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THE HERGLOTZ SPACE ASSOCIATED WITH THE DE BRANGES-ROVNYAK SPACE IN MULTISCALE SYSTEMS

Meehyea Yang

Department of Mathematics
Incheon National University
Yeonsoo-Gu Songdo-Dong 12-1
Incheon 406-772, Korea
e-mail: mhyang@incheon.ac.kr

Abstract

For a Schur multiplier S in a multiscale system and $\phi = (I + S)^{-1}(I - S)$, we obtain a multiscale linear system whose main transformation is found to be an isometry. The state space of the system is

$$\mathcal{E}(\Phi) = \{(F, G) : ((I + \Phi)F, -(I + \Phi^*)G) \in \mathcal{D}(S)\}.$$

1. Introduction

This paper continues the study of the de Branges-Rovnyak space in stationary multiscale systems based on the complementation theory (see [6, 7]). As in non-stationary systems, we derive the analogue Herglotz representation theorem in the stationary systems (see [2]). We first state some of notations and results as mentioned in [3] and [4].

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Let $\mathcal{T}$ be a homogeneous tree of order $q \geq 2$ and let $l_2(\mathcal{T})$ be the Hilbert space of square-summable sequences indexed by the nodes of $\mathcal{T}$ and $\mathbf{X}(\mathcal{T})$ the C^*-algebra of bounded linear operators on $l_2(\mathcal{T})$. Let $\bar{\gamma} : \mathcal{T} \to \mathcal{T}$ be the primitive upward shift. Define the upward shift operator $\bar{\gamma}$ on $l_2(\mathcal{T})$ by

$$\bar{\gamma} f(t) = \frac{1}{\sqrt{q}} f(t\bar{\gamma}).$$

Then $\bar{\gamma}$ is an isometry but not a unitary.

Define

$$\sigma_n = \bar{\gamma}^n \gamma^n, \quad \sigma_0 = I, \quad \omega_0 = I - \sigma_1, \quad \text{and} \quad \omega_n = \sigma_n - \sigma_{n+1}, \qquad n \in \mathbb{Z}_+.$$

Define the commutative $\mathbf{C}^*$ algebra

$$\mathbb{K} = \left\{ \mathbf{c} = \sum_{k=0}^{\infty} c_k \omega_k : c_k \in \mathbb{C}, \sup_k |c_k| < \infty \right\}$$

with the operator norm $\| \mathbf{c} \|_{\text{op}} = \sup_k |c_k|$ and define the Banach algebra

$$\mathbf{U}(\mathcal{T}) = \left\{ S = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{s}_k : \mathbf{s}_k \in \mathbb{K} \text{ and } \| \mathbf{s}_k \|_{\text{op}} \leq \| S \|_{\text{op}} \right\}.$$

In [1] it was shown that $S \in \mathbf{U}(\mathcal{T})$ if and only if

$$S = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{s}_k, \quad \mathbf{s}_k \in \mathbb{K}.$$

In this case, the operators $\mathbf{s}_k$ are uniquely determined by

$$\omega_n \mathbf{s}_k = \gamma^k \omega_{n+k} S \omega_n, \quad n, k \in \mathbb{Z}_+.$$

In [3] it was shown that the multiscale system is both stationary and casual if S is of the form

$$S = \sum_{k, n \in \mathbb{Z}_+} \bar{\gamma}^n \omega_k s_{k, n} \text{ with } s_{k, n} \in \mathbb{C}.$$

An operator $S \in \mathbf{U}(\mathcal{T})$ with $\| S \|_{\text{op}} \leq 1$ is called a *Schur multiplier*.

As in the non-stationary system, define Hilbert spaces

$$\mathbb{K}_2 = \left\{ \mathbf{c} = \sum_{k=0}^{\infty} c_k \omega_k \in \mathbb{K} : \sum_{k=0}^{\infty} | c_k |^2 < \infty \right\}$$

and

$$\mathbf{H}_2 = \left\{ F = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{f}_k : \mathbf{f}_k \in \mathbb{K}_2, \| F \|_{\mathbf{H}_2}^2 = \sum_{k=0}^{\infty} \| \mathbf{f}_k \|_{\mathbb{K}_2}^2 < \infty \right\}$$

with the scalar product

$$\langle \mathbf{c}, \mathbf{d} \rangle_{\mathbb{K}_2} = \sum_{\mathbf{k}=0}^{\infty} \bar{\mathbf{d}}_{\mathbf{k}} \mathbf{c}_{\mathbf{k}}, \text{ and } \langle \mathbf{F}, \mathbf{G} \rangle_{\mathbf{H}_2} = \sum_{\mathbf{k}=0}^{\infty} \langle \mathbf{f}_{\mathbf{k}}, \mathbf{g}_{\mathbf{k}} \rangle_{\mathbb{K}_2}.$$

