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FIXED POINT RESULTS OF Ψ_α-CONTRACTION SELF MAPS IN α-NON-ARCHIMEDEAN FUZZY METRIC SPACES

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Abstract

In this paper, by relaxing the triangular condition in a non-Archimedean fuzzy metric space further, a notion of an α-non-Archimedean fuzzy metric space is introduced and some of its properties are investigated. Existence of fixed point results of Ψ_α-contraction self map in a complete α-non-Archimedean fuzzy space is proved.

1. Introduction

In 1975, Kramosil and Michalek [7] introduced the celebrated notion of a fuzzy metric space, which can be considered as a generalization of

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the statistical (probabilistic) metric space (see [17]). Obviously, their work provides a fundamental basis for the construction of fixed point theory in fuzzy metric spaces.

In 1988, Grabiec [3] defined the completeness of the fuzzy metric space introduced by Kramosil and Michalek known as *G*-complete fuzzy metric space in order to extend the Banach's contraction theorem to *G*-complete fuzzy metric spaces. Following Grabiec's work, Fang [1] further established some new fixed point theorems for contractive type mappings in *G*-complete fuzzy metric spaces. Soon after, Mishra et al. [14] also obtained several common fixed point theorems for asymptotically commuting maps in the same space, which generalize several fixed point theorems in metric, Menger, fuzzy and uniform spaces.

Besides these works based on the *G*-complete fuzzy metric space, George and Veeramani [2] modified the definition of fuzzy metric space introduced by Kramosil and Michalek and defined a Hausdorff and first countable topology. Meanwhile, they introduced an alternative notion of the Cauchy sequence introduced by Grabiec [3]. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani has emerged as another characterization of completeness and fixed point theorems have also been constructed on the basis of this metric space.

In 2013, Rano and Bag in [16] proved some fixed point results in dislocated quasi-fuzzy metric space in a sense of Kramosil and Michalek in [7]. Following Radu's remark in [15], Mihet in [11] introduced the collection of fuzzy sets in unit interval, $I = [0, 1]$ to prove some fixed point theorems in a non-Archimedean fuzzy metric space answering partially the questions forwarded by Grogeri and Sapena in [4].

From the above analysis, we can see that there are many studies related to fixed point theory based on the above two kinds of complete fuzzy metric spaces (see [4, 6, 8, 10, 12, 18-24]).

The purpose of this work is to introduce a notation of an α-non-Archimedean fuzzy space and to investigate fixed point results of Ψ_α-

contraction (where Ψ = set of operators) self maps in it. More importantly, we prove the existence of fixed point results of Ψ_α-contraction self maps with respect to a one-parameter in a complete α-non-Archimedean fuzzy metric space.

2. Preliminaries

In this section, we review briefly the existing results.

Throughout this paper, we use the following notations: $\mathbb{N} =$ the set of positive integers; $\mathbb{R} =$ the set of real numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; given a non-empty set X and a $T : X \to X$: $T(x) = Tx$, $T^{[1]}x = Tx$, $T^{[0]} = I_X$ and $T^{[n]}x = T \overbrace{\cdots}^{(n-1)\,\text{times}} Tx$; $I = [0, 1]$, $I^I := \{f : I \to I\}$.

Definition 2.1 [17]**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if $*$ satisfies the following conditions:

(T_1) $a * (b * c) = (a * b) * c$ for all $a, b, c \in [0, 1]$;

(T_2) $a * b = b * a$ for all $a, b \in [0, 1]$;

(T_3) $a * 1 = a$ for all $a \in [0, 1]$;

(T_4) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

A t-norm is continuous if it is continuous with respect to both variables.

Definition 2.2 [7]**.** A 3-tuple $(X, D, *)$ is said to be a *fuzzy metric space* if X is an arbitrary non-empty set, $*$ is a continuous t-norm and D is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions:

(D_1) $D(x, y, 0) = 0$ for all $x, y \in X$;

(D_2) $D(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(D_3) $D(x, y, t) = D(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(D_4) $D(x, y, t) * D(y, z, s) \leq D(x, z, t + s)$ for all $x, y, z \in X$ and for all $s, t \in (0, \infty)$;

(D_5) $D(x, y, \_) : (0, \infty) \to (0, 1]$ is left continuous for every fixed $x, y \in X$.

In 1994, George and Veeramani [2] slightly modified the notion of fuzzy metric introduced by Kramosil and Michalek as following definition:

Definition 2.3. A 3-tuple $(X, D, *)$ is said to be a *fuzzy metric space* if X is an arbitrary non-empty set, $*$ is a continuous t-norm and D is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:

(D_1) $D(x, y, t) > 0$ for all $x, y \in X$ and for all $t > 0$;

(D_2) $D(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(D_3) $D(x, y, t) = D(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(D_4) $D(x, y, t) * D(y, z, s) \leq D(x, z, t + s)$ for all $x, y, z \in X$ and $s, t \in (0, \infty)$;

(D_5) $D(x, y, \_) : (0, \infty) \to (0, 1]$ is continuous for every fixed $x, y \in X$.

The following notion of Cauchy sequence, convergent sequence, and a complete fuzzy metric space are recorded from the work of George and Veeramani [2].

