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COMPLETE ANALYSIS OF THE NONLINEAR PENDULUM FOR AMPLITUDES IN ALL REGIMES USING NUMERICAL INTEGRATION

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Abstract

A complete analysis is presented for the nonlinear pendulum covering the low, medium and high regimes of the pendulum motion. In addition, the behavior of the pendulum velocity, potential and kinetic energies is investigated in all regimes. Numerical integration in the form of Gaussian quadrature with 9 quadrature points proved sufficient in solving the elliptic integral of the first kind which represents the behavior of the nonlinear pendulum. Also, the power series solution is calculated here for all regimes and used as the exact solution for comparison purposes. A regression based equation for the period in the low and medium regimes is given in terms of the amplitude and length of the pendulum. The equation was successfully used to calculate the gravitational acceleration in terms of the measured values of the pendulum length, period and amplitude.

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1. Introduction

Pendulums have played an important role in physics, mathematics and engineering. A pendulum is a rigid body suspended from a fixed point (hinge) which is offset with respect to the body's center of mass. If all the mass is assumed to be concentrated at a point, we obtain the idealized simple pendulum. The simple pendulum is an idealized model for the real pendulum, consisting of a rod of length L and a bob of mass m, with the following assumptions: (1) The rod/string/cable is not extensible and its weight is negligible. (2) There is no loss of energy and no friction and (3) The motion is under the gravitational field and takes place in a vertical plane.

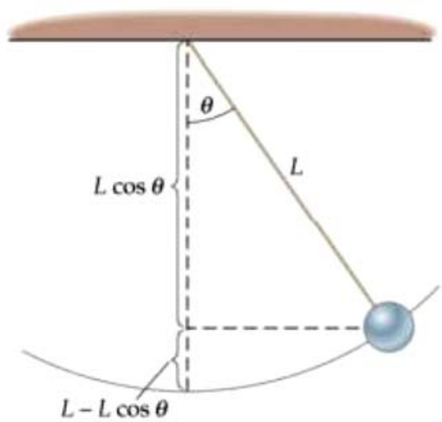

Figure 1. Schematic of the pendulum.

The angular displacement of the bob from the vertical line is represented by θ, Figure 1. From the law of conservation of energy, we get, Edwards and Penney [1]:

$$E = mgL(1 - \cos\theta_0) = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1 - \cos\theta), \tag{1}$$

where E is the total energy, m is the mass of pendulum, g is the acceleration due to gravity, L is the pendulum length, θ_0 is amplitude or the maximum angular displacement at time zero measured from the vertical line, and θ is the angle at time t. The first term on the right hand side of equation (1) is the kinetic energy and the second term is the potential energy. We measure the potential energy in such a way that when $\theta = 0$, the potential is equal to zero. We assume that $0 < \theta_0 < \pi$ such that the pendulum does not go around the pivot and bob is never directly above the pivot. The interval from 0 to π can be divided into the following three regimes:

Small regime: $0^\circ < \theta_0 \leq 15^\circ$

Medium regime: $15^\circ < \theta_0 \leq 90^\circ$

High regime: $90^\circ < \theta_0 < 180^\circ$.

By equating the forces acting on the bob, we have the following differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0. \tag{2}$$

In a usual course of physics or calculus, θ is assumed small and $\sin\theta \approx \theta$. Edwards and Penney [1] stated that sin θ and θ agree to two decimal digits when $|\theta|$ is at most $\pi/12$ (that is less than 15°). This gives the linear model or the simple harmonic motion model as:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{3}$$

Straightforward solution of equation (3) yields the well-known cosine function solution as

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\,t\right). \tag{4}$$

The period of oscillation of the linear pendulum, T_0, can be found from equation (4) as:

$$T_0 = 2\pi\sqrt{\frac{L}{g}}. \tag{5}$$

Students of physics in colleges and universities are often taught the simple pendulum motion as a simple harmonic motion in that case the amplitude of the pendulum should be less than 15°, Giancoli [2]. As seen in equation (5), the period of the simple pendulum depends only on the length of pendulum. Students' scientific curiosity requires knowing the period and profile motion beyond this limit of low amplitudes. In addition, velocity of the pendulum bob, potential and kinetic energies transformations are usually not addressed in standard physics text books. In this paper, a complete analysis is given of the simple pendulum for amplitudes up to 180° (the medium and high regimes). To analyze the pendulum in all regimes, the nonlinear equation (1) is rewritten as

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{L}}\sqrt{(\cos\theta - \cos\theta_0)}, \tag{6}$$

where + (-) sign is for counter-clockwise (clockwise) motion, Lima and Arun [3]. They integrated $d\theta/dt$ from θ_0 to 0 (thus choosing the negative sign in equation (6)) which corresponds to a time equal to one quarter of the exact period T yielding:

$$T = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_0} \frac{1}{\sqrt{(\cos\theta - \cos\theta_0)}}\,d\theta. \tag{7}$$

The definite integral in equation (7) cannot be solved analytically and it is an improper integral as it has a vertical asymptote at $\theta = \theta_0$. Edwards and Penney [1], Lima and Arun [3] and Benacka [4] are among others who in order to circumvent this difficulty substituted $\cos\theta = 1 - 2\sin^2(\theta/2)$ while making a change of variables given implicitly by $\sin\varphi = \sin(\theta/2)/\sin(\theta_0/2)$. In this way, equation (7) becomes

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{(1 - k^2\sin^2\varphi)}}\,d\varphi, \tag{8}$$

where $k = \sin(\theta_0/2)$. The definite integral in equation (8) is the complete elliptic integral of the first kind, which is not improper because $k < 1$ for $|\theta_0| < \pi$. Now different approaches have dealt with evaluating the complete elliptic integral of the first kind in equation (8) as in the following.

