zeta(d(z_n, u)) + \xi_n \\ &\le (1+\mu_n M)d(z_n, u) + \xi_n, \quad \forall n \ge 1, \qquad (3.11)\end{aligned}$$

by (3.1), (3.8) and (3.9), we have

$$\limsup_{n\to\infty} d(T^n y_n, u) \le r \quad \text{and} \quad \limsup_{n\to\infty} d(T^n z_n, u) \le r. \qquad (3.12)$$

Besides, by (3.4) and (3.5), we get

$$d(x_{n+1}, u) = d(W(T^n y_n, T^n z_n, \alpha_n), u) \le (1+\omega_n)d(x_n, u) + \psi_n$$

which yields that

$$\lim_{n\to\infty} d(x_{n+1}, u) = \lim_{n\to\infty} d(W(T^n y_n, T^n z_n, \alpha_n), u) = r. \qquad (3.13)$$

By (3.12), (3.13) and Lemma 2.4, we have

$$\lim_{n\to\infty} d(T^n y_n, T^n z_n) = 0. \tag{3.14}$$

Next,

$$\begin{aligned} d(x_{n+1}, u) &= d(W(T^n y_n, T^n z_n, \alpha_n), u) \\ &\le (1-\alpha_n) d(T^n y_n, u) + \alpha_n d(T^n z_n, u) \\ &\le (1-\alpha_n) d(T^n y_n, u) + \alpha_n d(T^n z_n, T^n y_n) + \alpha_n d(T^n y_n, u) \\ &\le d(T^n y_n, u) + \alpha_n d(T^n z_n, T^n y_n) \\ &\le (1+\mu_n M) d(y_n, u) + \xi_n + \alpha_n d(T^n z_n, T^n y_n), \end{aligned} \tag{3.15}$$

we have $\liminf_{n\to\infty} d(y_n, u) \ge r$ and from (3.7), $\limsup_{n\to\infty} d(y_n, u) \le r$, it yields that $\lim_{n\to\infty} d(y_n, u) = r$. This implies that

$$\lim_{n\to\infty} d(W(T^n x_n, T^n z_n, \beta_n), u) = r. \tag{3.16}$$

Since $\limsup_{n\to\infty} d(T^n x_n, u) \le r$ and $\limsup_{n\to\infty} d(T^n z_n, u) \le r$, it follows from Lemma 2.4 that

$$\lim_{n\to\infty} d(T^n x_n, T^n z_n) = 0. \tag{3.17}$$

Also,

$$\begin{aligned} d(y_n, u) &= d(W(T^n x_n, T^n z_n, \beta_n), u) \\ &\le (1-\beta_n) d(T^n x_n, u) + \beta_n d(T^n z_n, u) \\ &\le (1-\beta_n)[d(T^n x_n, T^n z_n) + d(T^n z_n, u)] + \beta_n d(T^n z_n, u) \\ &\le (1-\beta_n) d(T^n x_n, T^n z_n) + d(T^n z_n, u) \\ &\le (1-\beta_n) d(T^n x_n, T^n z_n) + (1+\mu_n M) d(z_n, u) + \xi_n. \end{aligned} \tag{3.18}$$

By virtue of (3.17) that

$$\liminf_{n\to\infty} d(z_n, u) \geq r,$$

and since (3.1),

$$d(z_n, u) \leq (1 + \gamma_n \mu_n M) d(x_n, u) + \gamma_n \xi_n,$$

we have $\limsup_{n\to\infty} d(z_n, u) \leq r$. Therefore,

$$\lim_{n\to\infty} d(z_n, u) = \lim_{n\to\infty} d(W(x_n, T^n x_n, \gamma_n), u) = r. \tag{3.19}$$

Thus, from Lemma 2.4, we obtain

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{3.20}$$

By (3.14), we have

$$d(x_{n+1}, y_n) \leq d(W(T^n y_n, T^n z_n, \alpha_n), y_n) \leq \alpha_n d(T^n y_n, T^n z_n) \to 0 \quad \text{as } n \to \infty.$$