Then $\mathbf{H}_2$ is a left ideal in $\mathbf{U}(\mathcal{T})$ and for a Schur multiplier S, the multiplication operator $M_S F = SF$ is a contraction in $\mathbf{H}_2$ (see [3]).

Now review the following notations to define the left point evaluation on $\mathbf{H}_2$ (see [1]).

For $\mathbf{c} \in \mathbb{K}$ and $n \in \mathbb{Z}$, define

$$\mathbf{c}^{[n]} = (\mathbf{c}\gamma)^n \bar{\gamma}^n, \quad \mathbf{c}^{\langle n \rangle} = \bar{\gamma}^n (\gamma \mathbf{c})^n, \quad \mathbf{c}^{[0]} = \mathbf{c}^{\langle 0 \rangle} = I,$$

$$\mathbf{c}^{(n)} = \gamma^n \mathbf{c} \bar{\gamma}^n, \quad \mathbf{c}^{(-n)} = \bar{\gamma}^{-n} \mathbf{c} \gamma^{-n} \text{ and } \mathbf{c}^{(0)} = \mathbf{c}.$$

Define the unit open disk in the multiscale system by

$$\mathbb{D}(\mathcal{T}) = \{ \mathbf{c} \in \mathbb{K} : \rho(c) = \limsup_{n \to \infty} \| \mathbf{c}^{[n]} \|^{\frac{1}{n}} < 1 \}.$$

For $F = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{f}_k \in \mathbf{U}(\mathcal{T})$ and $\mathbf{c} \in \mathbb{D}(\mathcal{T})$, the left point evaluation of F

at $\mathbf{c}$ is defined by

$$F^{\wedge}(\mathbf{c}) = \sum_{k=0}^{\infty} \mathbf{c}^{[k]}\mathbf{f}_k = \sum_{k=0}^{\infty} (\mathbf{c}\gamma)^k \bar{\gamma}^k \mathbf{f}_k$$

and the right point evaluation of F at $\mathbf{c}$ is defined by

$$F^{\triangle}(\mathbf{c}) = \sum_{k=0}^{\infty} \mathbf{f}_k^{(-k)}\mathbf{c}^{\langle k\rangle} = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{f}_k (\gamma\mathbf{c})^k.$$

Then both $F^{\wedge}(\mathbf{c})$ and $F^{\triangle}(\mathbf{c})$ are absolutely convergent in $\mathbb{K}$ and for any $\mathbf{k} \in \mathbb{K}_2$,

$$\langle F,\, K_{\wedge}^{\mathbf{c}}\mathbf{k}\rangle_{\mathbf{H}_2} = \langle F^{\wedge}(\mathbf{c}),\, \mathbf{k}\rangle_{\mathbb{K}_2} \quad \text{and} \quad \langle F,\, K_{\triangle}^{\mathbf{c}}\mathbf{k}\rangle_{\mathbf{H}_2} = \langle F^{\triangle}(\mathbf{c}),\, \mathbf{k}\rangle_{\mathbb{K}_2},$$

where

$$K_{\wedge}^{\mathbf{c}} = (I - \bar{\gamma}\mathbf{c}^*)^{-1} \quad \text{and} \quad K_{\triangle}^{\mathbf{c}} = (I - \mathbf{c}^*\bar{\gamma})^{-1}.$$

2. The de Branges-Rovnyak Space $\mathcal{H}(S)$

Let S be a Schur multiplier. The Hilbert space

$$\mathcal{H}(S) = \{F \in \mathbf{H}_2 : \sup_{G\in\mathbf{H}_2} \{\| F + SG \|_{\mathbf{H}_2}^2 - \| G \|_{\mathbf{H}_2}^2\} < \infty\}$$

with the scalar product

$$\| F \|_{\mathcal{H}(S)}^2 = \sup_{G\in\mathbf{H}_2} \{\| F + SG \|_{\mathbf{H}_2}^2 - \| G \|_{\mathbf{H}_2}^2\} < \infty$$

was introduced by de Branges and Rovnyak (see [5]). The de Branges-Rovnyak space $\mathcal{H}(S)$ was constructed based on the complementation theory. $\operatorname{Ran}(I - M_S M_S^*)$ is a dense in $\mathcal{H}(S)$ and $K_S^{\mathbf{c}} = (I - SS^{\wedge}(\mathbf{c})^*)K_{\wedge}^{\mathbf{c}}$ is a reproducing kernel function of $\mathcal{H}(S)$.

