**Far East Journal of Mathematical Sciences (FJMS)**

Published Online: December 2015
http://dx.doi.org/10.17654/FJMSDec2015_977_998
Volume 98, Number 8, 2015, Pages 977-998 **ISSN: 0972-0871**

# A QUADRATIC FUNCTIONAL EQUATION AND ITS STABILITY IN FELBIN'S TYPE SPACES

**K. Ravi and S. Sabarinathan**

Department of Mathematics
Sacred Heart College
Tirupattur 635 601
Tamil Nadu, India
e-mail: shckravi@yahoo.co.in
ssabarimaths@gmail.com

**Abstract**

In this paper, we introduce a new quadratic functional equation, obtain the general solution and investigate the Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability for the quadratic functional equations in Felbin's type fuzzy normed linear spaces. A counter-example for singular case is also provided in the space of real numbers.

## 1. Introduction and Preliminaries

In 1940, Ulam [13] raised the following question:

Let $(G, \cdot)$ be a group and let $H$ be a metric group with metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to H$

Received: May 27, 2015; Accepted: August 5, 2015

2010 Mathematics Subject Classification: 39B55, 39B52, 39B82.

Keywords and phrases: generalized quadratic functional equation, Hyers-Ulam-Rassias stability, Felbin's type fuzzy normed linear spaces.

Communicated by E. Thandapani

satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $a : G \to H$ exists with $d(f(x), a(x)) < \varepsilon$ for all $x \in G$?

In other words, under what condition, does there exist a homomorphism near an approximate homomorphism? In 1941, Hyers [7] gave a first affirmative answer to the question of Ulam for Banach spaces.

**Theorem 1.1** (Hyers [7]). *Let $f : E \to E'$ be a mapping between Banach spaces such that*

$$\| f(x+y) - f(x) - f(y) \| \leq \delta \tag{1.1}$$

*for all $x, y \in E$ and for some $\delta > 0$. Then there exists a unique additive mapping $A : E \to E'$ such that*

$$\| f(x) - a(x) \| \leq \delta \tag{1.2}$$

*for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$, then a is linear*.

The famous Hyers stability result that appeared in [7] was generalized in the stability involving a sum of powers of norms by Aoki [1]. Th. M. Rassias [11] and J. M. Rassias [9] provided generalizations of Hyers' theorem which allows the Cauchy difference to be unbounded.

**Theorem 1.2** (Th. M. Rassias [11]). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space $E'$ subject to the inequality*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon(\| x \|^p + \| y \|^p) \tag{1.3}$$

*for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.4}$$

*exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\| f(x) - L(x) \| \leq \frac{2\varepsilon}{2 - 2^p} \| x \|^p \tag{1.5}$$

*for all* $x \in E$. *If* $p < 0$, *then inequality* (1.3) *holds for* $x, y \neq 0$ *and* (1.5) *for* $x \neq 0$. *Also, if for each* $x \in E$, *the function* $f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then L is linear*.

**Theorem 1.3** (J. M. Rassias [9]). *Let X be a real normed linear space and Y be a real complete normed linear space. Assume that* $f : X \to Y$ *is an approximately additive mapping for which there exist constants* $\theta \geq 0$ *and* $p, q \in \mathbb{R}$ *such that* $r = p + q \neq 1$ *and f satisfies inequality*

$$\| f(x + y) - f(x) - f(y) \| \leq \theta \| x \|^p \| y \|^p \tag{1.6}$$

*for all* $x, y \in X$. *Then there exists a unique additive mapping* $L : E \to E'$ *satisfying*

$$\| f(x) - L(x) \| \leq \frac{\theta}{| 2^r - 2 |} \| x \|^r \tag{1.7}$$

*for all* $x \in X$. *If, in addition,* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then L is an* $\mathbb{R}$*-linear mapping*.

A function $f : X \to Y$ is called *quadratic* if $f$ satisfies the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.8}$$

for all $x, y \in X$.

I. S. Chang and H. M. Kim [2] established the general solution and the generalized Hyers-Ulam stability for the functional equations

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x), \tag{1.9}$$

$$f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y) \tag{1.10}$$

and showed that equations (1.9) and (1.10) are equivalent to (1.8).

During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [1, 2, 6, 8-10].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.11}$$

is called a *quadratic functional equation*. In fact, every solution of the quadratic equation (1.11) is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation (1.11) was discussed by Skof [12], Cholewa [3] and Czerwik [4] in different settings.

Recently, Zivari-Kazempour and Eshaghi Gordji [16] proved the general solution of the following quadratic functional equation:

$$f(x+2y)+f(y+2z)+f(z+2x)$$

$$=2f(x+y+z)+3[f(x)+f(y)+f(z)] \tag{1.12}$$

and investigate the Hyers-Ulam stability in Banach space.

In [5], Felbin introduced the concept of fuzzy normed linear space (FNLS); Xiao and Zhu [14] studied its linear topological structures and some basic properties of a fuzzy normed linear space. It is known that theories of classical normed space and Menger probabilistic normed spaces are special cases of fuzzy normed linear spaces.

Before we proceed to the main theorems, we will introduce some basic definitions which helpful to our theorem.