2. Previous Work

Edwards and Penney [1], Fulcher and Davis [5] and Benacka [6] gave the power series solution to the pendulum equation, equation (8), as follows:

$$T = T_0\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \cdots\right] \tag{9}$$

for the period T of the nonlinear pendulum released from rest with initial angle θ_0 in terms of the linear period T_0 given in equation (5) and $k = \sin(\theta_0/2)$. The same equation was used by Benacka [4] and Benacka [6] for amplitudes between 0° and 90°. Benacka [6] used spread sheet capabilities for evaluating the terms in the series appearing in the brackets of equation (9) and the first 12 terms were sufficient for relative error 0.1% and amplitudes up to 90°. The same solution (graph) was also obtained by Benacka [4] using numerical integration in the form of the trapezoidal rule. The number of subintervals in the trapezoidal rule was less than 170. Qureshi et al. [7] used a generalized hypergeometric function to solve the problem exactly. Their analysis is presented for maximum amplitude of 90°. They reported that their hypergeometric functions are intended for graduate students who are familiar with hypergeometric functions.

Several approximations were made for either integrating the elliptic integral equation or considering few terms of the power series solution as in the following. A perturbation analysis with second-order approximation found by Bernoulli in 1749 was applied to equation (8) yielded the most famous formula, Smith [8], for the large angle period as:

$$T_2 = T_0\left(1 + \frac{\theta_0^2}{16}\right). \tag{10}$$

According to Nelson and Olsson [9], the following expression can be deduced:

$$T = T_0\left(1 + \sum_{n=1}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 \sin^{2n}\left(\frac{\theta_0}{2}\right)\right) = T_0\left(1 + \frac{\theta_0^2}{16} + \frac{11}{3072}\theta_0^4 + \cdots\right). \quad (11)$$

Similar expression is given in pendulum (mathematics) in Wikipedia, the free encyclopedia [10], by expanding equation (11) up to seven terms:

$$T_7 = T_0\left(1 + \frac{\theta_0^2}{16} + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \frac{22931}{1321205760}\theta_0^8 \right.$$

$$\left. + \frac{1319183}{951268147200}\theta_0^{10} + \frac{233526463}{2009078326886400}\theta_0^{12} + \dots\right). \quad (12)$$

The Kidd-Fogg [11] formula has attracted much interest due its simplicity and is given as

$$T_{KF} = T_0 \frac{1}{\sqrt{\cos(\theta_0/2)}}. \quad (13)$$

Molina [12] used an interpolation-like linearization in equation (2) which results in

$$T_M = T_0\left(\frac{\sin\theta_0}{\theta_0}\right)^{-3/8}. \quad (14)$$

Lima and Arun [3] made linear interpolation to the function $\sqrt{(1 - k^2\sin^2\varphi)}$ and obtained

$$T_{\log} = -T_0 \frac{\ln a}{1 - a}, \quad (15)$$

where $a = \cos(\theta_0/2)$. They noted that $\ln a < 0$ and hence $T_{\log} > 0$ for $|\theta_0| < \pi$.

The focus was on the calculation of the period in the low and medium regimes with little attention to the profile motion. To the best of the author's knowledge, no attention has been given to other important variables such as pendulum velocity, potential energy and kinetic energy. Moreover, the period and profile motion in the high regime are rarely addressed.

In this paper, a complete analysis of the period, profile motion, velocity, potential energy and kinetic energy of the nonlinear pendulum is presented using numerical integration in the form of Gaussian quadrature technique. In addition, an expression is presented to calculate the gravitational acceleration for amplitudes in the medium regime. This analysis should be useful to students in physics, applied mathematics and mechanics.

3. The Present Approach

The elliptic integral in equation (8) for the calculation of the pendulum period is evaluated herein using numerical integration in terms of Gaussian quadrature. Gaussian quadrature, Burden and Faires [13], is applied to equation (8) as follows:

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{(1-k^2\sin^2\varphi)}}\,d\varphi = 4\sqrt{\frac{L}{g}}\,\frac{(\pi/2-0)}{2}\int_{-1}^{1} F(\varphi)\,d\varphi, \quad (16)$$

where $F(\phi) = \dfrac{1}{\sqrt{(1-k^2\sin^2\varphi)}}$. Gauss quadrature allows equation (16) to be written as:

$$T = 4\sqrt{\frac{L}{g}}\,\frac{(\pi/2-0)}{2}\sum_{i=1}^{n} W_i F(x_i), \quad (17)$$

where n is the number of Gaussian quadrature points used, W_i's are the weighting factors corresponding to the x_i's which are the coordinates of quadrature. Thompson [14] listed the Gaussian weights and coordinates for $n = 4,\ 7$ and 9. It should be noted that the quadrature formula is exact if the integrand function can be approximated by a polynomial of degree less than $2n-1.$ The profile motion is calculated from the following integral:

$$t = \sqrt{\frac{L}{g}} \int_{\phi}^{\theta_0} \frac{1}{\sqrt{(\cos\theta - \cos\theta_0)}}\, d\theta, \quad -\theta_0 \le \varphi \le \theta_0, \tag{18}$$

where θ is any angle between θ_0 and φ. A FORTRAN computer code is written to perform the numerical integrations needed in equation (17) and equation (18) using Gaussian quadrature. In equation (18), φ is taken in one degree increments, i.e., for $\theta_0 = 15^\circ$, the domain from $-\theta_0$ to θ_0 is traversed 30 times in steps of one degree. In addition, velocity is calculated from the following equation:

$$V = L\frac{d\theta}{dt} = \pm L\sqrt{\frac{2g}{L}}\sqrt{(\cos\theta - \cos\theta_0)}. \tag{19}$$

The total, kinetic and potential energies of the pendulum at any time are calculated from equation (1). The length of the pendulum is taken as 1.0m throughout this study unless stated otherwise.

