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A NOTE ON $*$-SIMPLE RINGS

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Abstract

It is known that if all proper ideals in a non-reduced ring have no zero-divisors, then the ring is simple. In this paper, we use the concept of $*$-reversible elements and then prove that if a ring with involution is not reduced and all $*$-proper ideals do not have any $*$-reversible elements, then R is $*$-simple. But if R is reduced and all $*$-proper ideals do not have any $*$-reversible element, then R is a direct sum of $*$-simple rings.

Throughout this note, we assume that all rings are associative. These rings may be without identity, but if identity is there, we will specifically mention it. Let us recall that an involution on a ring R is an additional unary operation $*$, such that

$$(a^* + b^*) = a^* + b^*,\ (ab)^* = b^*a^*,\ (a^*)^* = a,$$

for all $a, b \in R.$

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So, let R be a ring with an involution $*$. For any subset S of R, we assume that

$$S^* := \{x^* \in R : x \in S\}.$$

An ideal I of R is called a $*$-*ideal* if $I^* \subseteq I$. If S is a subring of R such that $SRS \subseteq S$, then S is called a *biideal* of R. If moreover, $S^* \subseteq S$, then S is called a $*$-*biideal* of R. A subset S of R is called a *principal* $*$-*biideal* [6], if for some $a \in R$,

$$S = \langle a \rangle^* = \mathbb{Z}a + \mathbb{Z}a^* + aRa + a^*Ra + aRa^* + a^*Ra^*.$$

An involution ring R is said to be $*$-*simple* if $R^2 \neq 0$, and R has no non-trivial $*$-ideals. A $*$-simple ring need not to be simple, although the converse is trivially true. Birkenmeier and Groenewald gave a characterization of $*$-simple rings that shows that they are simple rings or direct sum of $*$-simple rings (see [1, Proposition 2.1]). It will be proved in Corollary 5 below, that a $*$-simple ring with identity is a simple ring. An additive subgroup Q of R is called a *quasi-ideal* if $QR \cap RQ \subseteq Q$ [7].

In [5] Feigelstock proved that if all proper ideals in a non-reduced ring have no zero-divisors, then the ring is simple. The purpose of this note is to study the effect of the absence of the left (right) $*$-reversible elements in all proper $*$-ideals (or proper $*$-biideals) in the reduced and non-reduced $*$-rings, and in $*$-rings with unity, and to prove that if all proper $*$-ideals have no $*$-reversible elements in a non-reduced or a reduced ring, then the ring is $*$-simple, or a direct sum of $*$-simple rings, respectively. Thus, establishing involutive version of [5, Lemmas 4.1.8; 4.1.10; 4.1.11].

The concept of $*$-reversible elements has been introduced in [3]: an element $x \in R$ is right (left) $*$-reversible if there is a non-zero element $y \in R$, such that $xy = 0$ implies that $yx^* = 0$ $(y^*x = 0)$. It is a

$*$-reversible element if it is a left and right $*$-reversible element. If all elements of the ring R are right (left, two-sided) $*$-reversible, then we use the same term for the ring R (see details in [3]).

A non-zero element $x \in R$ is called *no left* (*right*) $*$-*reversible* if there is a $y \in R,$ such that $xy = 0$ and $y^*x = 0$ $(yx^* = 0),$ then $y = 0.$ It is clear that if x is a right (left) $*$-reversible, then $xy = 0 \Rightarrow y^*x^* = 0$ and $yx^* = 0 \Rightarrow xy^* = 0,$ so y^* is a left (right) reversible element.

Below are some examples that investigate the relationship between $*$-simple rings and the presence of $*$-reversible elements in its proper $*$-ideals.

Example 1. Consider the upper triangular matrix ring

$$R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | \, a, b, c \in \mathbb{Z}\right\}.$$

It is clear that R is not reduced. Consider the involution $*$ on R defined by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}.$$

The subset

$$A = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} | \, b \in \mathbb{Z}\right\}$$

of R is a $*$-ideal of R. Clearly, on the other hand for any $a, b \in \mathbb{Z},$ $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} = 0.$ Then

$$\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}^*\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -d \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Thus, A has non-zero $*$-reversible elements.

Example 2. Let $R = \mathbb{Z}_5 \oplus \mathbb{Z}_5$ be ring with the involution $(a, b)^* = (b, a)$. R is reduced, and its proper ideals have no $*$-reversible elements, and it is clear that R is $*$-simple.