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy metric space $(X, D, *)$ is said to be

(i) *Cauchy* iff $\lim_{n, m} D(x_n, x_m, t) = 1$ for each $t > 0$;

(ii) *convergent* to $x \in X$ iff $\lim_n D(x_n, x, t) = 1$ for each $t > 0$.

Definition 2.5. A fuzzy metric space $(X, D, *)$ is said to be *complete* iff every Cauchy sequence is convergent in X.

In 2008, Mihet [11] extended the notion of non-Archimedean probabilistic metric space (see Hadzic and Pap [13]) into non-Archimedean fuzzy metric space as following definition and proved some fixed point results.

Definition 2.6. A 3-tuple $(X, D, *)$ is said to be a *non-Archimedean fuzzy metric space* if X is an arbitrary non-empty set, $*$ is a continuous t-norm and D is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions:

(D_1) $D(x, y, 0) = 0$ for all $x, y \in X$;

(D_2) $D(x, y, t) = 1$ for all $t > 0$ iff $x = y$;

(D_3) $D(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(D_4) $D(x, y, t) * M(y, z, s) \leq D(x, z, \max(t + s))$ for all $x, y, z \in X$ and for $t, s \in (0, \infty)$;

(D_5) $D(x, y, \_) : (0, \infty) \to [0, 1]$ is left continuous for every fixed $x, y \in X$.

3. An α-non-Archimedean Fuzzy Metric Space and its Induced Topology

Now, we introduce a notion of α-non-Archimedean fuzzy metric space, a weaken version of a non-Archimedean fuzzy metric space and discuss its induced topology.

Definition 3.1. A 3-tuple $(X, D, *)$ is said to be an α-*non-Archimedean fuzzy metric space* if $X \neq \emptyset$, $*$ is a continuous t-norm and D is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions:

(D_1) $D(x, y, t) = 1$ for all $t > 0$ iff $x = y$;

(D_2) $D(x, y, t) = D(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(D_3) there exists $\alpha \geq 0$ such that

(i) $1 > D(x, y, \alpha) > 0$ for all $x, y \in X, x \neq y$;

(ii) $D(x, y, \alpha) * D(y, z, \alpha) \leq D(x, z, \alpha)$ for all $x, y, z \in X$;

(D_4) $D(x, y, \_) : (0, \infty) \to [0, 1]$ is left continuous for every fixed $x, y \in X$.

In Definition 3.1, if D satisfies one more condition,

(D_5) $D(x, y, 0) > 0$ for all $x, y \in X$,

then we call $(D, X, *)$ a *relaxed* α-*non-Archimedean fuzzy metric space*.

Example 3.2. Every stationary fuzzy metric space $(X, D, *)$ (for a stationary fuzzy metric space, refer [5]) with $D(x, y, t) \neq 0$ for all $x, y \in X$ is an α-non-Archimedean fuzzy metric space.

Example 3.3. $(X, D, *)$ is an α-non-Archimedean fuzzy metric space, where $a * b = ab$, D, X and α are as in Table 3.1.

Table 3.1. α-non-Archimedean fuzzy metric spaces

	X	$D(x, y, t)$	α
1	(X, d) is any metric space	$\begin{cases} 0 & \text{if } t = 0 \\ \dfrac{t}{t + d(x, y)} & \text{otherwise} \end{cases}$	1
2	$X = (0, \infty)$	$\dfrac{\min(x, y) + t}{\max(x, y) + t}$	0, 1
3	$X = (0, \infty)$	$\begin{cases} 0 & \text{if } t = 0 \\ \dfrac{\min(x, y)}{\max(x, y)} & \text{otherwise} \end{cases}$	$\alpha \in (0, \infty)$
4	$X \neq \emptyset$	$\begin{cases} 1 & \text{if } t > 0 \text{ and } x = y \\ t & \text{if } 0 < t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$	$\alpha \in (0, 1)$

Definition 3.4. Let $(X, D, *)$ be an α-non-Archimedean fuzzy metric space. An open ball centered at $x \in X$ and radius $r \in (0, 1)$ is defined as $\mathbb{B}(x, r, \alpha) = \{y \in X : D(x, y, \alpha) > 1 - r\}$.

Definition 3.5. Let $(X, D, *)$ be an α-non-Archimedean fuzzy metric space. A subset W of X is said to be *open set* in $(X, D, *)$ if for each $x \in W$, there exists an $r \in (0, 1[$ such that $\mathbb{B}(x, r, \alpha) \subset W$.

Proposition 3.6. *Let $(X, D, *)$ be an α-non-Archimedean fuzzy metric space. Every open ball is an open set.*

Proof. Consider an open ball $\mathbb{B}(x, r, t)$ with center $x \in X$, radius $r \in (0, 1)$. Now, $y \in \mathbb{B}(x, r, \alpha)$ implies $s = D(x, y, \alpha) > 1 - r$. Let $u \in (0, 1)$ be such that $s > 1 - u > 1 - r$. Hence, there exists $v \in (0, 1)$ such that $s * v \geq 1 - u$. We claim that $\mathbb{B}(y, 1 - v, \alpha) \subset \mathbb{B}(x, r, \alpha)$. If $z \in \mathbb{B}(y, 1 - v, \alpha)$, then $D(y, z, \alpha) > v$. Therefore,

$$D(x, z, \alpha) \geq D(x, y, \alpha) * D(y, z, \alpha) \geq s * v \geq 1 - u > 1 - r$$

and so $z \in \mathbb{B}(x, r, \alpha)$ and $\mathbb{B}(y, 1 - v, \alpha) \subset B(x, r, \alpha)$. □

The family of subsets of X given by

$$\tau = \{A \subset X : x \in A \text{ iff there exists } r \in (0, 1) \text{ such that } B(x, r, \alpha) \subset A\}$$

is a topology on X induced by an α-non-Archimedean fuzzy metric space.