The space $\mathcal{H}(S)$ is the state space of a linear system whose transfer function is a Schur multiplier S (see [1]).

Theorem 2.1. *Let S be a Schur multiplier. Then the linear system*

$$V_S = \begin{pmatrix} A_S & B_S \\ C_S & D_S \end{pmatrix} : \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{K}_2 \end{pmatrix}$$

defined by

$$A_S F = (F - F^\wedge(0))\gamma, \quad B_S \mathbf{k} = (S - S^\wedge(0))\mathbf{k}\gamma,$$

$$C_S F = F^\wedge(0) \ \ \textit{and} \ \ D_S \mathbf{k} = S^\wedge(0)\mathbf{k} \tag{2.1}$$

satisfies

$$V_S V_S^* = \begin{pmatrix} I - \hat{M}_{\omega_0} & 0 \\ 0 & I \end{pmatrix}, \tag{2.2}$$

where $\hat{M}_{\omega_0} F = F\omega_0$. *S can be written by*

$$S\mathbf{k} = D_S\mathbf{k} + \sum_{k=0}^{\infty} \bar{\gamma}^{k+1} (C_S A_S^k B_S \mathbf{k})^{(k+1)}.$$

The space

$$\mathcal{L}(S) = \{F \in \mathbb{U}(\mathcal{T}) : SF \in \mathcal{H}(S)\}$$

is a Hilbert space with the norm

$$\| F \|^2_{\mathcal{L}(S)} = \| SF \|^2_{\mathcal{H}(S)} + \| F \|^2_{\mathbb{H}_2}.$$

The space $\mathcal{L}(S)$ is called the *overlapping space* of $\mathcal{H}(S)$. The transformation $T : \mathcal{L}(S) \to \mathcal{L}(S)$ defined by $TF = (F - F^\wedge(0))\gamma$ is contractive and its adjoint is isometric (see [7]).

For a Schur multiplier S, $\Phi = (I + S)^{-1}(I - S)$ is a Carathéodory operator. The space $\mathcal{L}(\Phi) = \mathrm{Ran}(M_\Phi + M_\Phi^*)^{1/2}$ is the state space of a bounded linear system (see [7]).

Theorem 2.2. *Let* $\Phi = (I+S)^{-1}(I-S)$ *for a Schur multiplier S. Then*

$$\mathcal{L}(\Phi) = \{(I+\Phi)G : G \in \mathcal{H}(S)\}$$

and for $G \in \mathcal{H}(S)$,

$$\| (I+\Phi)G \|_{\mathcal{L}(\Phi)} = \sqrt{2} \| G \|_{\mathcal{H}(S)}. \tag{2.3}$$

The linear system

$$\begin{pmatrix} A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{pmatrix} : \begin{pmatrix} \mathcal{L}(\Phi) \\ \mathbb{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{L}(\Phi) \\ \mathbb{K}_2 \end{pmatrix}$$

defined by

$$A_\Phi F = (F - F^\wedge(0))\gamma, \quad B_\Phi \mathbf{k} = (\Phi - \Phi^\wedge(0))\mathbf{k}\gamma,$$

$$C_\Phi F = F^\wedge(0) \;\; \textit{and} \;\; D_\Phi \mathbf{k} = \Phi^\wedge(0)\mathbf{k} \tag{2.4}$$

satisfies

$$A_\Phi A_\Phi^*((I+\Phi)G) = (I+\Phi)(A_S A_S^* G + B_S B_S^* G)$$

for $G \in \mathcal{H}(S)$ *and*

$$A_\Phi C_\Phi^* = B_\Phi.$$

3. The Overlapping Space of $\mathcal{D}(S)$

From (2.1), we have that $A_S^* F = F\bar{\gamma} - SB_S^* F$ for $F \in \mathcal{H}(S)$ and

$$\| F\bar{\gamma} - SB_S^* F \|_{\mathcal{H}(S)}^2 = \| F \|_{\mathcal{H}(S)}^2 - \| B_S^* \mathbf{k} \|_{\mathbf{K}_2}^2.$$