Example 3. Let $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ be ring with involution $(a, b)^* = (b, a)$, R is non-reduced and the proper $*$-ideal $A = \{0, 2\} \times \{0, 2\}$ has a $*$-reversible element, it is clear that R is not $*$-simple.

Theorem 1. *Let R be a ring with the involution $*$ such that all proper $*$-ideals have no left $*$-reversible element. If R is not reduced, then R is $*$-simple.*

Proof. Because R is not reduced, it must have a non-zero nilpotent element. Let $0 \neq x \in R$ be a nilpotent element. Then x^* is also nilpotent. Without loss of generality, we assume that $x^2 = 0$ and so $(x^*)^2 = 0$. Let I be a proper $*$-ideal in R. For all $y \in I$, the product, $(x^*yx)(xy) = 0$ and simultaneously, $(xy)^*(x^*yx) = 0$. This means that either (i) $xy = 0$ or (ii) $x^*yx = 0$.

If (i) holds, then $(xy^*)x^* = 0$ and $x(xy^*) = 0 \Rightarrow xy^* = 0$ (since $x \neq 0$). But then $yx^* = 0$ and $xy = 0 \Rightarrow y = 0$. Hence $I = 0$.

If (ii) holds, then $x^*(yx) = (yx)x = 0 \Rightarrow yx = 0$ (since $x \neq 0$). Now $x^*(x^*y) = 0$ and $(x^*y)x = 0 \Rightarrow x^*y = 0 \Rightarrow y = 0$. Hence $I = 0$. Hence, R is $*$-simple. □

Corollary 2. *Let R be not $*$-reversible and not reduced. If the involution is anisotropic, then R is a simple ring.*

Proof. By Theorem 1 and [3, Proposition 6].

Theorem 3. *Let R be a ring with the involution $*$ such that all proper $*$-biideals have no left $*$-reversible element. If R is not reduced, then R has no proper $*$-biideals.*

Proof. Since R is not reduced, we assume that $0 \neq x \in R$, $x^2 = 0$ and $(x^*)^2 = 0$. Let B be a proper $*$-biideal in R and let $y \in B$. Then $\langle y \rangle^*$ is a proper $*$-ideal, so it has no left $*$-reversible element. On the other hand, the products,

$$(x^*yx)(xyx) = (xyx)^*(x^*yx) = (xyx^*)(x^*yx) = (x^*yx)^*(xyx^*)$$

$$= (x^*yx)(xyx^*) = (xyx^*)^*(x^*yx) = 0,$$

$$(xyx^*)(x^*yx^*) = (x^*yx^*)^*(xyx) = 0,$$

since $\langle y \rangle^*$ has no $*$-reversible element,

$$xyx = x^*yx = x^*y^*x = xyx^* = xy^*x^* = x^*yx^* = xy^*x = 0.$$

Similarly,

$$xryrx = x^*ryrx = x^*ry^*rx = xryrx^* = xry^*rx^* = x^*ryrx^* = xry^*rx = 0.$$

This yields that

$$\langle x \rangle^* y \langle x \rangle^* = 0.$$

Hence, the ideal $\langle x \rangle^*$ possesses a non-trivial left $*$-reversible element, so $\langle x \rangle^* = R$, and therefore $RBR = 0$. Hence, $BBB^* = BB^*B = 0$, so y is a left $*$-reversible element, which yields that $B = 0$, i.e., R has no proper $*$-biideals. □

Theorem 4. *Let R be a ring with involution $*$. Then the following are equivalent*:

(1) *R has no non-trivial proper $*$-biideals.*

(2) *R has no non-trivial left and right ideals.*

(3) *Either R is a division ring or R is the zero-ring on $\mathbb{Z}_p$, where p is some prime.*

Proof. (1) $\Leftrightarrow$ (2) Suppose that R is an involution ring and has no non-trivial proper $*$-biideals but R has a non-trivial left (right) ideal I. Then I^* is a right (left) ideal in R. If $I \cap I^* \neq 0$, then it is a quasi-ideal of R [7, Proposition 2.3]. Thus, $I \cap I^*$ is a biideal of R. But $I \cap I^*$ is closed under involution, hence it is a $*$-biideal, which is a contradiction to our assumption. Now assume that $I \cap I^* = 0$, then $I^*I = 0$. But then

$$(IR + RI^*)^2 \subseteq IR + RI^*,$$

$$(IR + RI^*)R(IR + RI^*) = IR + RI^*.$$

Hence, we conclude that $IR + RI^* \neq 0$ and it is a $*$-biideal, again a contradiction.