Similarly, we have

$$d(y_n, z_n) \leq d(W(T^n x_n, T^n z_n, \beta_n), z_n) \leq (1 - \beta_n) d(T^n x_n, T^n z_n) \to 0 \quad \text{as } n \to \infty$$

and

$$d(z_n, x_n) \leq d(W(x_n, T^n x_n, \gamma_n), x_n) \leq \gamma_n d(T^n x_n, x_n) \to 0 \text{ as } n \to \infty.$$

It follows that

$$d(x_{n+1}, x_n) \leq d(x_{n+1}, y_n) + d(y_n, z_n) + d(z_n, x_n) \to 0 \text{ as } n \to \infty. \tag{3.21}$$

Since $T$ is uniformly $L$-Lipschitzian, we get

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\ &\quad + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\leq (1 + L) d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + Ld(T^n x_n, x_n). \end{aligned}$$

Therefore, (3.20) and (3.21) imply that

$$d(x_n, Tx_n) = 0. \tag{3.22}$$

The proof is completed. □

**Theorem 3.3.** *Let $C$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T : C \to C$ be a uniformly $L$-Lipschitzian and totally asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.5) *and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$,

(ii) *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b]$,

(iii) *there exists a constant* $M > 0$ *such that* $\zeta(r) \leq Mr, \ \forall r > 0$.

*Then the sequence $\{x_n\}$ $\Delta$-converges to a fixed point of $T$.*

**Proof.** By Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, u)$ exists. This implies that the sequence $\{x_n\}$ is bounded. Therefore, by Lemma 2.3, $\{x_n\}$ has a unique asymptotic center $A_C(\{x_n\}) = \{x\}$. Let $\{y_n\}$ be any subsequence of $\{x_n\}$ such that $A_C(\{y_n\}) = \{y\}$. By Theorem 3.2, we have

$$\lim_{n\to\infty} d(y_n, Ty_n) = 0. \tag{3.23}$$

We claim that $y \in F(T)$, we define a sequence $\{a_m\}$ in $C$ by $a_m = T^m y$. Then one has

$$d(a_m, y_n) \leq d(T^m y, T^m y_n) + d(T^m y_n, T^{m-1} y_n) + \cdots + d(Ty_n, y_n)$$

$$\leq d(y,\ y_n) + \mu_n \zeta(d(y,\ y_n)) + \xi_n + \sum_{j=1}^{m} d(T^j y_n,\ T^{j-1} y_n)$$

$$\leq (1 + \mu_n M) d(y,\ y_n) + \xi_n + \sum_{j=1}^{m} d(T^j y_n,\ T^{j-1} y_n). \quad (3.24)$$

By (3.23), one gets

$$\limsup_{n\to\infty} d(a_n,\ y_n) \leq \limsup_{n\to\infty} d(y,\ y_n) = r(y,\ \{y_n\})$$

which yields that

$$|\, r(a_m,\ \{y_n\}) - r(y,\ \{y_n\})\,| \to 0 \text{ as } m \to 0.$$

It follows from Lemma 2.4 that $\lim_{m\to\infty} T^m y = y$. Utilizing the uniform continuity of $T$, $Ty = T(\lim_{m\to\infty} T^m y) = \lim_{m\to\infty} T^{m+1} y = y$. That is, $y \in F(T)$. By the uniqueness of asymptotic centers, $Tx = x$. Similarly, we can prove $Ty = y$. Suppose that $x \neq y$. Then by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\lim_{n\to\infty} d(y_n,\ y) &\leq \limsup_{n\to\infty} d(y_n,\ x) \\
&\leq \limsup_{n\to\infty} d(x_n,\ x) \\
&< \limsup_{n\to\infty} d(x_n,\ y) \\
&= \limsup_{n\to\infty} d(y_n,\ y), \qquad (3.25)
\end{aligned}$$

a contradiction. Therefore, $x = y$. Hence, $A(\{y_n\}) = \{y\}$ for all subsequences $\{y_n\}$ of $\{x_n\}$, this is, $\{x_n\}$ $\Delta$-converges to a fixed point of $T$. The proof is completed. □

**Theorem 3.4.** *Let $C$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T : C \to C$ be a uniformly $L$-Lipschitzian and totally asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.5) *and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$,

(ii) *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b]$,

(iii) *there exists a constant* $M > 0$ *such that* $\zeta(r) \leq Mr, \forall r > 0$.