Let $\eta$ be a fuzzy subset on $\mathbb{R}$, i.e., a mapping $\eta : \mathbb{R} \to [0; 1]$ associating with each real number *t* its grade of membership $\eta(t)$.

**Definition 1.4** [5]**.** A fuzzy subset $\eta$ on $\mathbb{R}$ is called a *fuzzy real number*, whose $\alpha$-level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:

$(N_1)$ There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

$(N_2)$ For each $\alpha \in (0, 1]$, $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, where $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$.

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then $\eta$ is called a *non-negative fuzzy real number* and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers.

**Definition 1.5** [5]**.** Fuzzy arithmetic operations $\oplus$, $\ominus$, $\otimes$, $\oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

(1) $(\eta \oplus \delta)(t) = \sup\limits_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(t - s)\}$, $t \in \mathbb{R}$,

(2) $(\eta \ominus \delta)(t) = \sup\limits_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(s - t)\}$, $t \in \mathbb{R}$,

(3) $(\eta \otimes \delta)(t) = \sup\limits_{s \in \mathbb{R}, s \neq 0} \{\eta(s) \wedge \delta(t/s)\}$, $t \in \mathbb{R}$,

(4) $(\eta \oslash \delta)(t) = \sup\limits_{s \in \mathbb{R}} \{\eta(st) \wedge \delta(s)\}$, $t \in \mathbb{R}$.

The additive and multiplicative identities in $F(\mathbb{R})$ are $\overline{0}$ and $\overline{1}$, respectively. Let $\ominus\eta$ be defined as $\overline{0} - \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus\eta)$.

**Definition 1.6** [5]**.** For $k \in \mathbb{R}/0$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta)(t) = \eta(t/k)$ and $0 \odot \eta$ is defined to be $\overline{0}$.

**Definition 1.7** [5]**.** Define a partial ordering $\preceq$ in $F(\mathbb{R})$ by $\eta \preceq \delta$ if and only if $\eta_\alpha^- \leq \delta_\alpha^-$ and $\eta_\alpha^+ \leq \delta_\alpha^+$ for all $\alpha \in (0, 1]$. The strict inequality in $F(\mathbb{R})$ is defined by $\eta < \delta$ if and only if $\eta_\alpha^- < \delta_\alpha^-$ and $\eta_\alpha^+ < \delta_\alpha^+$ for all $\alpha \in (0, 1]$.

**Definition 1.8** [14]**.** Let $X$ be a real linear space; $L$ and $R$ (respectively, left norm and right norm) be symmetric and non-decreasing mappings from $[0; 1] \times [0; 1]$ into $[0; 1]$ satisfying $L(0; 0) = 0$, $R(1; 1) = 1$. Then $\|\cdot\|$ is called a *fuzzy norm* and $(X, \|\cdot\|, L, R)$ is a fuzzy normed linear space (abbreviated to FNLS) if the mapping $\|\cdot\|$ from $X$ into $F^*(R)$ satisfies the following axioms, where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $x \in X$ and $\alpha \in (0; 1]$:

(N1) $\|x\| = \overline{0}$ if and only if $x = 0$,

(N2) $\|rx\| = |r| \odot \|x\|$ for all $x \in X$ and $r \in (-\infty, \infty)$,

(N3) for all $x, y \in X$:

(N3L) If $s \le \|x\|_1^-$, $t \le \|y\|_1^-$ and $s + t \le \|x + y\|_1^-$, then

$$\|x + y\|(s + t) \ge L(\|x\|(s), \|y\|(t)).$$

(N3R) If $s \ge \|x\|_1^-$, $t \ge \|y\|_1^-$ and $s + t \ge \|x + y\|_1^-$, then

$$\|x + y\|(s + t) \le R(\|x\|(s), \|y\|(t)).$$

**Lemma 1.9** [15]. *Let* $(X, \|\cdot\|, L, R)$ *be an FNLS and suppose that*

(R-1) $R(a, b) \le \max(a, b)$,

(R-2) $\forall\alpha \in (0, 1]$, $\exists\beta(0, \alpha]$ *such that* $R(\beta, y) \le \alpha$ *for all* $y \in (0, \alpha)$,

(R-3) $\lim_{a\to 0^+} R(a, a) = 0$.

*Then* (R-1) $\Rightarrow$ (R-2) $\Rightarrow$ (R-3), *but not conversely*.

**Lemma 1.10** [15]. *Let* $(X, \|\cdot\|, L, R)$ *be an FNLS. Then we have the following*:

(1) *If* $R(a, b) \le \max(a, b)$, *then for all* $\alpha \in (0, 1]$, $\|x + y\|_\alpha^+ \le \|x\|_\alpha^+ + \|y\|_\alpha^+$ *for all* $x, y \in X$.

(2) *If* (R-2), *then for each* $\alpha \in (0, 1]$, *there is* $\beta \in (0, \alpha]$ *such that* $\|x + y\|_\alpha^+ \le \|x\|_\beta^+ + \|y\|_\beta^+$ *for all* $x, y \in X$.