Conversely, suppose that R has no non-trivial left (right) ideals. As stated above, R is a division ring or R is a zero-ring on $\mathbb{Z}_p$, where p is a prime (see [5, Lemma 4.1.1]). If R is a division ring, then there is no biideal in R. If R is a zero-ring on $\mathbb{Z}_p$, it is clear that R has no biideal.

(2) $\Leftrightarrow$ (3) This follows from [5, Lemma 4.1.1]. □

Corollary 5. *Let R be a ring with the involution $*$ such that all proper $*$-biideals have no left $*$-reversible element. If R is not reduced, then R is simple.*

Proof. This follows from Theorem 3 and Theorem 4. □

Corollary 6. *Let R be a $*$-simple ring with unity. Then R is simple.*

Proof. Let R be $*$-simple. So R has no proper $*$-ideal. Let R have a proper $*$-biideal B. Since $1 \in R$, $RBR \neq 0$ and $RBR \neq R$, so RBR is a proper $*$-ideal, which is a contradiction. Hence, R has no proper $*$-biideals, and so by Theorem 4, R is simple. □

Corollary 7. *Let R be an involution ring with unity $*$ such that all proper $*$-biideals have no left $*$-reversible element. If R is not reduced, then there exists a prime p, R is the zero-ring on $\mathbb{Z}_p$.*

Proof. Since R has a proper $*$-biideal without any left $*$-reversible element, R has a proper $*$-ideal without any left $*$-reversible element. Then by Theorem 1, R is $*$-simple. By the proof of Corollary 6, R has no proper $*$-biideal. Hence, R is the zero-ring on $\mathbb{Z}_p$ by Theorem 4. $\square$

Lemma 8. *Let R be a ring with involution $*$. Then R has a proper* (*non-trivial*) *subring with involution if and only if R has a proper* (*non-trivial*) *subring*.

Proof. One way is clear. Conversely, assume that an involution ring R has a proper subring $S \neq 0$. Then $S^* \neq 0$. If $S \cap S^* \neq 0$, then $S \cap S^*$ is a proper $*$-subring of R and hence the proof is finished. So let $S \cap S^* = 0$. Then we have to deal with three cases.

Case I. $SS^* = 0$ and $S^*S = 0$. Then $(S + S^*)^2 \subseteq S + S^*$ and $S + S^*$ is a $*$-subring. If $S + S^*$ is not proper, then $S \oplus S^* = R$, and so S is a proper ideal of R. By Theorem 4 we conclude that R has a proper $*$-biideal, thus, R has a $*$-subring.

Case II. $SS^* = 0$ and $S^*S \neq 0$. Then $0 = (S^*S)^2 \subset S^*S$ and $S^*S \neq R$. If not, then for some $0 \neq s \in S \subset R$, $s = s_1^* s_1$, and so $s^* = s_1^* s_1$, a contradiction with $S \cap S^* = 0$. Thus, SS^* is a proper $*$-subring.

Case III. Finally if $SS^* \neq 0$ and $S^*S \neq 0$, then there exist $0 \neq s \in S$ and $0 \neq s^* \in S^*$, such that $0 \neq s^*s \in RS$, so $RS \neq 0$ and $ss^* \notin RS$, so $RS \neq R$. Hence, RS is a proper left ideal of R and again by Theorem 4, R has a proper $*$-subring. $\square$

Corollary 9. *An involution ring has no non-trivial proper $*$-subrings if and only if $R^+ = \mathbb{Z}_p$.*

Proof. This follows from [5, Corollary 4.1.2] and Lemma 8 above. $\square$

Lemma 10. *Let R be an involution ring which is also reduced and let $x \in R$ be a right $*$-reversible element. Then there exists a non-zero element $y \in R$, such that the following relations are satisfied*:

$$x^*Ry^* = yRx = x^*Ry = y^*Rx = 0,$$

$$Rx \cap Ry = xR \cap yR = x^*R \cap y^*R = Rx^* \cap Ry^* = xR \cap y^*R$$

$$= x^*R \cap yR = Rx \cap Ry^* = Rx^* \cap Ry = 0,$$

and

$$RxR \cap RyR = Rx^*R \cap Ry^*R = RxR \cap Ry^*R = Rx^*R \cap RyR = 0.$$

Proof. Because $x \in R$ is right $*$-reversible, $0 \neq y \in R$ exists, such that, $xy = 0 \Rightarrow yx^* = 0.$ Then, it follows that $xy^* = 0.$