Let $F_0 = F \in \mathcal{H}(S)$, $F_{n+1} = A_S^* F_n$ and $\mathbf{g}_n = B_S^* F_n$ for each non-negative integer *n*. Then for each *n*,

$$F\bar{\gamma}^n - S(\mathbf{g}_0 \bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1}) \in \mathcal{H}(S)$$

and

$$\| F\bar{\gamma}^{n} - S(\mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1})\|^2_{\mathcal{H}(S)} = \| F \|^2_{\mathcal{H}(S)} - \sum_{k=0}^{n-1}\| \mathbf{g}_k \|^2_{\mathbf{K}_2}.$$

Define the extension space $\mathcal{D}(S)$ associated with $\mathcal{H}(S)$ by the set of pairs (F, G), where $F \in \mathcal{H}(S)$ and $G = \sum_{k=0}^{\infty} \mathbf{g}_k\gamma^k$ and for each non-negative integer *n*,

$$F\gamma^{n} - S(\mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1}) \in \mathcal{H}(S).$$

Then the sequence

$$\| F\bar{\gamma}^{n} - S(\mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1})\|^2_{\mathcal{H}(S)} + \sum_{k=0}^{n-1}\| \mathbf{g}_k \|^2_{\mathbf{K}_2}$$

is bounded and non-decreasing. $\mathcal{D}(S)$ becomes a Hilbert space with the scalar product

$$\| (F, G)\|^2_{\mathcal{D}(S)} = \lim_{n\to\infty}\left[\| F\bar{\gamma}^{n} - S(\mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1})\|^2_{\mathcal{H}(S)} + \sum_{k=0}^{n-1}\| \mathbf{g}_k \|^2_{\mathbf{K}_2}\right].$$

There is a partial isometry from $\mathcal{D}(S)$ onto $\mathcal{H}(S)$.

The extended space $\mathcal{D}(S)$ is the state space of a linear system (see [6]).

Theorem 3.1. *Let S be a Schur multiplier. The linear system*

$$\hat{V}_S = \begin{pmatrix} \hat{A}_S & \hat{B}_S \\ \hat{C}_S & \hat{D}_S \end{pmatrix} : \begin{pmatrix} \mathcal{D}(S) \\ \mathbb{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{D}(S) \\ \mathbb{K}_2 \end{pmatrix} \tag{3.1}$$

defined by

$$\hat{A}_S((F, G)) = ((F - F^\wedge(0))\gamma,\ G\gamma - S^*F^\wedge(0)),$$

$$\hat{B}_S\mathbf{k} = ((S - S^\wedge(0))\mathbf{k}\gamma,\ (I - S^*S^\wedge(0))\mathbf{k}),$$

$$\hat{C}_S((F, G)) = F^\wedge(0) \quad \textit{and} \quad \hat{D}_S\mathbf{k} = S^\wedge(0)\mathbf{k}$$

satisfies

$$\hat{V}_S\hat{V}_S^*\begin{pmatrix}(F, G)\\ \mathbf{k}\end{pmatrix} = \begin{pmatrix}(F\bar{\gamma}\gamma, (G - G^\vee(0))\bar{\gamma}\gamma + G^\vee(0))\\ \mathbf{k}\end{pmatrix}.$$

Define

$$\mathcal{E}(S) = \{(F, G) : (SF, -G) \in \mathcal{D}(S)\}.$$

Then $\mathcal{E}(S)$ is a Hilbert space with the scalar product

$$\| (F, G) \|_{\mathcal{E}(S)}^2 = \| F \|_{\mathbf{H}_2}^2 + \| (SF, -G) \|_{\mathcal{D}(S)}^2.$$

The space $\mathcal{E}(S)$ is a Herglotz space and there is a relation between $\mathcal{E}(S)$ and $\mathcal{L}(S)$.