(3) *If* $\lim_{a\to 0^+} R(a, a) = 0,$ *then for each* $\alpha \in (0, 1],$ *there is* $\beta \in (0, \alpha]$ *such that* $\| x + y \|_\alpha^+ \le \| x \|_\beta^+ + \| y \|_\beta^+$ *for all* $x, y \in X.$

**Lemma 1.11.** *Let* $(X, \| \cdot \|, L, R)$ *be an FNLS and suppose that*

$(L\text{-}1)$ $L(a, b) \ge \min(a, b),$

$(L\text{-}2)$ $\forall \alpha \in (0, 1],$ $\exists \beta(\alpha, 1]$ *such that* $L(\beta, \gamma) \ge \alpha$ *for all* $\gamma \in (\alpha), 1,$

$(L\text{-}3)$ $\lim_{a\to 1^-} L(a, a) = 1.$

*Then* $(L\text{-}1) \Rightarrow (L\text{-}2) \Rightarrow (L\text{-}3)$, *but not conversely*.

**Lemma 1.12.** *Let* $(X, \| \cdot \|, L, R)$ *be an FNLS. Then we have the following*:

(1) *If* $L(a, b) \ge \min(a, b),$ *then* $\forall \alpha \in (0, 1],$ $\| x + y \|_\alpha^- \le \| x \|_\alpha^- + \| y \|_\alpha^-$ *for all* $x, y \in X.$

(2) *If* $(L\text{-}2)$, *then for each* $\alpha \in (0, 1],$ *there is* $\beta \in (\alpha, 1]$ *such that* $\| x + y \|_\alpha^- \le \| x \|_\beta^- + \| y \|_\alpha^-$ *for all* $x, y \in X.$

(3) *If* $\lim_{a\to 1^-} L(a, a) = 1,$ *then for each* $\alpha \in (0, 1],$ *there is* $\beta \in (\alpha, 1]$ *such that* $\| x + y \|_\alpha^- \le \| x \|_\beta^- + \| y \|_\beta^-$ *for all* $x, y \in X.$

**Lemma 1.13.** *Let* $(X, \| \cdot \|, L, R)$ *be an FNLS satisfying* $(R\text{-}2)$. *Then*:

(1) *For each* $\alpha \in (0, 1],$ $\| \cdot \|_\alpha^+$ *is a continuous mapping from X into* $\mathbb{R}$ *at* $x \in X.$

(2) *For any* $n \in \mathbb{Z}^+$ *and* $\{x_i\}_{i=1}^n,$ *we have*

$$\forall \alpha \in (0, 1], \quad \exists \beta \in (0, \alpha]; \quad \left\| \sum_{i=1}^n x_i \right\|_\alpha^+ \le \sum_{i=1}^n \| x_i \|_\beta^+.$$

**Definition 1.14** [14]**.** Let $(X, \|\cdot\|, L, R)$ *be an FNLS and* $\lim_{a\to 0^+} R(a, a) = 0$. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \| x_n - x \|_{\alpha}^{+} = 0$ for every $\alpha \in (0, 1]$ and is called a *Cauchy sequence* if $\lim_{m, n\to\infty} \| x_m - x_n \|_{\alpha}^{+} = 0$ for every $\alpha \in (0, 1]$. A subset $A \subseteq X$ is said to be *complete* if every Cauchy sequence in *A*, converges in *A*. The fuzzy normed space $(X, \|\cdot\|, L, R)$ is said to be a *fuzzy Banach space* if it is complete.

**Lemma 1.15.** *Let* $(X, \|\cdot\|, L, R)$ *be an FNLS satisfying* (*R*-2). *Then*:

(1) *For each* $\alpha \in (0, 1]$, $\|\cdot\|_{\alpha}^{+}$ *is a continuous mapping from X into* $\mathbb{R}$ *at* $x \in X$.

(2) *For any* $n \in \mathbb{Z}^{+}$ *and* $\{x_i\}_{i=1}^{n}$, *we have*

$$\forall\alpha \in (0, 1], \quad \exists\beta \in (0, \alpha]; \quad \left\| \sum_{i=1}^{n} x_i \right\|_{\alpha}^{+} \leq \sum_{i=1}^{n} \| x_i \|_{\beta}^{+}.$$

In this paper, we introduce a new quadratic functional equation of the form

$$9[f(3x+y)+f(x+3y)]-52f(x+y)+2f(x-y)=40[f(x)+f(y)] \quad (1.13)$$

and discuss its general solutions in Section 2. In Section 3, we investigate the Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability for functional equation (1.13) in Felbin's type fuzzy normed linear spaces.

Now we proceed to find the general solution of the functional equation (1.13).

## 2. General Solution to Quadratic Functional Equation (1.13)

In this section, we obtain the general solution of the functional equation (1.13). Throughout this section, let *X* and *Y* be vector spaces.

**Theorem 2.1.** *A function* $f : X \to Y$ *is a solution of the functional equation* (1.13) *for all* $x, y \in X$ *if and only if it satisfies the quadratic functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.1}$$

*for all* $x \in X$.