Proposition 3.7. *Every α-non-Archimedean fuzzy metric space $(X, D, *)$ is Hausdorff.*

Proof. If $x, y \in X$ with $x \neq y$, then $D(x, y, \alpha) = r \in (0, 1)$.

Let $r < s < 1$ and $u \in (0, 1)$ be such that $u * u \geq s$. Consider the open balls $\mathbb{B}(x, 1 - u, \alpha)$ and $\mathbb{B}(y, 1 - u, \alpha)$.

We claim that $\mathbb{B}(x, 1 - u, \alpha) \cap \mathbb{B}(y, 1 - u, \alpha) = \emptyset$.

For if we assume that $z \in B(x, 1-u, \alpha) \cap B(y, 1-u, \alpha)$, then we get a contradiction

$$r \geq D(x, z, \alpha) * D(y, z, \alpha) \geq u * u \geq s > r.$$

Therefore, $\mathbb{B}(x, 1-u, \alpha) \cap \mathbb{B}(y, 1-u, \alpha) = \varnothing$. $\square$

Definition 3.8. A sequence $\{x_n\}$ in an α-non-Archimedean fuzzy metric space $(X, D, *)$ is convergent to $x \in X$ if given $r \in (0, 1)$, there exists $n_0 \in N$ such that $x_n \in B(x, r, \alpha)$ for all $n \geq n_0$.

Proposition 3.9. *A sequence* $\{x_n\}$ *in an* α*-non-Archimedean fuzzy metric space* $(X, D, *)$ *is convergent to* $x \in X$ *if and only if* $\lim_n D(x_n, x, \alpha) = 1$.

Proof. If $\lim_n x_n = x$, then for $r \in (0, 1)$, there exists $n_0 \in N$ such that $x_n \in \mathbb{B}(x, r, \alpha)$ for all $n \geq n_0$. It follows that $D(x_n, x, \alpha) > 1 - r$ and hence $\lim_n D(x_n, x, \alpha) = 1$. Conversely, if $\lim_n D(x_n, x, \alpha) = 1$, then for $r \in (0, 1)$, there exists $n_0 \in N$ such that $D(x_n, x, t) > 1 - r$ for all $n \geq n_0$. Thus, $x_n \in \mathbb{B}(x, r, t)$ for all $n \geq n_0$ and hence $\lim_n x_n = x$. $\square$

Proposition 3.10. *Let* $\{x_n\}$ *be a sequence in an* α*-non-Archimedean fuzzy metric space* $(X, D, *)$. *If* $\{x_n\}$ *is convergent in* X, *then its limit is unique*.

Proof. Suppose $x, y \in X$ with $x \neq y$ and $\lim_n x_n = x$. Since $(X, D, *)$ is a Hausdorff space, there exist $r_0, s_0 \in (0, 1)$ such that $B(x, r_0, \alpha) \cap B(y, s_0, \alpha) = \varnothing$. Since $\lim_n x_n = x$, there exists $n(r_0) \in \mathbb{N}$ such that $x_n \in B(x, r_0, \alpha)$ for all $n \geq n(r_0)$. Consequently, $x_n \notin B(y, s_0, \alpha)$ for all $n \geq n(r_0)$. Thus, $\lim_n x_n \neq y$. Therefore, the limit of sequence (x_n) (if exists) is unique. $\square$

Definition 3.11. A sequence $\{x_n\}$ in α-non-Archimedean fuzzy metric space $(X, D, *)$ is a Cauchy sequence, if for each $r \in (0, 1)$, there exists $n_0(r) \in \mathbb{N}$ such that $x_m \in \mathbb{B}(x_n, r, \alpha)$ for all $n, m \geq n_0(r)$.

An α-non-Archimedean fuzzy metric space $(X, D, *)$ is said to be *complete* if every Cauchy sequence is convergent.

Definition 3.12. Let $(X, D, *)$ be an α-non-Archimedean fuzzy metric space. A mapping $T : X \to X$ is a continuous mapping if for every sequence $\{x_n\}$ such that $\lim_n x_n = x$ implies $\lim_n Tx_n = Tx$.

