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WEIGHTED SYMMETRIC ESTIMATORS OF AUTOREGRESSIVE MODELS

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Abstract

In this paper, the estimators of the parameters of AR(2) model with (without) constant term will be derived using weighted symmetric (WS) method of estimation. The linearization and the unbiasedness of the estimated parameter of AR(1) model will be proven which is an extension of Park and Fuller [5].

1. Introduction

Dickey et al. [1] proposed the simple weighted symmetric estimator (SWS) for AR(1) model constructed with $w_t = 0.5$. Park and Fuller [5] derived the WSE for AR(1) model without constant term. Fuller [2] suggested a modification of the WSE for AR(1) model approximated for

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$\rho \in (-1, 1]$. So and Shin [6] applied a recursive mean adjustment to the OLS estimator (R-OLS) and they concluded that the mean squares error of the R-OLS estimator, is smaller than the OLS estimator for $\rho \in (0, 1)$. They also showed that the estimator has a coverage probability that is close to the nominal value. Panichkitkosolkul [3] suggested an estimator for an unknown mean Gaussian AR(1) process with additive outliers by applying the recursive median adjustment to the weighted symmetric estimator (Rmd-W). Youssef et al. [7] suggested an OLS estimator for AR(2) model with constant and the properties of the estimated parameters have been derived. Also, a closed form of the variance of the estimated parameters has been obtained.

2. WS Estimator for Autoregressive Model

In this section, the parameter of AR(1) model without constant and its properties will be derived. Moreover, the estimators for AR(2) model with (without) constant will be obtained by using the weighted symmetric method.

2.1. WS estimator for AR (1) without constant

Park and Fuller [5] introduced a weighted symmetric estimator of AR(1) model that has the form:

$$y_t = \rho y_{t-1} + \varepsilon_t, \quad t = 2, 3, ..., n, \tag{1}$$

where $\{\varepsilon_t\}$ are IID $(0, \sigma^2)$ random variables, and y_0 is a fixed. Under the stationarity condition, $|\rho| < 1$, a backward representation of model (1) can be represented as:

$$y_t = \rho y_{t+1} + \varepsilon_t, \quad t = 2, 3, ..., n. \tag{2}$$

The weighted symmetric estimator of ρ is the estimator that minimizes:

$$Q_{ws}(\rho) = \sum_{t=2}^{n} w_t (y_t - \rho y_{t-1})^2 + \sum_{t=1}^{n-1} (1 - w_{t+1})(y_t - \rho y_{t+1})^2. \tag{3}$$

By differentiating equation (3) with respect to ρ and equal to zero to get,

$$\sum_{t=2}^{n} w_t (y_t - \rho y_{t-1}) y_{t-1} + \sum_{t=1}^{n-1} (1 - w_{t+1})(y_t - \rho y_{t+1}) y_{t+1} = 0. \tag{4}$$

By solving equation (4) to obtain:

$$\hat{\rho}_{ws} = \frac{\sum_{t=2}^{n} w_t y_t y_{t-1} + \sum_{t=1}^{n-1} y_t y_{t+1} - \sum_{t=1}^{n-1} w_{t+1} y_t y_{t+1}}{\sum_{t=1}^{n-1} y_{t+1}^2 + \sum_{t=2}^{n} w_t y_{t-1}^2 - \sum_{t=1}^{n-1} w_{t+1} y_{t+1}^2}, \tag{5}$$

where $w_t (0 \le w_t \le 1)$, $t = 1, 2, \ldots, n$ had the form:

$$w_t = \begin{cases} 0, & t = 1, 2, \ldots, p, \\ (n - 2p + 2)^{-1}(t - p), & t = p + 1, p + 2, \ldots, n - p + 1, \\ 1, & t = n - p + 2, n - p + 3, \ldots, n, \end{cases} \tag{6}$$

where p is the number of the estimated parameters.