Theorem 3.2. *Let S be a Schur multiplier. Then* $\mathcal{E}(S)$ *is the set of pairs* (F, G) *with* $G = \sum_{k=0}^{\infty} \mathbf{g}_k\gamma^k$ *such that for each non-negative integer n,*

$$F\bar{\gamma}^n + \mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1} \in \mathcal{L}(S).$$

There is a partial isometry from $\mathcal{E}(S)$ *onto* $\mathcal{L}(S)$. *Furthermore, the transformation* $\hat{T} : \mathcal{E}(S) \to \mathcal{E}(S)$ *defined by*

$$\hat{T}(F, G) = ((F - F^\wedge(0))\gamma, G\gamma + F^\wedge(0))$$

is an isometry.

Proof. Let $F \in \mathcal{L}(S)$. If $T : \mathcal{L}(S) \to \mathcal{L}(S)$ is defined by $TF = (F - F^\wedge(0))\gamma$, then $T^*F = F\bar{\gamma} - B_S^*(SF)$ so T^* is isometric (see [7]).

Let $F_0 = F \in \mathcal{L}(S)$, $F_{n+1} = T^*F_n$ and $\bar{\mathbf{f}}_n = -B_S^*(SF_n)$. For $\tilde{F} = \sum_{k=0}^{\infty} \tilde{\mathbf{f}}_k\gamma^k$,

$$F\bar{\gamma}^n + \tilde{\mathbf{f}}_0\bar{\gamma}^{n-1} + \cdots + \tilde{\mathbf{f}}_{n-1} \in \mathcal{L}(S)$$

so we have $(SF, -\tilde{F}) \in \mathcal{D}(S)$, $(F, \tilde{F}) \in \mathcal{E}(S)$ and

$$\|(F, \tilde{F})\|_{\mathcal{E}(S)} = \| F \|_{\mathcal{L}(S)}.$$

Let $\mathcal{E}$ be the set of all elements in $\mathcal{E}(S)$ of the form $(F, \tilde{F})$. If $(F, \tilde{F}) \in \mathcal{E}$ and $(0, Q) \in \mathcal{E}(S)$ with $Q = \sum_{k=0}^{\infty} \mathbf{q}_k \gamma^k$, then the identities

$$\langle F\bar{\gamma}^n + \tilde{\mathbf{f}}_0\bar{\gamma}^{n-1} + \cdots + \tilde{\mathbf{f}}_{n-1}, \mathbf{q}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{q}_{n-1}\rangle_{\mathcal{L}(S)}$$

$$= \langle F\bar{\gamma}^{n-1} + \tilde{\mathbf{f}}_0\bar{\gamma}^{n-2} - \cdots - \tilde{\mathbf{f}}_{n-2}, (\mathbf{q}_0\bar{\gamma}^{n-2} + \cdots + \mathbf{q}_{n-2})\bar{\gamma}\gamma\rangle_{\mathcal{L}(S)}$$

$$= \langle F\bar{\gamma} + \tilde{\mathbf{f}}_0, \mathbf{q}_0\bar{\gamma}^{n-1}\gamma^{n-1}\rangle_{\mathcal{L}(S)} = 0$$

imply that $\mathcal{E}$ is the orthogonal complement of the set of elements of the form $(0, Q) \in \mathcal{E}(S)$. Hence the transformation which maps (F, G) to F is a partial isometry from $\mathcal{E}(S)$ onto $\mathcal{L}(S)$.

Let $(F, G) \in \mathcal{E}(S)$. Since $(SF, -G) \in \mathcal{D}(S)$, the linear system (3.1) implies that

$$((SF - S^\wedge(0)F^\wedge(0))\gamma, -G\gamma - S^*S^\wedge(0)F^\wedge(0)) \in \mathcal{E}(S).$$

Hence $((F - F^\wedge(0))\gamma, G\gamma + F^\wedge(0)) \in \mathcal{E}(S)$ from

$$S(F - F^\wedge(0))\gamma = (SF - S^\wedge(0)F^\wedge(0))\gamma - (S - S^\wedge(0))F^\wedge(0)\gamma$$

and

$$G\gamma + F^\wedge(0) = G\gamma + S^*S^\wedge(0)F^\wedge(0) + (I - S^*S^\wedge(0))F^\wedge(0).$$