4. Results for the Small Regime $(0^\circ < \text{Amplitude} \le 15^\circ)$

Figure 2 shows that there is no difference between the numerical solution and the cosine solution for amplitudes in the small regime such as at $\theta_0 = 15^\circ$. This shows that equations (2) and (3) can be made equivalent and neglecting the nonlinearity in equation (2) can be assumed. The calculated period is 2.01469s for amplitude equal to 15°. Table 1 shows that the calculated dimensionless period has a value of $T/T_0 = 1.0019$ at amplitude equal to 10° using 4, 7 and 9 quadrature points which matches the exact solution obtained from equation (9). Figure 3 shows the velocity profile follows in this case a typical sine curve which is the derivative of the cosine angular displacement curve. As shown in Figure 3, the maximum velocities are –0.0818m/s and +0.818m/s occurring at $\theta = 0$ with the corresponding times at about 0.5s $(t = T/4)$ and 1.5s .$(t = 3/4\,T)$, respectively.

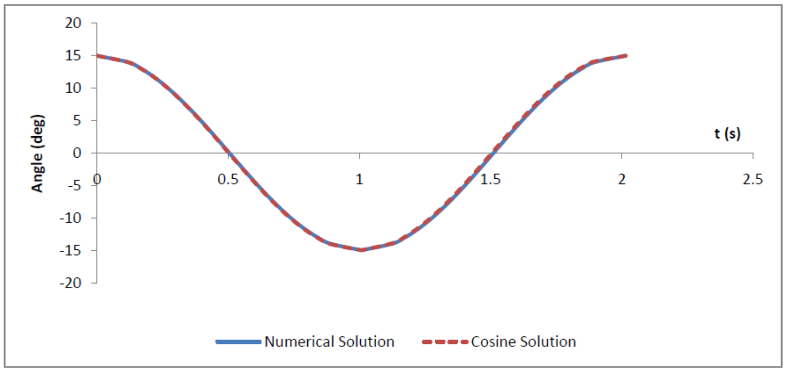

Figure 2. Motion profile for amplitude $= 15^{\circ}$.

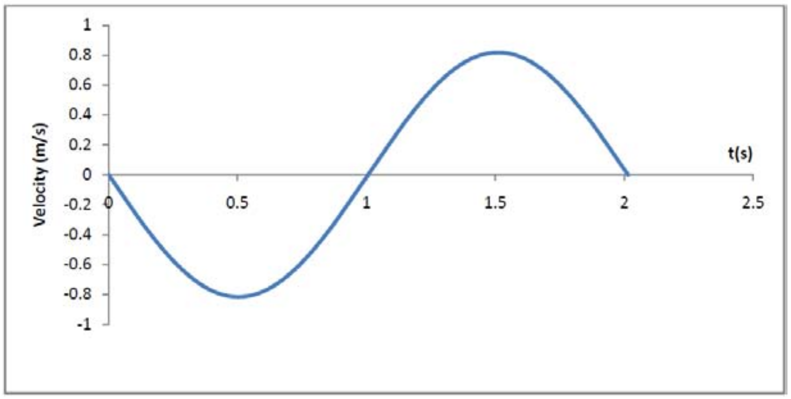

Figure 3. Velocity profile for amplitude = 15°.

Energy transformations from potential energy to kinetic energy and vice versa are evident from Figure 4 where the sum of the potential and kinetic energies remains constant as initially assumed. Initially at $t = 0$ and $\theta = \theta_0$, the potential energy is at its maximum value and gradually decreases till it reaches its minimum value of zero at $\theta = 0$ $(t = T/4 = 0.5\text{s})$ because it has been transforming completely into kinetic energy. The kinetic energy on the other hand starts to increase from zero till it reaches its maximum value at $\theta = 0$ $(t = T/4 = 0.5\text{s})$. This energy transformation graphical representation is very useful to physics and applied mathematics students.

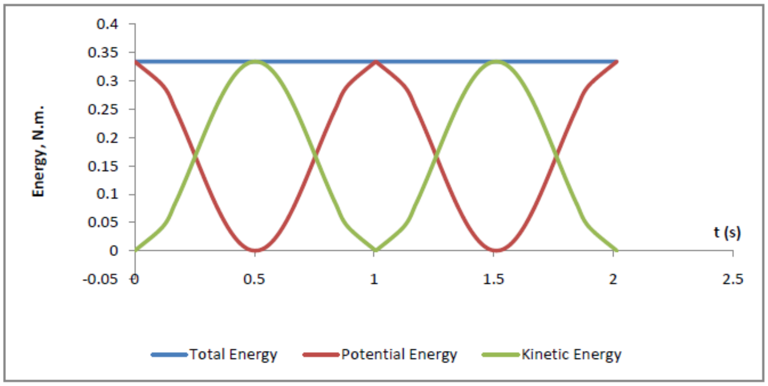

Figure 4. Total, potential and kinetic energies for amplitude = 15°.