**Proof.** Assume *f* satisfies the functional equation (1.13). Letting $(x, y)$ by $(0, 0)$ in (1.13), we get $f(0) = 0$. Setting $y = 0$ in (1.13), we obtain

$$f(3x) = 9f(x) \tag{2.2}$$

for all $x \in X$. Replacing *x* by $\frac{x}{3}$ and *y* by 0 in (1.13), we get

$$f\left(\frac{x}{3}\right) = \frac{1}{9} f(x) \tag{2.3}$$

for all $x \in X$. Thus, *f* is quadratic. Putting $x = 0$ in (1.13) and using (2.2), we obtain $f(-y) = f(y)$ for all $y \in X$. Thus, *f* is an even function. Setting *y* by $-x$ in (1.13) and using evenness of *f*, we arrive

$$f(2x) = 4f(x) \tag{2.4}$$

for all $x \in X$. Replacing $(x, y)$ by $(x+y, x-y)$ in (1.13), we get

$$9f(2x+y) + 9f(2x-y) - 52f(x) + 2f(y) = 10[f(x+y) + f(x-y)] \tag{2.5}$$

for all $x, y \in X$. Setting $(x, y)$ by $\left(\frac{x}{2}, 0\right)$ in (2.5), we get

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.6}$$

for all $x \in X$. Substituting *x* by $\frac{x}{2}$ in (2.5), we obtain

$$5[f(x+2y) + f(x-2y)] = 18[f(x+y) + f(x-y)] - 26f(x) + 4f(y) \tag{2.7}$$

for all $x, y \in X$. Replacing $x$ by $y$ in (2.7) and subtracting from (2.5), we arrive

$$f(2x+y)+f(2x-y)=12f(x)+6f(y)-2[f(x+y)+f(x-y)] \quad (2.8)$$

for all $x, y \in X$. Putting $y$ by $x+y$ in (2.8), we get

$$f(3x+y)+f(x-y)+2f(2x+y)+2f(y)=12f(x)+6f(x+y) \quad (2.9)$$

for all $x, y \in X$. Replacing $y$ by $-y$ in (2.9) and adding the resultant with (2.9) and using (2.8), we arrive

$$f(3x+y)+f(3x-y)=9[f(x+y)+f(x-y)]-16f(x) \quad (2.10)$$

for all $x, y \in X$. Setting $(x, y)$ by $(y, x)$ in (2.10), we get

$$f(x+3y)+f(x-3y)=9[f(x+y)+f(x-y)]-16f(x) \quad (2.11)$$

for all $x, y \in X$. Setting $(x, y)$ by $\left(y, \frac{2x+y}{3}\right)$ in (2.11) and using (2.3), we arrive

$$f(x+2y)=f(x+y)-f(x-y)+f(x)+4f(y) \quad (2.12)$$

for all $x, y \in X$. Replacing $x$ by $y$ in (2.12), we arrive

$$f(2x+y)=f(x+y)-f(x-y)+f(y)+4f(x) \quad (2.13)$$

for all $x, y \in X$. Replacing $y$ by $-y$ in (2.13) and adding the resultant with (2.13), we arrive

$$f(2x+y)+f(2x-y)=8f(x)+2f(y) \quad (2.14)$$

for all $x, y \in X$. Replacing $x$ by $\frac{x}{2}$ and using (2.6), we arrive (2.1).

Conversely, suppose that a function $f: X \to Y$ satisfies (2.1). Replacing $y$ by $x+y$ in (2.1), we get

$$f(2x+y)+f(y)=2f(x)+2f(x+y) \quad (2.15)$$

for all $x, y \in X$. Now, replacing $y$ by $x + y$ in (2.15), we obtain

$$f(3x + y) - 3f(x + y) = 6f(x) - 2f(y) \tag{2.16}$$

for all $x, y \in X$. Setting $(x, y)$ by $(y, x)$ in (2.16) and adding the resultant with (2.16), we arrive

$$f(3x + y) + f(x + 3y) - 6f(x + y) = 4f(x) + 4f(y) \tag{2.17}$$

for all $x, y \in X$. Setting $(x, y)$ by $(x + y, x - y)$ in (2.15), we get

$$f(3x + y) + f(x - y) = 2f(x + y) + 8f(x) \tag{2.18}$$

for all $x, y \in X$. Replacing $x$ by $y$ and $y$ by $x$ in (2.18) and adding the resultant with (2.18), we obtain

$$f(3x + y) + f(x + 3y) + 2f(x - y) = 4f(x + y) + 8f(x) + 8f(y) \tag{2.19}$$

for all $x, y \in X$. Multiply equation (2.17) by 8 and adding the resultant with (2.19), we arrive (1.13). □

## 3. Generalized Hyers-Ulam-Rassias Stability of (1.13)

Now, we prove the fuzzy stability of quadratic functional equation in the spirit of Hyers, Ulam and Rassias.

Let us denote

$$D_f(x, y) = 9[f(3x + y) + f(x + 3y)] - 52f(x + y)$$
$$+ 2f(x - y) - 40[f(x) + f(y)]$$

for all $x, y \in X$.