Define

$$\hat{T}(F, G) = ((F - F^\wedge(0))\gamma, G\gamma + F^\wedge(0)).$$

For $(F, G) \in \mathcal{E}(S)$ and $G = \sum_{k=0}^{\infty} \mathbf{g}_k \gamma^k$, write $G^\vee(0) = \mathbf{g}_0$. Then the identities

$$\begin{aligned}\langle \hat{T}(F, G), (P, Q)\rangle_{\mathcal{E}(S)} &= \langle ((F - F^{\wedge}(0))\gamma, G\gamma + F^{\wedge}(0)), (P, Q)\rangle_{\mathcal{E}(S)} \\ &= \langle (S(F - F^{\wedge}(0))\gamma, -G\gamma - F^{\wedge}(0)), (SP, -Q)\rangle_{\mathcal{D}(S)} \\ &\quad + \langle (F - F^{\wedge}(0))\gamma, P\rangle_{\mathbf{H}_2} \\ &= \langle (\hat{A}_S(SF, -G) - \hat{B}_S F^{\wedge}(0)), (SP, -Q)\rangle_{\mathcal{D}(S)} \\ &\quad + \langle F, P\bar{\gamma}\rangle_{\mathbf{H}_2} \\ &= \langle (SF, -G), (SP\bar{\gamma} + SQ^{\vee}(0), -(Q - Q^{\vee}(0))\bar{\gamma})\rangle_{\mathcal{D}(S)} \\ &\quad + \langle F^{\wedge}(0), Q^{\vee}(0)\rangle_{\mathbf{H}_2} + \langle F, P\bar{\gamma}\rangle_{\mathbf{H}_2} \\ &= \langle (F, G), (P\bar{\gamma} + Q^{\vee}(0)), (Q - Q^{\vee}(0))\bar{\gamma}\rangle_{\mathcal{E}(S)}\end{aligned}$$

hold for each $(P, Q) \in \mathcal{E}(S)$. Hence

$$\hat{T}^*(F, G) = (F\bar{\gamma} + G^{\vee}(0), (G - G^{\vee}(0))\bar{\gamma})$$

and $\hat{T}$ is isometric. □

Let $\Phi = (I + S)^{-1}(I - S)$ for a Schur multiplier S. Define $\mathcal{E}(\Phi)$ by the set of pairs (F, G) such that for each non-negative integer n,

$$F\bar{\gamma}^n + \mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1} \in \mathcal{L}(\Phi)$$

and $G = \sum_{k=0}^{\infty} \mathbf{g}_k\gamma^k$.

The relation (2.3) implies that

$$A_{\Phi}^* F = F\bar{\gamma} - (I + S^{*\vee}(0))^{-1} B^*((I + S)F).$$

Let $F_0 = F \in \mathcal{L}(\Phi)$, $F_{n+1} = A_{\Phi}^* F_n$ and $\hat{\mathbf{f}}_n = -(I + S^{*\vee}(0))^{-1} B^*((I + S)F_n)$. Then for $\hat{F} = \sum_{k=0}^{\infty} \hat{\mathbf{f}}_k\gamma^k$,

$$F\bar{\gamma}^n + \bar{\mathbf{f}}_0\bar{\gamma}^{n-1} + \cdots + \hat{\mathbf{f}}_{n-1} \in \mathcal{L}(\Phi)$$

so $(F, \hat{F}) \in \mathcal{E}(\Phi)$. Since

$$\langle F\bar{\gamma}^n + \bar{\mathbf{f}}_0\bar{\gamma}^{n-1} + \cdots + \hat{\mathbf{f}}_{n-1}, \mathbf{q}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{q}_{n-1}\rangle_{\mathcal{L}(\Phi)} = 0$$

for any $Q = \sum_{k=0}^{\infty}\mathbf{q}_k\gamma^k \in \mathcal{L}(\Phi)$, $\mathcal{E}(\Phi)$ becomes a Hilbert space with the scalar product

$$\|(F, G)\|_{\mathcal{E}(\Phi)} = \lim_{n\to\infty}\|F\bar{\gamma}^n + \mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1}\|_{\mathcal{L}(\Phi)}$$

and

$$\|(F, G)\|_{\mathcal{E}(\Phi)} = \|F\|_{\mathcal{L}(\Phi)}. \tag{3.2}$$

There is a relation between $\mathcal{E}(\Phi)$ and $\mathcal{D}(S)$.