Proposition 3.13. *Let* $X = (0, \infty)$, $a * b = ab$ *for all* $a, b \in [0, 1]$. *If* $D : X \times X \times [0, \infty) \to [0, 1]$ *be defined by* $D(x, y, t) = \dfrac{\min(x, y) + t}{\max(x, y) + t}$, *then* $(X, D, *)$ *is a complete* α*-non-Archimedean fuzzy metric space.*

Proof. Clearly, Definition 3.1 $(D_1, D_2 \text{ and } D_4)$ holds. It is straightforward to verify that $1 > D(x, y, 0) > 0$, $x, y \in X$, $x \neq y$ and $D(x, y, 0) \geq D(x, z, 0) * D(z, y, 0)$ for all $x, y \in X$. So, $(X, D, *)$ is a 0-non-Archimedean fuzzy metric space. Now, we need to show that $(X, D, *)$ is a complete 0-non-Archimedean fuzzy metric space. If $\{x_n\}$ is a Cauchy sequence in $(X, D, *)$, then given $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$D(x_n, x_m, 0) > 1 - r \text{ for all } n, m \geq n_0. \tag{3.1}$$

But $D(x_n, x_m, 0) > 1 - r$ for all $n, m \geq n_0 \Rightarrow \dfrac{\min(x_n, x_m)}{\max(x_n, x_m)} > 1 - r$, $\forall n, m \geq n_0 \Rightarrow \dfrac{|x_n - x_m|}{\max(x_n, x_m)} < r$, $\forall n, m \geq n_0 \Rightarrow \lim_{n,m} |x_n - x_m| = 0$. Thus, $\{x_n\}$ is a Cauchy sequence in $(Y = [0, \infty), |\cdot|)$, where $|\cdot|$ is a usual metric on $\mathbb{R}$ restricted to $Y = [0, \infty)$. Since $(Y, |\cdot|)$ is complete, $\lim_n x_n = x \in Y$.

Claim. $x \neq 0$. For if $x = 0$, then (since $\{x_n\} \subset (0, \infty)$) there exists a decreasing sub-sequence $\{x_{n_j}\}$ of $\{x_n\}$ such $\lim_{n_j} x_{n_j} = 0$ and $D(x_{n_j}, x_{n_i}, 0) > 1 - r$, $\forall n_j, n_i \geq n_{i_0} \geq n_0$, where n_0 is as in (3.1). In particular (fixing n_{i_0}), $D(x_{n_j}, x_{n_{i_0}}, 0) > 1 - r$, $\forall n_j, n_i \geq n_{i_0} \geq n_0$. Since

$\{x_{n_j}\}$ is decreasing, $\lim_{n_j} x_{n_j} = 0$ and $D(x_{n_j}, x_{n_{i_0}}, 0) > 1 - r$, $\forall n_j, n_i \geq n_{i_0} \geq n_0$, we get a contradiction $0 = \lim n_j \left(\frac{x_{n_j}}{x_{n_{i_0}}} \right) \geq 1 - r$. Hence, the claim.

Thus, every Cauchy sequence $\{x_n\}$ in $(x, D, *)$ is convergent in X. Therefore, $(X, D, *)$ is a complete 0-non-Archimedean fuzzy metric space.□

Remark 3.14. $(X, D, *)$ in Proposition 3.13 is 1-non-Archimedean fuzzy metric space but not complete 1-Archimedean fuzzy space.

4. Fixed Point Results of Ψ_α-contraction Self Maps in α-non-Archimedean Fuzzy Metric Spaces

In this section, we present some fixed point results of Ψ_α-contraction self mapping in α-non-Archimedean fuzzy metric spaces.

First we define a collection of fuzzy sets in $I = [0.1]$ comparison mappings, similar to that of Mihet's in [11].

Definition 4.1. Let $\psi : [0, 1] \to [0, 1]$ be a mapping satisfying the following conditions:

(1) ψ is a non-decreasing and left continuous function;

(2) $\psi(r) > 0$ for all $r > 0$;

(3) $\lim_n \psi^{[n]}(r) = 1$ for all $r \neq 0$.

Define $\Psi = \{\psi \in I^I : \psi \text{ satisfies (1) to (3)}\}$. We call every $\psi \in \Psi$ a Ψ-*mapping*.

Example 4.2. Let $k \in (0, 1)$ be arbitrarily fixed. A mapping $\psi : [0, 1] \to [0, 1]$ defined by $\psi(r) = \dfrac{r}{k + (1 - k)r}$ is a Ψ-mapping.

Lemma 4.3. *If* $F \in \Psi$, *then* $F(1) = 1$ *and* $F(r) > r$ *for all* $r \in (0, 1)$.

Proof. If $F \in \psi$ and $r \in (0, 1]$ are arbitrary, then $F(r) \leq 1$ (since $F([0, 1]) \subset [0, 1]$) and $F(r) \leq F(1)$ (since F is non-decreasing). Therefore, $F(r) \leq F(1) \leq 1 \Rightarrow F^{[2]}(r) \leq F^{[2]}(1) \leq F(1) \leq 1$. Repeating the same procedure, we obtain that $F^{[n]}(r) \leq F(1) \leq 1$ for all $n \in \mathbb{N}$. Thus, $1 = \lim_n F^{[n]}(r) \leq F(1) \leq 1$. That is, $F(1) = 1$. Similarly, if $\psi(r) \leq r$ for some $r \in (0, 1)$, then we get a contradiction $1 = \lim_n \psi^{[n]}(r) \leq r$. Thus, $F(r) > r$ for all $r \in (0, 1)$. □

Definition 4.4. Let $(X, D, *)$ be a fuzzy metric space. A self mapping $T : X \to X$ is said to be

(1) a Ψ_α-*contraction* if there exists an $\alpha \in (0, \infty)$ and a $\psi \in \Psi$ such that

$$D(Tx, Ty, \alpha) \geq \psi(D(x, y, \alpha)), \quad \forall x, y \in X; \tag{4.1}$$