**Theorem 3.1.** *Let $X$ be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying* ($R$-2). *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to F^*(\mathbb{R})$ such that*

$$\sum_{i=0}^{\infty} \frac{(\phi(3^i x,\ 3^i y))_{\alpha}^{+}}{9^i} < \infty \tag{3.1}$$

*for all* $x,\ y \in X$.

$$\| D_f(x,\ y) \| \preceq \phi(x,\ y) \tag{3.2}$$

*for all* $x,\ y \in X$ *and all* $\alpha \in (0,\ 1]$. *Then there exists a unique quadratic mapping* $T : X \to Y$ *such that* $\forall \alpha \in (0,\ 1]$, $\exists \beta \in (0,\ \alpha]$ *s.t.*

$$\| f(x) - T(x) \|_{\alpha}^{+} \leq \frac{1}{81} \sum_{i=0}^{\infty} \frac{(\phi(3^i x,\ 0))_{\beta}^{+}}{9^i}, \quad \forall x \in X. \tag{3.3}$$

**Proof.** Setting $y = 0$ in (3.2), we get

$$\| 9 f(3x) - 81 f(x) \| \preceq \phi(x,\ 0) \tag{3.4}$$

for all $x \in X$. Multiply both sides of equation (3.4) by $\frac{1}{9}$ in the fuzzy scalar multiplication sense, then we have

$$\| f(3x) - 9 f(x) \| \preceq \frac{1}{9} \odot \phi(x,\ 0). \tag{3.5}$$

Replacing $x$ by $3^n x$ and multiplying by $\frac{1}{9^{n+1}}$ in equation (3.5) in the fuzzy scalar multiplication sense, we obtain

$$\left\| \frac{f(3^{n+1} x)}{9^{n+1}} - \frac{f(3^n x)}{9^n} \right\| \preceq \frac{1}{81} \frac{1}{9^n} \odot \phi(3^n x,\ 0) \tag{3.6}$$

for all $x \in X$ and all non-negative integers $n \in \mathbb{N}$. By Lemma 1.13 and inequality (3.6), we conclude that for all $\alpha \in (0,\ 1]$, there exists $\beta \in (0,\ \alpha]$ such that

$$\left\| \frac{f(3^{n+1} x)}{9^{n+1}} - \frac{f(3^m x)}{9^m} \right\|_{\alpha}^{+} \leq \frac{1}{81} \sum_{i=m}^{n} \frac{1}{9^i} (\phi(3^i x,\ 0))_{\beta}^{+} \tag{3.7}$$

for all $x \in X$ and all non-negative integers $m$ and $n$ with $n \geq m$. Now (3.1) and (3.7) imply that $\left\{\frac{f(3^n x)}{9^n}\right\}$ is a fuzzy Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is a fuzzy Banach space, the sequence $\left\{\frac{f(3^n x)}{9^n}\right\}$ converges for all $x \in X$. So we can define the mapping $T : X \to Y$ by

$$T(x) := \lim_{n\to\infty} \frac{f(3^n x)}{9^n}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.7), by continuity of $\|\cdot\|_\alpha^+$, we have

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{1}{81} \sum_{i=0}^{\infty} \frac{(\phi(3^i x, 0))_\beta^+}{9^i} \tag{3.8}$$

for all $x \in X$. Therefore, we obtain (3.3). Now we show that $T$ is quadratic and unique. Applying (3.1), (3.2) and the continuity of $\|\cdot\|_\alpha^+$, we have

$$\| 9[T(3x + y) + T(x + 3y)] - 52T(x + y) + 2T(x - y) - 40[T(x) + T(y)] \|_\alpha^+$$

$$\leq \lim_{n\to\infty} \frac{(\phi(3^n x, 3^n y))_\alpha^+}{9^n} = 0$$

for all $x, y \in X$. Therefore, the mapping $T : X \to Y$ is quadratic. To prove the uniqueness of $T$, let $T' : X \to Y$ be a quadratic mapping satisfying (3.3). Since by Lemma 1.13,

$$\| T(x) - T'(x) \| \leq \lim_{n\to\infty} \frac{1}{9^n} \frac{1}{81} \sum_{i=0}^{\infty} \frac{(\phi(3^i 3^n x, 0))_\beta^+}{9^i}$$

$$\leq \lim_{n\to\infty} \frac{1}{81} \sum_{i=n}^{\infty} \frac{(\phi(3^i x, 0))_\beta^+}{9^i} = 0$$

for all $x \in X$, $T = T'$. Hence, $T$ is unique. This completes the proof of Theorem 3.1. □

**Remark 3.2.** The above theorem is also true if $\|\cdot\|_\alpha^+$ is replaced by $\|\cdot\|_\alpha^-$ in (3.1) and the fuzzy Banach space $Y$ satisfies ($L$-2) and ($R$-2).

The following theorem is an alternative result of Theorem 3.1.