Theorem 3.3. *If* $(F, G) \in \mathcal{E}(\Phi)$, *then* $((I + S)F, -(I + S^*)G) \in \mathcal{D}(S)$ *and*

$$\|(F, G)\|_{\mathcal{E}(\Phi)} = \frac{1}{\sqrt{2}}\|((I + S)F, -(I + S^*)G\|_{\mathcal{D}(S)}.$$

Proof. Let $(F, G) \in \mathcal{E}(\Phi)$ and $G = \sum_{k=0}^{\infty}\mathbf{g}_k\gamma^k$. For $\mathbf{c} \in \mathbf{K}_2$,

$$M_S^*(\mathbf{c}\bar{\gamma}^k) = \sum_{j=0}^{k}\bar{\gamma}^j\mathbf{s}_{k-j}^{(j)*}\mathbf{c}^{(k)} = \sum_{j=0}^{k}\mathbf{s}_{k-j}^*\mathbf{c}^{(k-j)}\bar{\gamma}^j.$$

So we have $S^*G = \sum_{k=0}^{\infty}\mathbf{p}_k\gamma^k$ and

$$M_S^*(\mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1}) = \mathbf{p}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{p}_{n-1},$$

where $\mathbf{p}_k = \sum_{j=0}^{k}\mathbf{s}_{k-j}^*\mathbf{g}_j^{(k-j)}$.

Let $G_{n-1} = \mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1}$ and $M_S^*G_{n-1} = P_{n-1}$ for each positive integer n. The relation (2.2) implies that

$$(I+S)(F\bar{\gamma}^n + \mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1}) = (I+S)(F\bar{\gamma}^n + G_{n-1}) \in \mathcal{H}(S).$$

Hence $((I+S)F, -(I+S^*)G) \in \mathcal{D}(S)$ since

$$\begin{aligned} &(I+S)F\bar{\gamma}^n + S(G_{n-1} + P_{n-1}) \\ &= (I+S)F\bar{\gamma}^n + S(I+M_S^*)G_{n-1} \\ &= (I+S)(F\bar{\gamma}^n + G_{n-1}) - (I - SM_S^*)G_{n-1} \in \mathcal{H}(S). \end{aligned}$$

From (3.2) and the identities

$$\begin{aligned} &\| (I+S)F\bar{\gamma}^n + S(G_{n-1} + P_{n-1})\|^2_{\mathcal{H}(S)} \\ &= \| (I+S)(F\bar{\gamma}^n + G_{n-1}) - (I - SM_S^*)G_{n-1}\|^2_{\mathcal{H}(S)} \\ &= \| (I+S)(F\bar{\gamma}^n + G_{n-1})\|^2_{\mathcal{H}(S)} - \| G_{n-1} + P_{n-1}\|^2_{\mathbf{H}_2}, \end{aligned}$$

we have

$$\begin{aligned} &\| ((I+S)F, -(I+S^*)G)\|^2_{\mathcal{D}(S)} \\ &= \lim_{n\to\infty} [\| (I+S)F\bar{\gamma}^n + S(G_{n-1} + P_{n-1})\|^2_{\mathcal{H}(S)} + \| G_{n-1} + P_{n-1}\|^2_{\mathbf{H}_2}] \\ &= \lim_{n\to\infty} \| (I+S)(F\bar{\gamma}^n + G_{n-1})\|^2_{\mathcal{H}(S)} \\ &= 2\lim_{n\to\infty} \| F\bar{\gamma}^n + G_{n-1}\|^2_{\mathcal{L}(\Phi)} \\ &= 2\| (F, G)\|^2_{\mathcal{E}(\Phi)}. \end{aligned}$$ □

The space $\mathcal{E}(\Phi)$ is a Herglotz space.

Theorem 3.4. *Define the linear system*

$$\begin{pmatrix} \hat{A}_\Phi & \hat{B}_\Phi \\ \hat{C}_\Phi & \hat{D}_\Phi \end{pmatrix} : \begin{pmatrix} \mathcal{E}(\Phi) \\ \mathbb{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{E}(\Phi) \\ \mathbb{K}_2 \end{pmatrix}$$

by

$$\hat{A}_\Phi((F, G)) = ((F - F^\wedge(0))\gamma, G\gamma + F^\wedge(0)),$$

$$\hat{B}_\Phi\mathbf{k} = ((\Phi - \Phi^\wedge(0))\mathbf{k}\gamma, (\Phi^* + \Phi^\wedge(0))\mathbf{k}),$$

$$\hat{C}_\Phi((F, G)) = F^\wedge(0) \quad \textit{and} \quad \hat{D}_\Phi\mathbf{k} = \Phi^\wedge(0)\mathbf{k}.$$