And as a result $\hat{\rho}_{ws}$ will take the form:

$$\hat{\rho}_{ws} = \frac{n^{-1}\sum_{t=2}^{n} (t-1) y_t y_{t-1} + \sum_{t=1}^{n-1} y_t y_{t+1} - n^{-1}\sum_{t=1}^{n-1} t y_t y_{t+1}}{\sum_{t=1}^{n-1} y_{t+1}^2 + n^{-1}\sum_{t=2}^{n} (t-1) y_{t-1}^2 - n^{-1}\sum_{t=1}^{n-1} t y_{t+1}^2}, \tag{7}$$

which can be simplified by using Hoang [4] to the form:

$$\hat{\rho}_{ws} = \frac{\sum_{t=2}^{n} y_t y_{t-1}}{\sum_{t=2}^{n-1} y_t^2 + n^{-1}\sum_{t=1}^{n} y_t^2}. \tag{8}$$

2.2. Properties of the Weighted Symmetric Estimator (WS)

(a) The linearity of (WS) estimator of AR(1)

By reformulated equation (8), $\hat{\rho}_{ws}$ can be rewritten as:

$$\hat{\rho}_{ws} = \sum_{t=2}^{n} G_t y_t, \tag{9}$$

where

$$G_t = \frac{y_{t-1}}{\left[\sum_{t=2}^{n-1} y_t^2 + n^{-1} \sum_{t=1}^{n} y_t^2\right]}.$$

This provided the linearity of the estimator.

(b) Unbiasedness of (WS) estimator of AR (1)

By using the values of y_t of (1) and (2) in (9) to get,

$$\hat{\rho}_{ws} = \frac{\sum_{t=2}^{n} w_t y_{t-1}(\rho y_{t-1} + \varepsilon_t) + \sum_{t=1}^{n-1} y_{t+1}(\rho y_{t+1} + \varepsilon_t) - \sum_{t=1}^{n-1} w_{t+1} y_{t+1}(\rho y_{t+1} + \varepsilon_t)}{\sum_{t=1}^{n-1} y_{t+1}^2 + \sum_{t=2}^{n} w_t y_{t-1}^2 - \sum_{t=1}^{n-1} w_{t+1} y_{t+1}^2}. \tag{10}$$

It can be shown that, equation (11) can be rewritten as:

$$\hat{\rho}_{ws} = \rho + \frac{\sum_{t=2}^{n} w_t y_{t-1}\varepsilon_t + \sum_{t=1}^{n-1} y_{t+1}\varepsilon_t - \sum_{t=1}^{n-1} w_{t+1} y_{t+1}\varepsilon_t}{\sum_{t=1}^{n-1} y_{t+1}^2 + \sum_{t=2}^{n} w_t y_{t-1}^2 - \sum_{t=1}^{n-1} w_{t+1} y_{t+1}^2}. \tag{11}$$

By taking the expectation of equation (11) and since $E(\varepsilon_t) = 0$ the property of unbiasedness will be proved.

2.3. WS estimator for AR(2) model without constant

Let, the second order autoregressive model can be defined as:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t, \quad t = 3, 4, \ldots, n, \tag{12}$$

where $\{\varepsilon_t\}$ are IID $(0, \sigma^2)$ random variables, and (y_0, y_{-1}) is a fixed. A backward representation of the model (12) can be written as

$$y_t = \rho_1 y_{t+1} + \rho_2 y_{t+2} + \varepsilon_t, \quad t = 3, 4, \ldots, n. \tag{13}$$

The weighted symmetric estimators of (12) and (13) are the estimators that minimize:

$$Q_{ws}(\rho_1, \rho_2) = \sum_{t=3}^{n} w_t (y_t - \rho_1 y_{t-1} - \rho_2 y_{t-1})^2 + \sum_{t=1}^{n-2} (1 - w_{t+1})(y_t - \rho_1 y_{t+1} - \rho_2 y_{t+1})^2. \tag{14}$$

By differentiating equation (14) with respect to ρ_1 and ρ_2 and equal to zero, to get

$$\sum_{t=3}^{n} w_t (y_t - \hat{\rho}_1 y_{t-1} - \hat{\rho}_2 y_{t-1}) y_{t-1} + \sum_{t=1}^{n-2} (1 - w_{t+1})(y_t - \hat{\rho}_1 y_{t+1} - \hat{\rho}_2 y_{t+1}) y_{t+1} = 0.$$

So, $\hat{\rho}_1$ can be obtained as:

$$\hat{\rho}_1 = H_1 - \hat{\rho}_2 M_1, \tag{15}$$

where

$$H_1 = \frac{\sum_{t=3}^{n} w_t y_t y_{t-1} + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_t y_{t+1}}{\sum_{t=3}^{n} w_t y_{t-1}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_{t+1}^2},$$

$$M_1 = \frac{\sum_{t=3}^{n} w_t y_{t-1} y_{t-2} + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_{t+1} y_{t+2}}{\sum_{t=3}^{n} w_t y_{t-1}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_{t+1}^2}.$$