*Then the sequence $\{x_n\}$ converges strongly to a point $u \in F(T)$ if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

**Proof.** Since $0 \leq d(x_n, F(T)) \leq d(x_n, u)$. If $\{x_n\}$ converges to $u \in F(T)$, then $\lim_{n\to\infty} d(x_n, u) = 0$. That is, we get $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. By Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, F(T))$ exists and hence by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence, it follows from (3.5),

$$d(x_{n+1}, u) \leq (1 + \omega_n)d(x_n, u) + \psi_n, \quad \forall n \geq 1 \quad \text{and} \quad u \in F(T),$$

where $\sum_{n=1}^{\infty} \omega_n < \infty$ and $\sum_{n=1}^{\infty} \psi_n < \infty$. Thus, for any positive integers $m, n$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, u) + d(u, x_n) \\ &\leq (1 + \omega_{n+m-1})d(x_{n+m-1}, u) + \psi_{n+m-1} + d(x_n, u). \end{aligned}$$

For each $x \geq 0$, $1 + x \leq e^{x}$, we have

$$
\begin{aligned}
& d(x_{n+m}, x_n) \\
\leq\ & e^{\omega_{n+m-1}} d(x_{n+m-1}, u) + \psi_{n+m-1} + d(x_n, u) \\
\leq\ & e^{\omega_{n+m-1} + \omega_{n+m-2}} d(x_{n+m-2}, u) + e^{\omega_{n+m-1}} \psi_{n+m-2} \psi_{n+m-1} + d(x_n, u) \\
\leq\ & \cdots \\
\leq\ & e^{\sum_{i=n}^{m+n-1} \omega_i} d(x_n, u) + e^{\sum_{i=n+1}^{m+n-1} \omega_i} \psi_n + e^{\sum_{i=n+2}^{m+n-2} \omega_i} \psi_{n+1} + \cdots \\
& + e^{\omega_{n+m-1}} \psi_{m+n-2} + \psi_{m+n-1} + d(x_n, u) \\
\leq\ & (1 + A) d(x_n, u) + A \sum_{i=n}^{n+m-1} \psi_i,
\end{aligned}
$$

where $A = e^{\sum_{i=n}^{\infty} \omega_i} < \infty$. Hence, we have

$$
d(x_{n+m}, x_n) \leq (1 + A) d(x_n, F(T)) + A \sum_{i=n}^{n+m-1} \psi_i \to 0 \text{ as } m, n \to \infty.
$$

Thus, $\{x_n\}$ is a Cauchy sequence in a closed subset of a complete hyperbolic space $X$, it is complete. We assume that $\{x_n\}$ converges strongly to a point $u^* \in C$. Now, $d(x_n, F(T)) = 0$ gives that $d(u^*, F(T)) = 0$. Since $F(T)$ is closed, we obtain $u^* \in F(T)$. The proof is completed. □

**Theorem 3.5.** *Let $C$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T : C \to C$ be a uniformly L-Lipschitzian and totally asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.5) *and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$,

(ii) *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b]$,

(iii) *there exists a constant* $M > 0$ *such that* $\zeta(r) \leq Mr, \forall r > 0$.

*If T is semi-compact, then the sequence* $\{x_n\}$ *converges strongly to a fixed point of T.*

**Proof.** By Theorem 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and since $T$ is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to u \in C$. Therefore, $d(u, Tu) = \lim_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$. Hence, $u \in F(T)$. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, u)$ exists and $x_{n_k} \to u \in F(T)$. The proof is completed. □

**Theorem 3.6.** *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity* $\eta$. *Let* $T : C \to C$ *be a uniformly L-Lipschitzian and totally asymptotically nonexpansive mapping with* $F(T) \neq \emptyset$. *Suppose that* $\{x_n\}$ *is defined by the iteration process* (1.5) *and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$,

(ii) *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b]$,

(iii) *there exists a constant* $M > 0$ *such that* $\zeta(r) \leq Mr, \forall r > 0$.

*If T satisfies condition* (I), *then the sequence* $\{x_n\}$ *converges strongly to a fixed point of T.*

**Proof.** By Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, u)$ exists. From condition (I) and Theorem 3.2, we have $\lim_{n\to\infty} f(d(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0.$ Since $f$ is a non-decreasing function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty),$ it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0.$ Now, Theorem 3.4 implies that $\{x_n\}$ converges strongly to a point $u \in F(T).$ The proof is completed. □

## Acknowledgement

The author thanks the referee for comments and suggestions on this manuscript.