(2) a weak Ψ_α-*contraction* if there exists an $\alpha \in (0, \infty)$ and a $\psi \in \Psi$ such that

$$D(T^{[2]}x, Tx, \alpha) \geq \psi(D(Tx, x, \alpha)), \quad \forall x \in X; \tag{4.2}$$

(3) a Ψ-*contraction* if there exists $\psi \in \Psi$ such that

$$D(Tx, Ty, t) \geq \psi(D(x, y, t)), \ \forall x, y \in X \text{ and for each } t > 0; \tag{4.3}$$

(4) a *weak* Ψ-*contraction* if there exists $\psi \in \Psi$ such that

$$D(T^{[2]}x, Tx, t) \geq \psi(D(Tx, x, t)), \ \forall x \in X \text{ for each } t > 0. \tag{4.4}$$

Definition 4.5. A sequence $\{x_n\}$ in fuzzy metric space $(X, D, *)$ is said to be

(1) a Ψ-*contraction sequence* in X if there exists $\psi \in \Psi$ such that

$$D(x_{n+1}, x_{n+2}, t) \geq \psi(D(x_n, x_{n+1}, t)) \text{ for all } n \in \mathbb{N}_0, \ t > 0; \tag{4.5}$$

(2) a Ψ_α-*contraction sequence* in X if there exist an $\alpha \in (0, \infty)$ and a $\psi \in \Psi$ such that

$$D(x_{n+1}, x_{n+2}, \alpha) \geq \psi(D(x_n, x_{n+1}, \alpha)) \text{ for all } n \in \mathbb{N}_0, \ t > 0. \tag{4.6}$$

Lemma 4.6. *Let $(X, D, *)$ be an α-non-Archimedean fuzzy metric space. If $\{x_n\} \subset X$ is a Ψ_α-contraction sequence in X with respect to $\psi \in \Psi$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. If $\{x_n\}$ is a Ψ_α-contraction sequence with respect to $\psi \in \Psi$, then (4.6) holds and

$$D(x_1, x_2, \alpha) \geq \psi(D(x_0, x_1, \alpha))$$

$$D(x_2, x_3, \alpha) \geq \psi(D(x_1, x_2, \alpha)) \geq \psi^{[2]}(D(x_0, x_1, \alpha))$$

$$\vdots$$

$$D(x_{n+1}, x_{n+2}, \alpha) \geq \psi^{[n+1]}(D(x_0, x_1, \alpha)). \tag{4.7}$$

From (4.7) and the property of $\psi \in \Psi$, we have

$$\lim_n D(x_{n+1}, x_{n+2}, \alpha) \geq \lim_n \psi^{[n]}(D(x_0, x_1, \alpha)) = 1.$$

Similarly, $\lim_n D(x_{n+2}, x_{n+3}, p) = 1.$ That is,

$$\lim_n D(x_n, x_{n+1}, \alpha) = 1. \tag{4.8}$$

We claim $\{x_n\}$ is a Cauchy sequence.

If $\{x_n\}$ is not a Cauchy sequence, then given $k \in \mathbb{N}$ and for each $l \in \mathbb{N}$ with $l \geq k$, there exist $m(l), n(l) \in \mathbb{N}$ such that

$$m(l) \geq n(l) \geq l \text{ and } D(x_{m(l)}, x_{n(l)}, \alpha) \leq 1 - r \text{ for some } r \in (0, 1).$$

By (4.8), $\lim_l D(x_{n(l)}, x_{n(l)+1}, \alpha) = 1.$ From

$$1 - r \geq D(x_{m(l)}, x_{n(l)}, \alpha) \geq D(x_{m(l)}, x_{n(l)+1}, \alpha) * D(x_{n(l)+1}, x_{n(l)}, \alpha),$$

we get

$$1 - r \geq \limsup_l \{D(x_{m(l)}, x_{n(l)+1}, \alpha) * D(x_{n(l)+1}, x_{n(l)}, \alpha)\}$$

$$\geq \limsup_l D(x_{m(l)}, x_{n(l)+1}, \alpha).$$

Similarly,

$$1 - r \geq \limsup_l D(x_{m(l)+1}, x_{n(l)}, \alpha)$$

and

$$1 - r \geq \limsup_l D(x_{m(l)+1}, x_{n(l)+1}, \alpha).$$

Without loss of generality, assume that $m(l)$ is odd and $n(l)$ is even for all l and $1 - r \geq D(x_{m(l)}, x_{n(l)}, p)$ for all l.