Table 1. The dimensionless period (T/T_0) from Gaussian quadrature and the exact solution

θ_0 angle°	T/T_0 4 quad. points	T/T_0 7 quad. points	T/T_0 9 quad. points	T/T_0 exact solution, equation (9)
10	1.00191	1.00191	1.00191	1.0019
20	1.00767	1.00767	1.00767	1.0077
30	1.01741	1.01741	1.01741	1.0174
40	1.03134	1.03134	1.03134	1.0313
50	1.04978	1.04978	1.04978	1.0498
60	1.07318	1.07318	1.07318	1.0732
70	1.10215	1.10214	1.10214	1.1021
80	1.13749	1.13749	1.13749	1.1375
90	1.18031	1.18034	1.18034	1.1803

5. Results for the Medium Regime $(15^\circ < \text{Amplitudes} \leq 90^\circ)$

Since most published data are for amplitudes up to 90°, comparisons are then made using these data with the present approach which is based on numerical integration and Gaussian quadrature. Table 1 shows that the exact periods from equation (9) match up to four decimal digits the periods calculated from the present approach using numerical integration with 4, 7 and 9 Gaussian quadrature points for amplitudes up to 90°. It should be noted that equation (9) was calculated independently by writing a FORTRAN code to sum the power series terms in the equation and the calculation coincided with the published ones by Edwards and Penney [1].

It is apparent from Table 1 that up to four decimal digits, four quadrature points are sufficient for matching with the exact solution. A regression equation using Microsoft Office Excel 2007 was constructed with the numerically generated calculated periods versus the pendulum amplitude angles in radians. The equation reads as

$$\frac{T}{T_0} = 0.081\theta_0^2 - 0.016\theta_0 + 1.002 \text{ for } 0 < \theta_0 \leq \pi/2. \tag{20}$$

Equation (20) has regression coefficient $R^2 = 0.999$. Table 2 shows the periods calculated from equation (20) and the relative error. The maximum absolute relative error is less than 0.2% at all amplitudes except at amplitude $= \pi/2$ where it reaches 0.308%. The average value of the absolute error for $0 < \theta_0 \leq \pi/2$ is about 0.1% which shows a very good accuracy by equation (20). At amplitude $= \pi/2$, Kidd-Fogg (equation (13)) gives absolute relative error 0.8%, Molina (equation (14)) gives 0.4% while Lima and Arun (equation (15)) gives a value of about 0.25%.

Equation (20) can be used for estimation of the gravitational acceleration, g (m/s^2) after solving for g and rearranging to yield:

$$g = L\left(\frac{2\pi}{T}\right)^2 (0.081\theta_0^2 - 0.016\theta_0 + 1.002)^2. \tag{21}$$

It should be emphasized that equation (21) is valid for $0 < \theta_0 \leq \pi/2$ and θ_0 is expressed in radian measure. For small values of θ $(\theta \leq 15^\circ)$, the gravitational acceleration is usually calculated from equation (5) in the form

$$g = L\left(\frac{2\pi}{T}\right)^2. \tag{22}$$

Table 2. The period calculated from equation (20) and its error

θ_0 angle (rad.)	θ_0 angle (degrees)	T/T_0 exact, equation (9), [1]	T/T_0 equation (20)	Relative error %
0.17453	10	1.00191	1.00167	–0.023
0.34907	20	1.00767	1.00628	–0.137
0.52360	30	1.01741	1.01583	–0.155
0.69813	40	1.03134	1.03031	–0.100
0.87266	50	1.04978	1.04972	–0.006
1.04720	60	1.07318	1.07407	+0.083
1.22173	70	1.10214	1.10335	+0.110
1.39626	80	1.13749	1.13757	+0.007
1.57080	90	1.18034	1.17673	–0.306

Benacka [4] reported that the error in calculating g using equation (22) passes 1% at amplitude 17° and increases afterwards to pass 25% at amplitude = 90°.

The similarities and differences between equation (21) and equation (22) are clear. To illustrate the use of equation (21) in the calculation of g, consider a pendulum of length $L = 1.0\text{m}$, an amplitude = 5° and a period = 2.0s (a roughly measured value); substituting these values in equation (21) yields $g = 9.8937\text{m/s}^2$. If a more accurate value for the period (2.006s) is used, then $g = 9.8346\text{m/s}^2$ after substitution in equation (21). Although equation (21) is derived for medium amplitudes, it still yields close predictions compared to the commonly known value of $g = 9.81\text{m/s}^2$. Table 3 shows prediction of the gravitational acceleration using equation (21) and measured data by Lima and Arun [3] who used pendulum of length = 1.5m.

They used more sophisticated experiments in which both time keeping and position detection were done automatically to reduce the instrumental error to milliseconds. They gave values for T/T_0 which with the help of equation (5) for T_0 and using $L = 1.5\text{m}$ enables the calculation of T which is shown in Table 3 in the third column.