**Theorem 3.3.** *Let $X$ be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \leq \max(a, b)$ and $L(a, b) \geq \min(a, b)$. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to F^*(\mathbb{R})$ satisfying* (3.1) *and* (3.2) *for all $x, y \in X$ and all $\alpha \in (0, 1]$. Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\| f(x) - T(x) \| \preceq \overline{\varphi}(x, 0) \tag{3.9}$$

*for all $x \in X$, where $\overline{\varphi}(x, 0)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[a_\alpha, b_\alpha]$ such that*

$$a_\alpha = \frac{1}{81}\sum_{i=0}^{\infty} \frac{(\varphi(3^i x, 0))_\alpha^-}{9^i},$$

$$b_\alpha = \frac{1}{81}\sum_{i=0}^{\infty} \frac{(\varphi(3^i x, 0))_\alpha^+}{9^i}$$

*for all $x \in X$.*

**Theorem 3.4.** *Let $X$ be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying* ($R$-2). *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to F^*(\mathbb{R})$ such that*

$$\sum_{i=1}^{\infty} 9^i \left(\phi\left(\frac{x}{3^i}, \frac{y}{3^i}\right)\right)_\alpha^+ < \infty \tag{3.10}$$

*for all $x, y \in X$.*

$$\| D_f(x, y) \| \preceq \phi(x, y) \tag{3.11}$$

*for all* $x, y \in X$ *and all* $\alpha \in (0, 1]$. *Then there exists a unique quadratic mapping* $T : X \to Y$ *such that* $\forall \alpha \in (0, 1]$, $\exists \beta \in (0, \alpha]$ *s.t.*

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{1}{81} \sum_{i=1}^{\infty} 9^i \left( \varphi\left( \frac{x}{3^i}, 0 \right) \right)_\beta^+ \tag{3.12}$$

*for all* $x \in X$.

**Proof.** Setting $y = 0$ in (3.11), we get

$$\| 9f(3x) - 81f(x) \| \preceq \varphi(x, 0) \tag{3.13}$$

for all $x \in X$. Multiply both sides of equation (3.13) by $\frac{1}{9}$ in the fuzzy scalar multiplication sense, then we have

$$\| f(3x) - 9f(x) \| \preceq \frac{1}{9} \odot \varphi(x, 0) \tag{3.14}$$

for all $x \in X$. Replacing $x$ by $\frac{x}{3^{n+1}}$ and multiplying both sides by $9^n$ in equation (3.14) in the fuzzy scalar multiplication sense, we obtain

$$\left\| 9^n f\left( \frac{x}{3^n} \right) - 9^{n+1} f\left( \frac{x}{3^{n+1}} \right) \right\| \preceq \frac{9^n}{9} \odot \varphi\left( \frac{x}{3^{n+1}}, 0 \right) \tag{3.15}$$

for all $x \in X$. By Lemma 1.13 and inequality (3.15), we conclude that for all $\alpha \in (0, 1]$, there exists $\beta \in (0, \alpha]$ such that

$$\left\| 9^{n+1} f\left( \frac{x}{3^{n+1}} \right) - 9^m f\left( \frac{x}{3^m} \right) \right\|_\alpha^+ \leq \sum_{i=m}^{n} \frac{9^i}{9} \left( \varphi\left( \frac{x}{3^{i+1}}, 0 \right) \right)_\beta^+ \tag{3.16}$$

for all $x \in X$ and all non-negative integers $m$ and $n$ with $n \geq m$. Now (3.10) and (3.16) imply that $\left\{ 9^n f\left( \frac{x}{3^n} \right) \right\}$ is a fuzzy Cauchy sequence in $Y$ for all

$x \in X$. Since $Y$ is a fuzzy Banach space, the sequence $\left\{9^n f\left(\frac{x}{3^n}\right)\right\}$ converges for all $x \in X$. The rest of this proof is similar to the proof of Theorem 3.1. □

The same discussion in Remark 3.2 does hold for the above theorem. Also, the following theorem is an alternative result of Theorem 3.4.

**Theorem 3.5.** *Let $X$ be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \le \max(a, b)$ and $L(a, b) \ge \min(a, b)$. Let $f: X \to Y$ be a mapping for which there exists a function $\varphi: X \times X \to F^*(\mathbb{R})$ satisfying* (3.10) *and* (3.11) *for all $x, y \in X$ and all $\alpha \in (0, 1]$. Then there exists a unique quadratic mapping $T: X \to Y$ such that*

$$\| f(x) - T(x) \| \preceq \overline{\varphi}(x, 0) \tag{3.17}$$

*for all $x \in X$, where $\overline{\varphi}(x, 0)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[a_\alpha, b_\alpha]$ such that*

$$a_\alpha = \frac{1}{81} \sum_{i=1}^{\infty} 9^i \left(\varphi\left(\frac{x}{3^i}, 0\right)\right)_\alpha^-,$$

$$b_\alpha = \frac{1}{81} \sum_{i=1}^{\infty} 9^i \left(\varphi\left(\frac{x}{3^i}, 0\right)\right)_\alpha^+$$

*for all $x \in X$.*

The following corollaries are the immediate consequences of Theorems 3.1 and 3.4 which give the Hyers-Ulam and generalized Hyers-Ulam stability of the functional equation (1.13).

**Corollary 3.6.** *Let $\varepsilon$ be a non-negative fuzzy real number, $X$ be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \le \max(a, b)$ and $L(a, b) \ge \min(a, b)$. Suppose that the mapping $f: X \to Y$ satisfies the inequality*

$$\| D_f(x, y) \| \preceq \varepsilon \tag{3.18}$$

*for all* $x, y \in X$. *Then there exists a unique quadratic mapping* $T : X \to Y$ *satisfying*

$$\| f(x) - T(x) \| \preceq \frac{\varepsilon}{72} \tag{3.19}$$

*for all* $x \in X$.