Put

$$\varepsilon(l) = \min\{m(l) : 1 - r \geq D(x_{m(l)}, x_{n(l)}, p), m(l) \text{ is an odd number}\}.$$

Now, for $\varepsilon(l)$, we have

$$\begin{aligned} 1 - r &= D(x_{\varepsilon(l)}, x_{n(l)}, \alpha) \geq D(x_{\varepsilon(l)-2}, x_{n(l)}, p) * D(x_{\varepsilon(l)-2}, x_{\varepsilon(l)}, \alpha) \\ &\geq D(x_{\varepsilon(l)-2}, x_{n(l)}, \alpha) * D(x_{\varepsilon(l)-2}, x_{\varepsilon(l)-1}, p) * D(x_{\varepsilon(l)-1}, x_{\varepsilon(l)}, \alpha) \\ &\geq (1 - r) * D(x_{\varepsilon(l)-2}, x_{\varepsilon(l)-1}, \alpha) * D(x_{\varepsilon(l)-1}, x_{\varepsilon(l)}, \alpha) \\ &\to 1 - r \text{ as } l \to \infty. \end{aligned}$$

Thus,

$$\lim_l D(x_{\varepsilon(l)}, x_{n(l)}, t) = 1 - r. \tag{4.9}$$

Using Definition 3.1, we have

$$\begin{aligned} &D(x_{\varepsilon(l)}, x_{n(l)}, \alpha) \\ &\geq D(x_{\varepsilon(l)+1}, x_{n(l)}, \alpha) * D(x_{\varepsilon(l)+1}, x_{\varepsilon(l)}, \alpha) \end{aligned}$$

$$\geq D(x_{\varepsilon(l)+1}, x_{n(l)+1}, \alpha) * D(y_{n(l)+1}, x_{n(l)}, \alpha) * D(x_{\varepsilon(l)+1}, x_{\varepsilon(l)}, \alpha)$$

$$\geq \psi(D(x_{\varepsilon(l)}, x_{n(l)}, \alpha)) * D(x_{n(l)+1}, x_{n(l)}, \alpha) * D(x_{\varepsilon(l)+1}, x_{\varepsilon(l)}, \alpha).$$

By letting $l \to \infty$ in (4.10) and using (4.8), (4.9), continuity of $*$ and the properties of $\psi \in \Psi$, we get a contradiction $1 - r \geq \psi(1 - r) > 1 - r$. Hence, $\{x_n\}$ is a Cauchy sequence. □

Lemma 4.7. *Let $(X, D, *)$ be an α-non-Archimedean fuzzy metric space and let T be a (weak) Ψ_α-contraction mapping with respect to $\psi \in \Psi$. For each $x_0 \in X$, a sequence $\{x_n : x_n = T^{[n]}x_0\}$ is a Cauchy sequence in $(X, D, *)$.*

Proof. (I) Assume that T is a Ψ_α-contraction mapping with respect to a $\psi \in \Psi$. Let $x_0 \in X$ be arbitrary and let $\{x_n : n \in \mathbb{N}_0\}$ be as stated in the lemma. By (4.1), $D(Tx, Ty, \alpha) \geq \psi(D(x, y, \alpha))$ for all $x, y \in X$. Hence, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x_{n+1}, x_{n+2}, \alpha) &= D(T^{[n+1]}x_0, T^{[n+2]}x_0, \alpha) \\ &= D(T(T^{[n]}x_0), T(T^{[n+1]}x_0), \alpha) \\ &\geq \psi(D(T^{[n]}x_0, T^{[n+1]}x_0, \alpha)) \\ &= \psi(D(x_n, x_{n+1}, \alpha)). \end{aligned}$$

Therefore, $\{x_n : n \in \mathbb{N}_0\}$ is a Ψ_α-contraction sequence in $(X, D, *)$.

Hence, by Lemma 4.6, $\{x_n : n \in \mathbb{N}_0\}$ is a Cauchy sequence.

(II) Assume that T is a weak Ψ_α-contraction mapping with respect to a $\psi \in \Psi$. Let $x_0 \in X$ be arbitrary and let $\{x_n : n \in \mathbb{N}_0\}$ be as stated in the lemma. By (4.2), $D(T^{[2]}x, Tx, \alpha) \geq \psi(D(Tx, x, \alpha))$ for all $x \in X$. Hence, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x_{n+2}, x_{n+1}, \alpha) &= D(T^{[n+2]}x_0, T^{[n+1]}x_0, \alpha) \\ &= D(T^{[2]}(T^{[n]}x_0), T(T^{[n]}x_0), \alpha) \\ &\geq \psi(D(T(T^{[n]}x_0), T^{[n]}x_0, \alpha)) \\ &= \psi(D(x_{n+1}, x_n, \alpha)). \end{aligned}$$

Therefore, $\{x_n : n \in \mathbb{N}_0\}$ is a Ψ_α-contraction sequence in $(X, D, *)$.

Hence, by Lemma 4.6, $\{x_n : n \in \mathbb{N}_0\}$ is a Cauchy sequence. □

Theorem 4.8. *Let* $(X, D, *)$ *be a complete* α*-non-Archimedean fuzzy metric space. If* $T : X \to X$ *is a* Ψ_α*-contraction mapping with respect to* $\psi \in \Psi$, *then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary and let $\{x_n : x_n = T^{[n]}x_0, n \in \mathbb{N}\}$. By Lemma 4.7, $\{x_n : x_n = T^{[n]}x_o, n \in \mathbb{N}\}$ is a Cauchy sequence in $(X, D, *)$. Since $(X, D, *)$ is a complete α-non-Archimedean fuzzy metric space, $\lim_n x_n = z \in X$.