Table 3. Prediction of the gravitational acceleration using equations (15), (21) and (22)

θ_0 (deg.)	Measured T/T_0 Lima and Arun [3]	Period (seconds) calculated from [2]	g (m/s^2) from equation (21)	% Relative error in g from equation (21)	g (m/s^2) From equation (22)	% Relative error in g from equation (22)	g (m/s^2) From equation (15)	% Relative error in g from equation (15)
40	1.030	2.531	9.8161	0.06	9.2471	–5.74	9.837	0.28
45	1.040	2.555	9.7986	–0.12	9.0699	–7.54	9.812	0.02
50	1.049	2.577	9.8236	0.14	8.9150	–9.12	9.829	0.19
55	1.060	2.604	9.8340	0.24	8.7311	–11.00	9.832	0.22
60	1.070	2.629	9.8848	0.76	8.5685	–12.66	9.877	0.68
65	1.085	2.666	9.8658	0.57	8.3329	–15.06	9.856	0.47
70	1.100	2.703	9.8700	0.61	8.1075	–17.35	9.865	0.56
75	1.115	2.740	9.8952	0.87	7.8906	–19.57	9.902	0.93
80	1.130	2.776	9.9421	1.35	7.6828	–21.68	9.970	1.63
90	1.170	2.875	9.9231	1.15	7.1663	–26.95	10.034	2.28

It is clear from Table 3 that the error in calculating g from equation (21) is relatively small and is less than 1% for amplitudes $\leq 75^\circ$ while equation (22) significantly under-predicts g. Equation (15) yields good predictions, however, it produces less accurate predictions at amplitudes $\geq 75^\circ$ when compared to equation (21) predictions. Equation (21) thus allows more flexibility in conducting pendulum experiments for measuring the gravitational acceleration through allowing large amplitudes for which the period is considerably large and consequently the error in its measurement is small. In addition, it has simple and attractive structure as compared for example with g calculated from equation (15). Because the gravitational acceleration is inversely proportional to the period (as seen from equations (21) and (22)), equation (21) can be used for calculation of gravitational accelerations in high gravitational planets, stars or asteroids where high amplitudes can be allowed to measure the periods more accurately. Also, in physics laboratory experiments for calculation of g by measuring the period of the pendulum, equation (21) allows large amplitudes to be considered and hence large measured periods which improve the accuracy.

Figure 5 shows the calculated motion profile for amplitude $\theta_0 = \pi/2$. This profile coincides with that of Benacka [4] based on power series solution and deviates from the cosine solution.

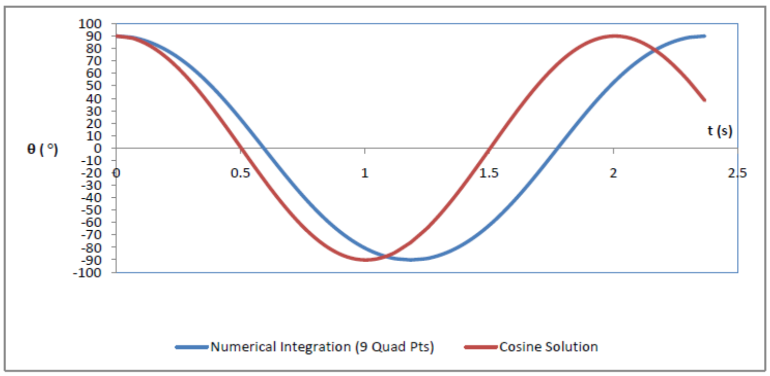

Figure 5. Pendulum motion profile for amplitude = 90°.

Velocity calculated from equation (19) is shown in Figure 6 which has a shape close to a sine curve.

The energy of the pendulum system calculated from equation (1) is shown in Figure 7. It can be seen that the total energy is conserved (constant) during the motion of the pendulum at any angle. Its value for a pendulum with $L = 1\text{m}$ happens to be equal to the numerical value of g, i.e., $E = 9.81\text{N.m}$. Figure 7 shows that the sum of the potential and kinetic energies is exactly equal to the total energy. As before, the potential energy is converted to kinetic energy and vice versa.

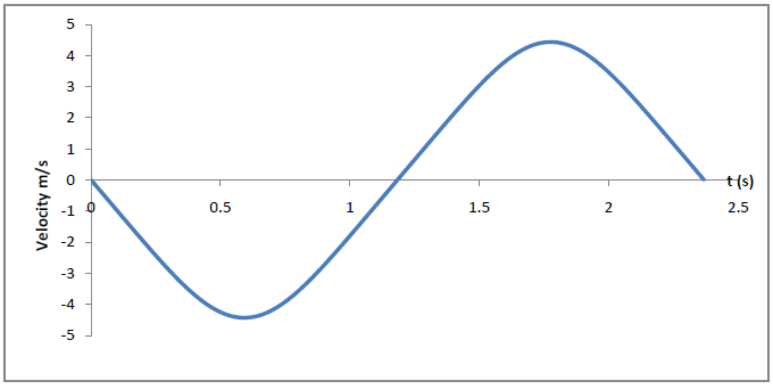

Figure 6. Velocity profile for amplitude = 90°.

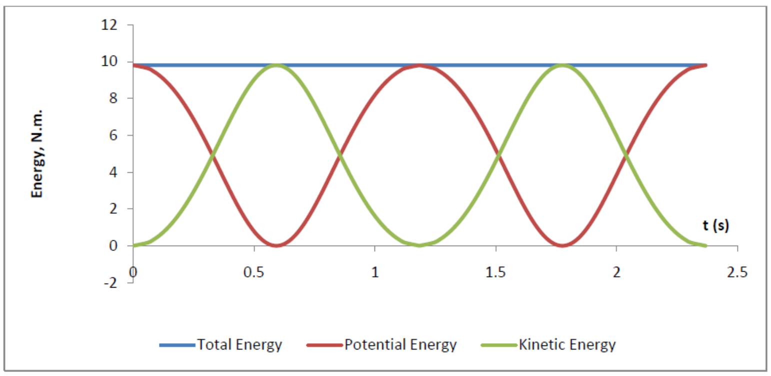

Figure 7. Total, potential and kinetic energies for amplitude = 90°.