**Proof.** Let $\varphi(x, y) := \varepsilon$ for all $x, y \in X$. By Theorem 3.1, we get the desired result. □

**Corollary 3.7.** *Let* $\varepsilon$ *be a non-negative fuzzy real number and* $p, q$ *be non-negative real numbers such that* $p, q > 2$ *or* $0 < p, q < 2$. *Let* $X$ *be a fuzzy normed linear space and* $(Y, \|\cdot\|, L, R)$ *be a fuzzy Banach space satisfying* ($R$-2). *Suppose that the mapping* $f : X \to Y$ *satisfies the inequality*

$$\| D_f(x, y) \| \preceq \varepsilon \otimes (\| x \|_X^p \oplus \| y \|_X^q) \tag{3.20}$$

*for all* $x, y \in X$. *Then there exists a unique quadratic mapping* $T : X \to Y$ *such that* $\forall \alpha \in (0, 1]$, $\exists \beta \in (0, \alpha]$ *s.t.*

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{\varepsilon_\beta^+ (\| x \|_\beta^+)^p}{9| 3^p - 3^2 |} \tag{3.21}$$

*for all* $x \in X$.

**Proof.** The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(x, y) := \varepsilon \otimes (\| x \|_X^p \oplus \| y \|_X^q)$$

for all $x, y \in X$. □

Now we will provide an example to illustrate that the functional equation (1.13) is not stable for $p = 2$ in Corollary 3.7.

**Example 3.8.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\phi(x) = \begin{cases} ax^2, & \text{if } |x| < 1, \\ a, & \text{otherwise,} \end{cases}$$

where $a > 0$ is a constant and a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(3^n x)}{9^n} \text{ for all } x \in \mathbb{R}.$$

Then *f* satisfies the functional inequality

$$\| 9f(x+3y) + 9f(3x+y) - 52f(x+y) + 2f(x-y) - 40f(x) - 40f(y) \|$$

$$\leq 1539a(|x|^2 + |y|^2) \tag{3.22}$$

for all $x, y \in \mathbb{R}$. Then there do not exist a quadratic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - C(x)| \leq \beta |x|^2 \text{ for all } x \in \mathbb{R}. \tag{3.23}$$

**Proof.** Now

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi(3^n x)|}{|9^n|} = \sum_{n=0}^{\infty} \frac{a}{9^n} = \frac{9a}{8}.$$

Therefore, we see that *f* is bounded. We are going to prove that *f* satisfies (3.22).

If $x = y = 0$, then (3.22) is trivial. If $|x|^2 + |y|^2 \geq \frac{1}{9}$, then the left hand side of (3.22) is less than $171a$. Now suppose that $0 < |x|^2 + |y|^2 < \frac{1}{9}$. Then there exists a positive integer *k* such that

$$\frac{1}{9^{k+1}} \leq |x|^2 + |y|^2 < \frac{1}{9^k}, \tag{3.24}$$

so that $9^{k-1}x^2 < \frac{1}{9}$, $9^{k-1}y^2 < \frac{1}{9}$ and consequently,

$$3^{k-1}(x),\ 3^{k-1}(y),\ 3^{k-1}(x+y),\ 3^{k-1}(x-y),\ 3^{k-1}(3x+y),\ 3^{k-1}(x+3y)$$

$\in (-1, 1)$.

Therefore, for each $n = 0, 1, \ldots, k-1,$ we have

$$3^n(x),\ 3^n(y),\ 3^n(x+y),\ 3^n(x-y),\ 3^n(3x+y),\ 3^n(x+3y) \in (-1, 1)$$

and

$$9\phi(3^n(x+3y)) + 9\phi(3^n(3x+y)) - 52\phi(3^n(x+y))$$

$$+ 2\phi(3^n(x-y)) - 40\phi(3^n(x)) - 40\phi(3^n(y)) = 0$$

for $n = 0, 1, \ldots, k-1.$ From the definition of $f$ and (3.24), we obtain that

$$|\, 9f(x+3y) + 9f(3x+y) - 52f(x+y) + 2f(x-y)$$

$$- 40f(x) - 40f(y)\,|$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{9^n} |\, 9\phi(3^n(x+3y)) + 9\phi(3^n(3x+y)) - 52\phi(3^n(x+y))$$

$$+ 2\phi(3^n(x-y)) - 40\phi(3^n(x)) - 40\phi(3^n(y))\,|$$

$$\leq \sum_{n=k}^{\infty} \frac{1}{9^n} |\, 9\phi(3^n(x+3y)) + 9\phi(3^n(3x+y)) + 52\phi(3^n(x+y))$$

$$+ 2\phi(3^n(x-y)) + 40\phi(3^n(x)) + 40\phi(3^n(y))\,|$$

$$\leq \sum_{n=k}^{\infty} \frac{1}{9^n} 152a = \frac{1368a}{8} \times \frac{1}{9^k} = 1539a(|\, x\,|^2 + |\, y\,|^2).$$

Thus, $f$ satisfies (3.22) for all $x, y \in \mathbb{R}$ with $0 < |\, x\,|^2 + |\, y\,|^2 < \frac{1}{9}$.