Now, we claim $z \in Fix(T) = \{y \in X : Ty = y\}$. Indeed, since $(X, D, *)$ is an α-non-Archimedean fuzzy metric space and T is a Ψ_α-contraction mapping, we have

$$\begin{aligned} D(z, Tz, \alpha) &\geq D(z, Tx_n, \alpha) * D(Tx_n, Tz, \alpha) \\ &\geq D(z, Tx_n, \alpha) * \psi(D(x_n, z, \alpha)) \\ &= D(z, x_{n+1}, \alpha) * \psi(D(x_n, z, \alpha)). \end{aligned}$$

By continuity of $*$ and left continuity of ψ, we conclude that

$$\begin{aligned} D(z, Tz, \alpha) &\geq \lim_n \{D(z, x_{n+1}, \alpha) * \psi(D(x_n, z, \alpha))\} \\ &= 1 \Rightarrow D(z, Tz, \alpha) = 1. \end{aligned}$$

Since $D(x, y, \alpha) \neq 1$ for all $x \neq y$ by Definition 3.1 $(D_3(\text{i}))$, we conclude that $Tz = z$.

Finally, we shall show uniqueness of a fixed point of T.

Suppose $z, w \in Fix(T)$. If $w \neq z$, then by Definition 3.1 $(D_3(\text{i}))$, $1 > D(w, z, \alpha) > 0$. Therefore, by Definition 4.1(1) and Lemma 4.3, we get a contradiction

$$D(w, z, \alpha) = D(Tz, Tw, \alpha) \geq \psi(D(z, w, \alpha)) > D(w, z, \alpha).$$

Therefore, T has a unique fixed point. □

Corollary 4.9. *Let $(X, D, *)$ be a complete non-Archimedean fuzzy metric space such that $1 > D(x, y, \alpha) > 0$ for some $\alpha \in (0, \infty)$ and for $x, y \in X, x \neq y$. If $T : X \to X$ is a Ψ-contraction mapping with respect to $\psi \in \Psi$, then T has a unique fixed point.*

Proof. Let $\alpha \in (0, \infty)$ be as stated in the theorem. Since $D(x, y, t)$ is non-Archimedean, $D(x, y, t) \geq D(x, z, t) * D(z, y, t)$ for all $x, y, z \in X$ and for all $t > 0$. Consequently, $D(x, y, \alpha) \geq D(x, z, \alpha) * D(x, y, \alpha)$ for all $x, y, z \in X$. Hence, $(X, D, *)$ is a complete α-non-Archimedean space. Therefore, the result follows by Theorem 4.8. □

By relaxing the contractive conditions imposed by Gregori and Sapena [4], we obtain the following corollary:

Corollary 4.10. *Let $(X, D, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a mapping. If there exist $\alpha \in (0, \infty)$ and $k \in (0, 1)$ such that*

$$0 < D(x, y, \alpha) < 1 \text{ for all } x, y \in X, \quad x \neq y, \tag{4.11}$$

$$D(x, y, \alpha) \geq D(x, z, \alpha) * D(z, y, \alpha), \quad \forall x, y, z \in X, \tag{4.12}$$

$$\frac{1}{D(Tx, Ty, \alpha)} - 1 \leq k\left(\frac{1}{D(x, y, \alpha)} - 1\right), \quad \forall x, y, z \in X, \tag{4.13}$$

then T has a unique fixed point.

Proof. Since a complete fuzzy metric space satisfying (4.11) and (4.12) is an α-non-Archimedean fuzzy metric space, it suffices to prove that T is a Ψ_α-contraction mapping. Observe that (4.13) can be rewritten as

$$\forall x, y \in X, \quad D(Tx, Ty, \alpha) \geq \frac{D(x, y, \alpha)}{k + (1-k)D(x, y, \alpha)}. \tag{4.14}$$

Now, define $\psi : [0, 1] \to [0, 1]$ by $\psi(r) = \frac{r}{k + (1-k)r}$, where $k \in (0, 1)$ is as in (4.14). Thus, (4.14) becomes that $\forall x, y \in X$,

$$D(Tx, Ty, \alpha) \geq \psi(D(x, y, \alpha)).$$

Hence, T is a Ψ_α-contraction mapping with respect to $\psi \in \Psi$. Therefore, by Theorem 4.8, T has a unique fixed point. □

Corollary 4.11. *Let $(X, D, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a Ψ-contraction mapping. If there exists $\alpha \in (0, \infty)$ such that*

$$1 > D(x, y, \alpha) > 0, \quad \forall x, y \in X, \quad x \neq y$$

and

$$D(x, y, \alpha) \geq D(x, y, \alpha) * D(z, y, \alpha)$$

for all $x, y, z \in X$, then T has a unique fixed point.

Proof. A complete fuzzy metric space $(X, D, *)$ with

$$1 > D(x, y, \alpha) > 0, \quad \forall x, y \in X, \quad x \neq y$$

and

$$D(x, y, \alpha) \geq D(x, y, \alpha) * D(z, y, \alpha)$$

for all $x, y, z \in X$ is a complete α-non-Archimedean fuzzy metric space. Hence, the result follows by Theorem 4.8. □

Theorem 4.12. *If* $(X, D, *)$ *is a complete* α*-non-Archimedean fuzzy metric space, then every continuous weak* Ψ_α *-contraction self mapping has a fixed point.*

Proof. Let T be a continuous weak Ψ_α -contraction self mapping of X with respect to ψ. If $x_0 \in X$ is an arbitrary point, then a sequence $\{x_n : x_n = T^{[n]}x_0, n \in \mathbb{N}\}$ is a Cauchy sequence (by Lemma 4.7). Since $(X, D, *)$ is a complete α-non-Archimedean fuzzy metric space, $\lim_n x_n = z \in X$.