In Figure 7, it is noted that the middle loop has grown in size compared to its left and right neighboring or adjacent loops while for small regime in Figure 4, the middle loop is exactly the same size as its neighboring loops. Thus, some change in energy transformation has begun.

6. Results for the High Regime $(90^\circ < \text{Amplitudes} < 180^\circ)$

Figure 8 shows that the dimensionless calculated periods from the numerical integration using 9 quadrature points tend to increase rapidly in the high regime region $(90^\circ < \text{Amplitudes} < 180^\circ)$. Moreover, at amplitudes of 150° and above, the increase is even more rapid approaching the theoretically infinite value at 180°. Such behavior will affect the pendulum motion profile and velocity as will be shown later. First, a trial is made to calculate the periods in this high regime from the power series solution, equation (9), as no tabulated values exist in the literature. A FORTRAN code was written to evaluate the terms of the series appearing in equation (9). With increase in the amplitude, it was noted that more terms are needed for the period to reach its exact value as seen in Table 4. For example, the second column shows that at amplitudes 160°, 170° and 179°, respectively, 111, 308 and 2611 terms are sufficient up to an accuracy of 4 decimal digits. However, for amplitude of 179° and 10 decimal digits accuracy, at least 100,000 terms are needed. The values of the period from the power series solution in equation (9), which can be considered as exact values, are shown in the last column of Table 5 up to an accuracy of 10 decimal digits. Table 5 shows also the period calculated from the various equations listed earlier for amplitudes ranging from 10° to 179°. Superiority of the present numerical integration approach with 9 quadrature points is evident over other methods. A closer look to the behavior of the period for amplitudes between 170° and 179° is shown in Table 6. It is noted that the difference between the 9-points numerical integration and the power series exact solution is about 1.1% except at 179° amplitude where it reaches about –6.1%. At amplitude of 179°, the period is about to approach infinity and some doubt is there on any predictive method.

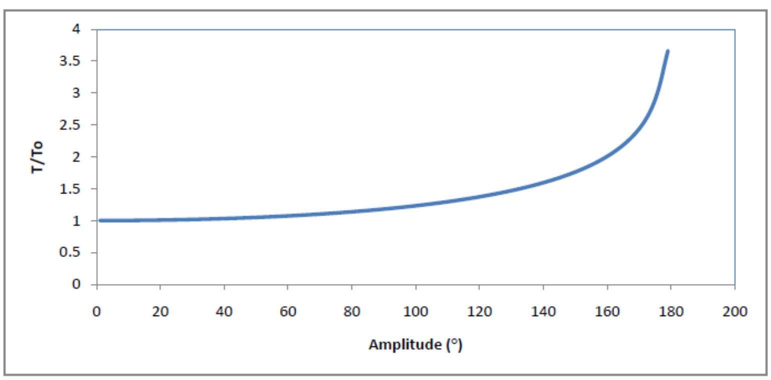

Figure 8. Numerical integration (9-quad pts.) results for (T/T_o) versus amplitudes.

Figure 9 shows that equation (10) ($T2$) clearly underestimates the period in the high regime while equation (13) (T_{KF}), equation (14) (T_M) and equation (15) $(T_{\log})$ overestimate the exact period. Figure 10 shows that equation (12) ($T7$) yields reasonable values up to amplitudes of 170° and that the seven quadrature points give also very close values to the exact solution. It is also clear from Table 5 that numerical integration using 9 quadrature points yields the same exact values for amplitudes up to 140°. At higher amplitudes, due to the very rapid increase in the period rate of change, deviations in the fifth decimal place take place. However, the numerical integration using 9 quadrature points still gives the best matches with the power series exact solution compared with other methods listed here.

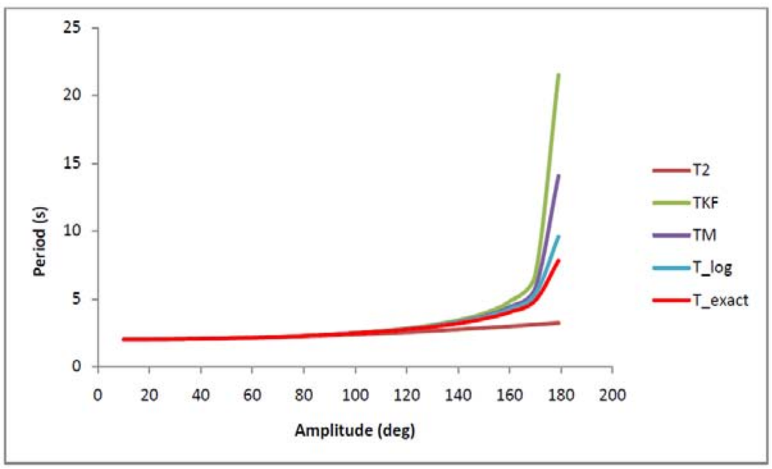

Figure 9. Period (in seconds) prediction by various formulas for amplitudes up to 179°.