We claim that the quadratic functional equation (1.13) is not stable for $p = 2$ in Corollary 3.7. Suppose on the contrary, there exist a quadratic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (3.23). Since $f$ is bounded and continuous for all $x \in \mathbb{R}$, $C$ is bounded on any open interval containing the origin and continuous at the origin. In view of Corollary 3.7, $C(x)$ must have the form $C(x) = kx^2$ for any $x$ in $\mathbb{R}$. Thus, we obtain that

$$|f(x)| \le (\beta + |k|)|x|^2. \tag{3.25}$$

But we can choose a positive integer $m$ with $ma > \beta + |k|$.

If $x \in \left(0, \frac{1}{3^{m-1}}\right)$, then $3^n x \in (0, 1)$ for all $n = 0, 1, ..., m-1$. For this $x$, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(3^n x)}{9^n} \ge \sum_{n=0}^{m-1} \frac{a(3^n x)^2}{9^n} = max^2 > (\beta + |k|)x^2$$

which contradicts (3.25). Therefore, the quadratic functional equation (1.13) is not stable in sense of Ulam, Hyers and Rassias if $p = 2$, assumed in the inequality (3.20). □

We obtain the following corollary for Theorems 3.1 and 3.4 for the functional equation (1.13).

**Corollary 3.9.** *Let* $\varepsilon$ *be a non-negative fuzzy real number and* $p$, $q$ *be non-negative real numbers such that* $p, q > 1$ *or* $0 < p, q < 1$. *Let* $X$ *be a fuzzy normed linear space and* $(Y, \|\cdot\|, L, R)$ *be a fuzzy Banach space satisfying* $(R\text{-}2)$. *Suppose that the mapping* $f : X \to Y$ *satisfies the inequality*

$$\| D_f(x, y) \| \preceq \varepsilon \otimes (\| x \|_X^p \otimes \| y \|_X^p \oplus [\| x \|_X^{2p} \oplus \| y \|_X^{2p}]) \tag{3.26}$$

*for all* $x, y \in X$. *Then there exists a unique quadratic mapping* $T : X \to Y$ *such that* $\forall \alpha \in (0, 1]$, $\exists \beta \in (0, \alpha]$ *s.t.*

$$\| f(x) - T(x) \|_{\alpha}^{+} \leq \frac{\varepsilon_{\beta}^{+}(\| x \|_{\beta}^{+})^{2p}}{9| 3^{2p} - 3^{2} |} \tag{3.27}$$

*for all* $x \in X$.

**Proof.** The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(x, y) := \varepsilon \otimes (\| x \|_{X}^{p} \otimes \| y \|_{X}^{p} \oplus [\| x \|_{X}^{2p} \oplus \| y \|_{X}^{2p}])$$

for all $x, y \in X$. □

## 4. Conclusion

In this paper, we introduced and achieved the general solution of new quadratic functional equation. Also, the fuzzy version of Hyers-Ulam stability, Hyers-Ulam-Rassias stability, generalized Hyers-Ulam-Rassias stability and Hyers-Ulam-J. M. Rassias stability problems are studied.

## References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950), 64-66.

[2] I. S. Chang and H. M. Kim, On the Hyers-Ulam stability of quadratic functional equations, J. Inequal. Pure Appl. Math. 3 (2002), Art. 33, pp. 12 (electronic).

[3] P. W. Cholewa, Remarks on the stability of functional equations, Aequationes Math. 27 (1984), 76-86.

[4] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg 62 (1992), 59-64.

[5] C. Felbin, Finite dimensional fuzzy normed linear space, Fuzzy Sets and Systems 48 (1992), 239-248.

[6] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431-436.

[7] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941), 222-224.

[8] Pl. Kannappan, Quadratic functional equation and inner product spaces, Results Math. 27 (1995), 368-372.

[9] J. M. Rassias, On approximation of approximately linear mappings by linear mapping, J. Funct. Anal. 46(1) (1982), 126-130.

[10] Matina J. Rassias, Generalised Hyers-Ulam product-sum stability of a Cauchy type additive functional equation, European J. Pure Appl. Math. 4(1) (2011), 50-58.

[11] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[12] F. Skof, Local properties and approximations of operators, Rend. Sem. Mat. Fis. Milano 53 (1983), 113-129.

[13] S. M. Ulam, A Collection of the Mathematical Problems, Interscience Publ., New York, 1960.

[14] J. Xiao and X. Zhu, On linearly topological structure and property of fuzzy normed linear space, Fuzzy Sets and Systems 125 (2002), 153-161.

[15] J. Xiao and X. Zhu, Topological degree theory and fixed point theorems in fuzzy normed space, Fuzzy Sets and Systems 147 (2004), 437-452.

[16] A. Zivari-Kazempour and M. Eshaghi Gordji, Generalized Hyers-Ulam stabilities of an Euler-Lagrange-Rassias quadratic functional equation, Asian-European Journal of Mathematics 5 (2012), Doi: 1250014.