Now, we claim $z \in Fix(T) = \{y \in X : Ty = y\}$. Indeed, since $(X, D, *)$ is an α-non-Archimedean fuzzy metric space and T is continuous for all $n \in \mathbb{N}_0$, we have

$$D(z, Tz, t) \geq D(z, x_{n+1}, \alpha) * D(x_{n+1}, Tz, \alpha)$$

$$= D(z, x_{n+1}, \alpha) * D(Tx_n, Tz, \alpha).$$

That is, $D(z, Tz, \alpha) \geq \lim_n \{D(z, x_{n+1}, \alpha) * D(Tx_n, Tz, \alpha)\} = 1.$

Thus, $Tz = z.$ □

Now we provide examples that support our results.

Example 4.13. Let $X = (0, \infty)$, $a * b = ab$, $\forall a, b \in [0, 1]$ and $D : X \times X \times [0, \infty) \to (0, 1]$ be defined by $D(x, y, t) = \dfrac{\min(x, y) + t}{\max(x, y) + t}$. A mapping $T : X \to X$ defined by $Tx = \sqrt{x}$ has a unique fixed point.

Proof. By Proposition 3.13, $(X, D, *)$ is a complete 0-non-Archimedean fuzzy metric space. So, it suffices to show that T is a Ψ_0 -contraction mapping for some $\psi \in \Psi$. Define $\psi : [0, 1] \to [0, 1]$ by $\psi(t) = \sqrt{t}$. Now, for all

$x, y \in X$, we have

$$D(Tx, Ty, 0) = \frac{\min(\sqrt{x}, \sqrt{y})}{\max(\sqrt{x}, \sqrt{y})} = \frac{\sqrt{\min(x, y)}}{\sqrt{\max(x, y)}} = \sqrt{\frac{\min(x, y)}{\max(x, y)}}$$

$$= \psi(D(x, y, 0)).$$

Therefore, T is a Ψ_0-contraction self mapping of X. Hence, the result follows by Theorem 4.8. □

Example 4.14. Let $X = [0, 1]$ and $a * b = ab$ for all $a, b \in [0, 1]$. If $D : X \times X \times (0, \infty) \to (0, 1]$ is defined by $D(x, y, t) = \frac{t}{t + |x - y|}$, then $(X, D, *)$ is a complete fuzzy metric space and a mapping $T : X \to X$ defined by

$$Tx = \begin{cases} \frac{1}{6} & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{2}{3} - x & \text{if } x \in \left(\frac{1}{2}, \frac{2}{3}\right] \\ 0 & \text{if } x \in \left(\frac{2}{3}, 1\right] \end{cases}$$

has a unique fixed point.

Proof. Clearly that $(X, D, *)$ is a complete α-non-Archimedean fuzzy metric space (with $\alpha = 1$). Let $\psi(k) = \frac{4k}{3 + k}$, $k \in [0, 1]$. Then ψ is a Ψ-mapping. Indeed, $\psi^{[n]}(k) = \frac{4^n k}{3^n + (4^n - 3^n)k} \Rightarrow \lim_n \psi^{[n]}(k) = 1$ for all $k \neq 0$. Clearly, ψ is increasing and $\psi(r) > 0$ for each $r \neq 0$. Therefore, ψ is a Ψ-mapping.

We can easily verify that $D(T^{[2]}x, Tx, 1) \geq \psi(D(Tx, x, 1))$ for each $x \in X$. Therefore, T is a continuous weak Ψ_α-contraction mapping with respect to ψ. So, the result follows by Theorem 4.12. □

5. Conclusion

In this work, we introduce the notion of α-non-Archimedean fuzzy metric by relaxing the triangular inequality of a non-Archimedean fuzzy metric space.

The well known fuzzy metric spaces such as $(X, D_d, *)$ (standard fuzzy metric spaces), $(X, D, *)$ with $X = (0, \infty)$, $D(x, y, t) = \frac{\min(x, y) + t}{\max(x, y) + t}$ and $a * b = ab$, every stationary fuzzy metric spaces $(X, D, *)$ with $D(x, y, .) \neq 0$ for all $x, y \in X$ and every non-Archimedean fuzzy metric spaces with $1 > D(x, y, \alpha) > 0$ for some $\alpha \in (0, \infty)$ and for all $x, y \in X$, $x \neq y$ belong to α-non-Archimedean fuzzy metric spaces.

Introduction of an α-non-Archimedean fuzzy metric spaces enables us to induce a topology on a given non-empty set X, using a "one-parameter" in fuzzy settings. In turn, a convergent sequence, a Cauchy sequence and continuity of a self map in α-non-Archimedean fuzzy space are precisely defined as usual. Consequently, some fixed point results of a Ψ_α-contraction self map in α-non-Archimedean fuzzy metric spaces are obtained. Since ψ_α-contraction self mapping deals with respect to one-parameter, contrary to other Ψ-contraction self mapping in fuzzy settings, which deal with all parameters, our results are helpful.

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