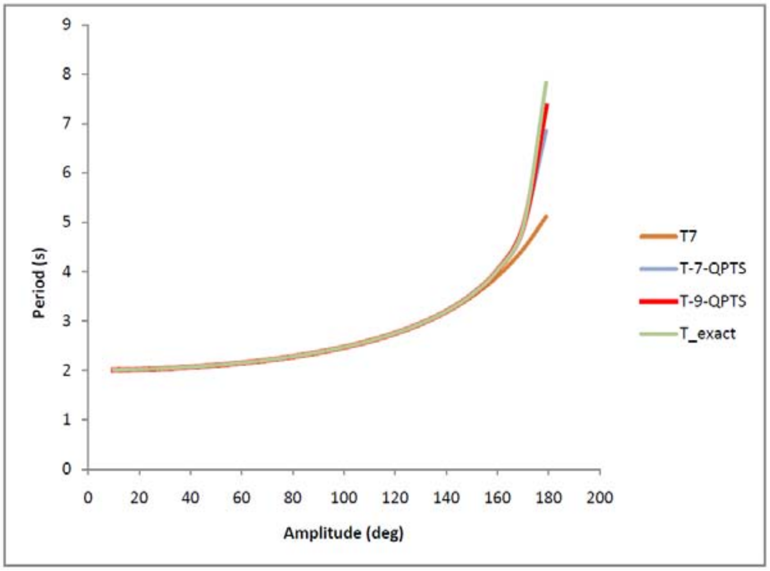

Figure 10. Period (in seconds) prediction by various formulas for amplitudes up to 179°.

Table 4. The number of terms used in the power series in equation (11)

Angle (deg)	No. of terms for accuracy = 1.0E-04	No. of terms for accuracy = 1.0E-05	No. of terms for accuracy = 1.0E-06	No. of terms for accuracy = 1.0E-07	No. of terms for accuracy = 1.0E-08	No. of terms for accuracy = 1.0E-09	No. of terms for accuracy = 1.0E-10
10	3	3	4	4	5	5	6
20	4	4	5	5	6	7	7
30	4	5	6	6	7	8	9
40	5	6	7	8	9	10	11
50	5	7	8	9	10	12	13
60	6	8	9	11	12	14	15
70	7	9	11	13	15	17	19
80	8	11	13	15	18	20	23
90	10	13	16	19	22	25	28
100	12	16	20	24	28	32	36
110	15	20	25	31	36	41	47
120	19	26	33	41	48	56	63
130	26	36	47	57	68	79	90
140	37	53	69	86	103	120	138
150	59	87	116	146	176	207	238
160	111	172	237	304	372	442	512
170	308	537	789	1053	1325	1602	1882
179	2611	12391	30709	53624	78804	100000	100000

Table 5. Period (T/T_0) prediction by various formulas with amplitudes up to 179°

Amplitude (deg)	$T2$, equation (10)	$T7$, equation (12)	TKF, equation (13)	TM, equation (14)	T_log, equation (15)	$T7$ QPTS	$T9$ QPTS	T exact, equation (9)
10	1.00190	1.00190	1.00190	1.00190	1.00190	1.00190	1.00190	1.00190
20	1.00761	1.00767	1.00768	1.00768	1.00767	1.00767	1.00767	1.00767
30	1.01713	1.01741	1.01748	1.01744	1.01743	1.01741	1.01741	1.01741
40	1.03046	1.03134	1.03159	1.03145	1.03142	1.03134	1.03134	1.03134
50	1.04760	1.04978	1.05042	1.05008	1.04999	1.04978	1.04978	1.04978
60	1.06854	1.07318	1.07457	1.07383	1.07364	1.07318	1.07318	1.07318
70	1.09329	1.10214	1.10489	1.10343	1.10306	1.10214	1.10214	1.10214
80	1.12185	1.13749	1.14254	1.13987	1.13917	1.13749	1.13749	1.13749
90	1.15421	1.18033	1.18921	1.18452	1.18327	1.18034	1.18034	1.18034
100	1.19038	1.23220	1.24728	1.23936	1.23719	1.23223	1.23223	1.23223
110	1.23037	1.29520	1.32039	1.30724	1.30355	1.29534	1.29534	1.29534
120	1.27415	1.37236	1.41421	1.39259	1.38629	1.37288	1.37288	1.37288
130	1.32175	1.46803	1.53824	1.50258	1.49171	1.46982	1.46981	1.46982
140	1.37315	1.58857	1.70991	1.64997	1.63057	1.59446	1.59445	1.59445
150	1.42836	1.74344	1.96563	1.86047	1.82361	1.76210	1.76222	1.76220
160	1.48739	1.94673	2.39974	2.19775	2.11862	2.00663	2.00747	2.00750
170	1.55022	2.21949	3.38728	2.89898	2.67302	2.44815	2.43853	2.43936
179	1.61001	2.55011	10.70480	6.99564	4.78312	3.41716	3.66321	3.90104

Table 6. Prediction of period (seconds) for pendulum $(L = 1.0\text{m})$ with amplitudes from 170° to 179°

Amplitude (deg)	9-Gauss-points solution T-9-QPTS	Power series solution T (exact), equation (9)	% Difference
170	4.8919	4.8935	-0.033
171	5.0265	5.0267	–0004
172	5.1789	5.1758	0.060
173	5.3547	5.3451	0.180
174	5.5618	5.5409	0.377
175	5.8113	5.7728	0.667
176	6.1178	6.0569	1.005
177	6.4945	6.4236	1.103
178	6.9334	6.9409	–0.109
179	7.3487	7.8258	–6.097