Then

$$\hat{A}_\Phi^*\hat{A}_\Phi = I, \quad \hat{A}_\Phi^*\hat{B}_\Phi = \hat{C}_\Phi^* \quad \textit{and} \quad \hat{B}_\Phi^*\hat{B}_\Phi = \hat{C}_\Phi\hat{C}_\Phi^* = \hat{D}_\Phi + \hat{D}_\Phi^*.$$

Proof. Let $(F, G) \in \mathcal{E}(\Phi)$ and $G = \sum_{k=0}^\infty \mathbf{g}_k\gamma^k$. Since

$$\begin{aligned} &F\bar{\gamma}^n + \mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1} \\ &= (F - F^\wedge(0))\gamma\bar{\gamma}^{n+1} + (F^\wedge(0)\bar{\gamma}^n + \mathbf{g}_0\bar{\gamma}^{n-1} + \cdots + \mathbf{g}_{n-1}) \\ &= (F\bar{\gamma} - \mathbf{g}_0)\bar{\gamma}^{n-1} + (\mathbf{g}_1\bar{\gamma}^{n-2} + \cdots + \mathbf{g}_{n-1}) \in \mathcal{L}(\Phi), \end{aligned}$$

we have

$$((F - F^\wedge(0))\gamma, G\gamma + F^\wedge(0)) \in \mathcal{E}(\Phi)$$

and

$$(F\bar{\gamma} + G^\vee(0), (G - G^\vee(0))\bar{\gamma}) \in \mathcal{E}(\Phi).$$

Define

$$\hat{A}_\Phi(F, G) = ((F - F^\wedge(0))\gamma, G\gamma + F^\wedge(0)).$$

Then

$$\hat{A}_\Phi^*(F, G) = ((F\bar{\gamma} + G^\vee(0)), (G - G^\vee(0))\bar{\gamma})$$

so $\hat{A}_\Phi$ is isometric.

Since $((S - S^\wedge(0))\mathbf{k}\gamma, (I - S^*S^\wedge(0))\mathbf{k})$ and $((I - SS^{*\vee}(0))\mathbf{k}, (S^* -$

$S^{*\vee}(0))\mathbf{k}\bar{\gamma})$ are elements of $\mathcal{D}(S)$ for $k \in \mathbb{K}_2$,

$$((I + \Phi)(S - S^{\wedge}(0))(I + S^{\wedge}(0))^{-1}\mathbf{k}\gamma,$$

$$-(I + \Phi^*)(I - S^*S^{\wedge}(0))(I + S^{\wedge}(0))^{-1}\mathbf{k})$$

$$= (-(\Phi + \Phi^{\wedge}(0))\mathbf{k}\gamma, -(\Phi^* + \Phi^{\wedge}(0))\mathbf{k})) \in \mathcal{E}(\Phi)$$

and

$$((I + \Phi)(I - SS^{*\vee}(0))(I + S^{*\vee}(0))^{-1}\mathbf{k},$$

$$-(I + \Phi)^*(S^* - S^{*\vee}(0))(I + S^{*\vee}(0))^{-1}\mathbf{k}\bar{\gamma})$$

$$= ((\Phi + \Phi^{*\vee}(0))\mathbf{k}, (\Phi^* - \Phi^{*\vee}(0)\mathbf{k}\bar{\gamma}) \in \mathcal{E}(\Phi).$$

Define

$$\hat{B}_{\Phi}\mathbf{k} = ((\Phi - \Phi^{\wedge}(0))\mathbf{k}\gamma, (\Phi^* + \Phi^{\wedge}(0))\mathbf{k})$$

and

$$\hat{C}^*_{\Phi}\mathbf{k} = ((\Phi + \Phi^{*\vee}(0))\mathbf{k}, (\Phi^* - \Phi^{*\vee}(0))\mathbf{k}\bar{\gamma}).$$

Then $\hat{B}^*_{\Phi}(F, G) = G^{\vee}(0)$ and $\hat{C}_{\Phi}(F, G) = F^{\wedge}(0)$. Hence we have $\hat{A}^*_{\Phi}\hat{B}_{\Phi} = \hat{C}^*_{\Phi}$ and $\hat{B}^*_{\Phi}\hat{B}_{\Phi} = \hat{C}_{\Phi}\hat{C}^*_{\Phi} = \hat{D}_{\Phi} + \hat{D}^*_{\Phi}$. □

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