Also, $\hat{\rho}_2$ can be obtained as:

$$\hat{\rho}_2 = H_2 - \hat{\rho}_1 M_2, \tag{16}$$

where

$$H_2 = \frac{\sum_{t=3}^{n} w_t y_t y_{t-2} + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_t y_{t+2}}{\sum_{t=3}^{n} w_t y_{t-2}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_{t+2}^2},$$

$$M_2 = \frac{\sum_{t=3}^{n} w_t y_{t-1} y_{t-2} + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_{t+1} y_{t+2}}{\sum_{t=3}^{n} w_t y_{t-2}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) y_{t+2}^2}.$$

By solving equations (15) and (16) to get

$$\begin{cases} \hat{\rho}_1 = \dfrac{H_1 - H_2 M_1}{(1 - M_2 M_1)}, \\ \hat{\rho}_2 = \dfrac{H_2 - H_1 M_2}{(1 - M_2 M_1)}. \end{cases} \tag{17}$$

2.4. WS estimator for AR(2) with constant

By adding the constant term to equation (12) and under the same assumptions above, taking the expectation of the model,

$$y_t = \alpha + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t, \quad t = 3, 4, \ldots, n. \tag{18}$$

To get

$$(1 - \rho_1 - \rho_2)\bar{y} = \alpha, \tag{19}$$

where

$$\bar{y} = \frac{1}{n-2}\sum_{t=3}^{n} y_t.$$

So, equation (18) can be written as:

$$\mathbf{z}_t = \rho_1 \mathbf{z}_{t-1} + \rho_2 \mathbf{z}_{t-2} + \varepsilon_t, \tag{20}$$

where $\mathbf{z}_t = (y_t - \bar{y})$, $\mathbf{z}_{t-1} = (y_{t-1} - \bar{y})$ and $\mathbf{z}_{t-2} = (y_{t-2} - \bar{y})$.

Similarly, the backward autoregressive models can be rewritten as:

$$\mathbf{z}_t = \rho_1 \mathbf{z}_{t+1} + \rho_2 \mathbf{z}_{t+2} + \varepsilon_t, \tag{21}$$

where $\mathbf{z}_{t+1} = (y_{t+1} - \bar{y})$ and $\mathbf{z}_{t+2} = (y_{t+2} - \bar{y})$.

The weighted symmetric estimators of model (20) are the estimators that minimizes,

$$Q_{ws}(\rho_1, \rho_2) = \sum_{t=3}^{n} w_t (\mathbf{z}_t - \rho_1 \mathbf{z}_{t-1} - \rho_2 \mathbf{z}_{t-1})^2$$

$$+ \sum_{t=1}^{n-2} (1 - w_{t+1})(\mathbf{z}_t - \rho_1 \mathbf{z}_{t+1} - \rho_2 \mathbf{z}_{t+1})^2. \tag{22}$$

Following the same procedures as above the estimators $(\hat{\rho}_1, \hat{\rho}_2)$, can be obtained as: $\hat{\rho}_1 = \dfrac{K_1 - K_2 F_1}{(1 - F_2 F_1)}$ and $\hat{\rho}_2 = \dfrac{K_2 - K_1 F_2}{(1 - F_2 F_1)}$, respectively.

Where

$$K_1 = \frac{\sum_{t=3}^{n} w_t \mathbf{z}_t \mathbf{z}_{t-1} + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_t \mathbf{z}_{t+1}}{\sum_{t=3}^{n} w_t \mathbf{z}_{t-1}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_{t+1}^2},$$

$$K_2 = \frac{\sum_{t=3}^{n} w_t \mathbf{z}_t \mathbf{z}_{t-2} + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_t \mathbf{z}_{t+2}}{\sum_{t=3}^{n} w_t \mathbf{z}_{t-2}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_{t+2}^2},$$

$$F_1 = \frac{\sum_{t=3}^{n} w_t \mathbf{z}_{t-1} \mathbf{z}_{t-2} + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_{t+1} \mathbf{z}_{t+2}}{\sum_{t=3}^{n} w_t \mathbf{z}_{t-1}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_{t+1}^2},$$

$$F_2 = \frac{\sum_{t=3}^{n} w_t \mathbf{z}_{t-1} \mathbf{z}_{t-2} + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_{t+1} \mathbf{z}_{t+2}}{\sum_{t=3}^{n} w_t \mathbf{z}_{t-2}^2 + \sum_{t=1}^{n-2} (1 - w_{t+1}) \mathbf{z}_{t+2}^2}.$$

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