## References

[1] M. Abbas and T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mathematicki Vesnik 106 (2013), 1-12.

[2] R. P. Agarwal, D. O'Regan and D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8(1) (2007), 61-79.

[3] H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, New York, NY, USA, 2011.

[4] V. Berinde, On the convergence of Ishikawa iterations in the class of quasi contractive operators, Acta Mathematica Universitatis Comenianae 73(1) (2004), 119-126.

[5] M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Springer, Berlin, 1999.

[6] S. S. Chang, L. Wang, H. W. J. Lee and C. Chan, Strong and Δ-convergence for mixed type total asymptotically nonexpansive mappings in CAT(0) spaces, Fixed Point Theory Appl. 2013, 2013:122, 16 pp.

[7] R. Chugh, V. Kumar and S. Kumar, Strong convergence of a new three step iterative scheme in Banach spaces, Am. J. Comp. Math. 2 (2012), 345-357. DOI: 10.42366/ajcm.2012.24048.

[8] S. Dhompongsa and B. Panyanak, On Δ-convergence theorems in CAT(0) spaces, Comp. Math. Appl. 56(10) (2008), 2572-2579.

[9] H. Fukhar-ud-din, A. Kalsoon and M. A. A. Khan, Existence and higher arity iteration for total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces (2013) arXiv:1312.2418v2 [math.FA].

[10] K. Goebel and W. A. Kirk, Iteration processes for nonexpansive mappings, S. P. Singh, S. Thomeier and B. Watson, eds., Topological Methods in Nonlinear Functional Analysis, Contemp. Math. Am. Math. Soc., Providence, Vol. 21, 1983, pp. 115-123.

[11] A. R. Khan, H. Fukhar-ud-din and M. A. A. Khan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, Fixed Point Theory Appl. 2012, 2012:54, 12 pp.

[12] S. H. Khan and M. Abbas, Strong and Δ-convergence of some iterative schemes in CAT(0) spaces, Comput. Math. Appl. 61(1) (2011), 109-116.

[13] U. Kohlenbach, Some logical metatheorems with applications in function analysis., Trans. Amer. Math. Soc. 357(1) (2004), 89-124.

[14] L. Leustean, Nonexpansive iterations in uniformly convex *W*-hyperbolic spaces, Contemp. Math. 513 (2010), 193-210.

[15] T. C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60 (1976), 179-182.

[16] W. R. Mann, Mean value methods in iteration, Proc. Am. Math. Soc. 4(3) (1953), 506-510. DOI: 10.1090/S0002-9939-1953-0054846-3.

[17] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251(1) (2000), 217-229. DOI: 10.1006/jmaa.2000.7042

[18] W. Phuenrattana and S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor, and SP iterations for continuous on an arbitrary interval, J. Comput. Appl. Math. 235(9) (2011), 3006-3014. DOI: 10.1016/j.cam.,2010.12.022.

[19] S. Reich and I. Shafrir, Nonexpansive iterations in hyperbolic spaces, Nonlinear Anal. 15 (1990), 537-558.

[20] A. Sahin and M. Basarir, Some convergence results for modified SP-iteration scheme in hyperbolic spaces, Fixed Point Theory Appl. 2014, 2014:133, 11 pp.

[21] H. F. Senter and W. G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44(2) (1974), 375-380.

[22] T. Shimizu and W. Takahashi, Fixed point of multivalued mappings in certain convex metric spaces, Topol. Methods Nonlinear Anal. 8 (1996), 197-203.

[23] W. Takahashi, A convexity in metric spaces and nonexpansive mappings, Kodai Mathematical Seminar Reports 22(2) (1970), 142-149.

[24] S. H. Khan, Fixed point approximation of nonexpansive mappings on a nonlinear domain, Abstr. Appl. Anal. 2014, Art. ID 401650, 5 pp.

[25] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984.

[26] E. Kopecka and S. Reich, A mean ergodic theorem for nonlinear semigroups on the Hilbert ball, J. Nonlinear Convex Anal. 11 (2010), 185-197.