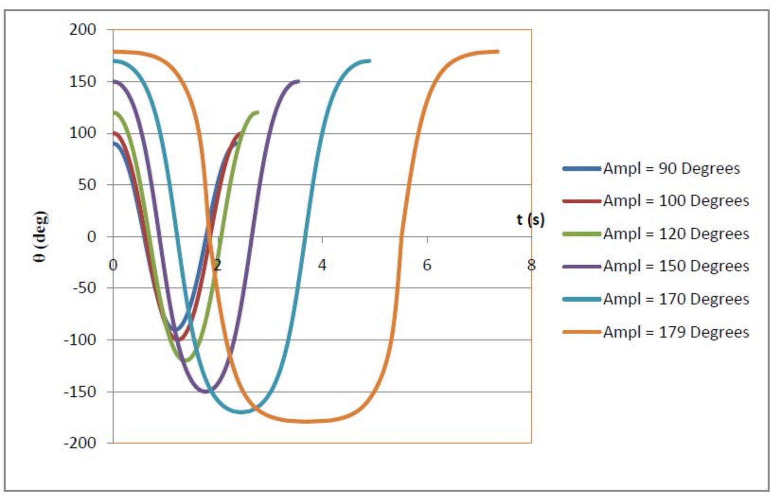

Figure 11. Motion profile for various amplitudes.

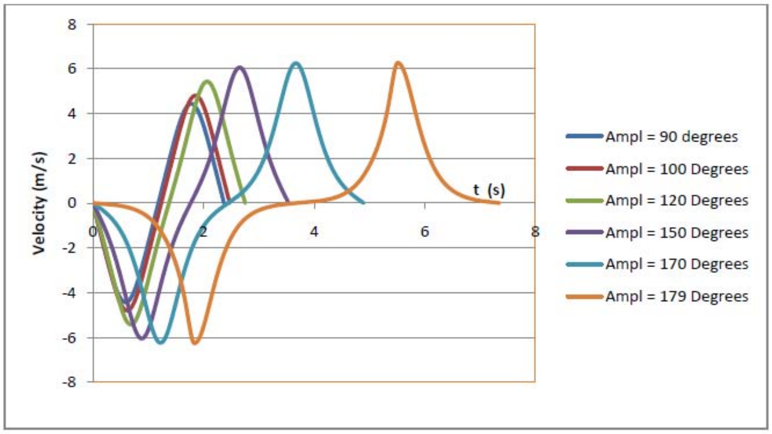

Figure 12. Velocity profiles for various amplitudes.

Figure 11 shows the pendulum profile motion by numerically integrating equation (18) with 9 quadrature points for several amplitudes ranging from 90° to 179°. The profiles for amplitudes from 90° up to 120° have similar shape with narrow trough (valley) region and short left and right edges. The profiles start to flatten at the left and right edges and the width of the valley region increases till it reaches its maximum width at 179°. The velocity profiles in Figure 12 exemplify these findings. The velocity profiles have the same nearly sine shape up to 120° amplitude. At the middle, a saddle point is clear at 179° amplitude which had started to grow at amplitude equal to 150°. Also, the nearly sine shape changes and the widths of the peaks and troughs are reduced in the velocity profiles for higher amplitudes such as at 170° and 179°.

Figure 13 exemplifies the flattening trend at amplitude = 179° in the energy profiles which was noted early in the profile motion and velocity profiles. The foregoing analysis suggests dividing further the high regime region into three sub-regions according to the rate of change of the pendulum variables in the high regime region. A mild high zone in which $90^\circ < \theta_0 \le 120^\circ$, a medium high zone in which $120^\circ < \theta_0 \le 150^\circ$ and a super high zone in which $150^\circ < \theta_0 < 180^\circ$. Further research is needed in this area especially experimental work.

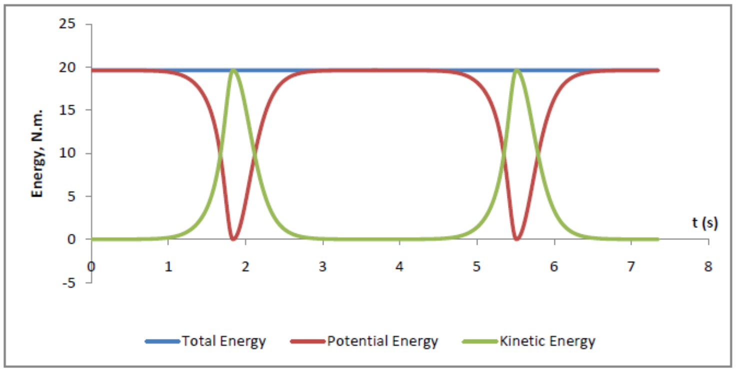

Figure 13. Total, potential and kinetic energies for amplitude = 179°.

7. Conclusions

It has been shown herein that computational methods such as numerical integration are very useful in analyzing physical phenomenon such as the motion of the nonlinear pendulum. The numerical integration approach with 9 Gaussian quadrature points proved successful in predicting the period, motion profile, velocity, potential energy and kinetic energy profiles for amplitudes in the low, medium and high regimes. The present study profile motion at amplitude equal to 90° is identical to the published results of the exact solution. The present study yields period predictions identical to values obtained from the power series solution calculated exactly for amplitudes up to 170°. In the high regime significant transformations of the profile motion, velocity, potential energy and kinetic energy profiles take place compared with the low and medium regimes. A regression based equation successfully predicts the acceleration of gravity for amplitudes in the medium regime. This equation can be used in physics laboratory experiments for more accurate determination of the gravitational acceleration by allowing longer periods to be measured. The presentation of the pendulum important variables such as period, profile motion, velocity, potential and kinetic energy not only for the low regime but also for the medium and high regimes adds to the understanding of the physics of the simple pendulum motion in its linear and nonlinear phases. It is hoped that this complete analysis of the nonlinear pendulum will benefit mathematics' and physics' college, university students and researchers too.

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