Now for any $r \in R,$ above relations yield that

$$(yrx)^2 = (x^*ry)^2 = (y^*rx)^2 = 0.$$

Since there are no non-zero nilpotent elements in R,

$$yrx = x^*ry = y^*rx = 0$$

$$\Rightarrow x^*r^*y^* = y^*r^*x = x^*r^*y^* = 0.$$

Hence, we conclude that:

$$yRx = x^*Ry = y^*Rx = x^*Ry^* = 0.$$

If $t \in Rx \cap Ry,$ then

$$t^2 \in (Ry)(Rx) = R(yRx) = 0,$$

$$\Rightarrow t = 0 \Rightarrow Rx \cap Ry = 0.$$

Similarly, it can be shown that

$$RxR \cap RyR = Rx^*R \cap Ry^*R = RxR \cap Ry^*R = Rx^*R \cap RyR = 0. \qquad \square$$

Theorem 11. *If an involution ring R is reduced and all proper $*$-ideals have no $*$-reversible element, then R satisfies one of the following conditions*:

(1) *R has no left $*$-reversible element*.

(2) $R = S \oplus T$, *where S, T are $*$-simple and have no $*$-reversible element*.

Proof. Suppose that R has a left $*$-reversible element, i.e., there exist $x,\ y \in R,\ \ x \neq 0,\ \ y \neq 0$ such that

$$xy = 0 \text{ and } y^* x = 0.$$

It follows from Lemma 10 that

$$\langle RxR\rangle \cap \langle RyR\rangle = 0 = \langle RxR\rangle^* \cap \langle RyR\rangle^*.$$

Then

$$R = \langle RxR\rangle^* \oplus \langle RyR\rangle^*.$$

Let

$$0 \neq S \triangleleft^* \langle RxR\rangle^*.$$

Then the equality

$$S \cap \langle RyR\rangle^* = 0$$

yields that

$$R = S \oplus \langle RyR\rangle^*.$$

Therefore

$$\langle RxR\rangle^* = \langle RxR\rangle^* \cap [S \oplus \langle RyR\rangle^*] = S \oplus [\langle RxR\rangle^* \cap \langle RyR\rangle^*] = S,$$

and so $\langle RxR\rangle^*$ is $*$-simple. Similarly $\langle RyR\rangle^*$ is $*$-simple. $\langle RxR\rangle^*$ and $\langle RyR\rangle^*$ are proper $*$-ideals in R, and they are no $*$-reversible element. □

Theorem 12. *Let R be reduced. If all proper $*$-biideal have no $*$-reversible element, then R satisfies one of the following conditions*:

(1) *R has no left $*$-reversible element.*

(2) $R = S \oplus T$ *with S, T division rings.*

Proof. Since all proper $*$-ideal have no $*$-reversible element, by Theorem 11, R satisfies one of the following conditions:

1. R has no left $*$-reversible element.

2. $R = S \oplus T$ with S, T $*$-simple rings. Let B be a $*$-biideal of R. Since B has no $*$-reversible element, $RBR \neq 0$. If not, B will be a $*$-biideal has $*$-reversible element, a contradiction.

$RBR \neq R$ because R has a $*$-reversible element. Hence, RBR is a proper $*$-ideal of R, and $RBR \cap S$ is a $*$-ideal of S, a contradiction, so R has no proper $*$-biideal. By Theorem 4, we choose S, T to be division rings. □

Corollary 13. *Let R be a ring with involution. If all $*$-subrings have no $*$-reversible element, then R satisfies one of the following conditions*:

(1) *R has no left $*$-reversible element.*

(2) $R \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, *where p and q are primes.*

Proof. Since all $*$-subrings have no $*$-reversible element, all proper $*$-biideals have no $*$-reversible element. So, by Theorem 12, R satisfies one of the following conditions:

(a) R has no $*$-reversible element,

(b) $R = S \oplus T$, where S, T are division rings.

By the proof of Theorem 11, we may suppose that $S = \langle RxR \rangle^*$ and $T = \langle RyR \rangle^*$. Now let H be a $*$-subring of S. Then $H \cap \langle RyR \rangle^* = 0$, and $H \oplus \langle RyR \rangle^* = R$. So

$$S = S \cap [H \oplus \langle RyR \rangle^*] = H \oplus [S \cap \langle RyR \rangle^*] = H.$$

Hence, S has no non-trivial proper $*$-subrings. By Corollary 6, $S^{+} = \mathbb{Z}_p$, and by similar arguments $T^{+} = \mathbb{Z}_q$, where p and q are some primes. $\square$

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