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LIE DERIVATIVE OF SHAPE OPERATOR ON REAL HYPERSURFACES IN A COMPLEX SPACE FORM

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Abstract

Let M be a real hypersurface with almost contact metric structure (ϕ, g, ξ, η) in a complex space form $M_n(c)$, $c \neq 0$. In this paper, we prove that $\mathcal{L}_\xi A = 0$ holds on M, then M is a Hopf hypersurface in $M_n(c)$, where A and $\mathcal{L}_\xi$ denote the shape operator and the Lie derivative with respect to the structure vector field ξ, respectively. We characterize such Hopf hypersurfaces of $M_n(c)$.

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1. Introduction

A complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kähler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [4] and that M is called a *Hopf hypersurface*.

Takagi [11] completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. Berndt [1] classified all homogeneous Hopf hypersurfaces in $H_n(\mathbb{C})$ as four model spaces which are said to be A_0, A_1, A_2 and B. A real hypersurface of A_1 or A_2 in $P_n(\mathbb{C})$ or A_0, A_1, A_2 in $H_n(\mathbb{C})$, then M is said to be a *type A* for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [9] for $c > 0$ and Montiel and Romero [7] for $c < 0$.

Theorem 1 [7, 9]**.** *Let M be a real hypersurface of* $M_n(c)$, $c \neq 0$, $n \geq 2$. *It satisfies* $\phi A - A\phi = 0$ *on M if and only if M is locally congruent to one of the model spaces of type A.*

We denote by $\mathcal{L}_\xi$ the Lie derivative with respect to ξ. As for Lie derivative, one of the interesting problems in the study of real hypersurfaces

M in $M_n(c)$ is to investigate a geometric characterization of these model spaces. Some characterizations of real hypersurfaces in $M_n(c)$ are determined by Lie derivative conditions of operators of real hypersurfaces and many important results on them have been obtained by many differential geometers (see [3, 4, 6, 10], etc.).

As for Lie derivative of shape operator, Ki et al. [3] showed the following.

Theorem 2 [3]**.** *Let M be a real hypersurface of* $P_n(\mathbb{C})$, $n \geq 3$. *If it satisfies*

$$\mathcal{L}_\xi A = 0, \tag{1.1}$$

where A denotes the shape operator, then M is locally a tube of radius r over one of the following Kähler submanifolds:

(A_1) *a hyperplane* $P_{n-1}C$, *where* $0 < r < \pi/2$,

(A_2) *a totally geodesic* $P_k C$ $(1 < k < n-2)$, *where* $0 < r < \pi/2$.

The holomorphic distribution T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$T_0(P) = \{X \in T_P(M) \mid g(X, \xi)_p = 0\}, \tag{1.2}$$

where $T_P(M)$ is the tangent space of M at $p \in M$.

On the other hand, Ki and Suh [2] studied real hypersurfaces M with T_0-Lie derivative of complex space form $M_n(c)$.

Theorem 3 [2]**.** *Let M be a real hypersurface of* $M_n(c)$, $c \neq 0$, $n \geq 3$. *Then*

$$g((L_\xi A)X, Y) = 0 \tag{1.3}$$

for any vector fields X and Y in the distribution T_0, *then M is of type A.*

With respect to Lie derivative of Ricci operator and Jacobi operator, Kimura and Maeda [6] and Pérez et al. [10] have proved the following.

Theorem 4 [6]**.** *Let M be a real hypersurface of* $P_n(\mathbb{C})$. *Then M satisfies* $\mathcal{L}_\xi S = 0$ *if and only if* ξ *is a principal curvature vector, in addition except for null set on which the focal map* ϕ_r *degenerates, M lies on a tube of radius r over one of the following Kähler submanifolds*:

(a) *totally geodesic* $P_k(C)(1 \leq k \leq n-1)$, *where* $0 < r < \frac{\pi}{2}$,

(b) *complex quadric* Q_{n-1}, *where* $0 < r < \frac{\pi}{4}$ *and* $\cot^2 2r = n-2$,

(c) $P_1(C) \times P_{(n-1)/2}(C)$, *where* $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 1/(n-2)$ *and* $n(\geq 5)$ *is odd*,

(d) *complex Grassmann* $G_{2,5}(C)$, *where* $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 3/5$ *and* $n = 9$,

(e) *Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 5/9$ *and* $n = 15$,

(f) *k-dimensional Kähler submanifold* $\tilde{N}$ *on which the rank of each shape operator is not greater than* 2 *with non-zero principal curvatures not equal to* $\pm\sqrt{(2k-1)/(2n-2k-1)}$ *and* $\cot^2 r = (2k-1)/(2n-2k-1)$, *where* $k = 1, 2, \ldots, n-1$.

Theorem 5 [10]**.** *Let M be a real hypersurface of* $P_n(\mathbb{C})$, $n \geq 3$ *such that the structure Jacobi operator* R_ξ *is invariant under the structure vector field* ξ, *that is,* $\mathcal{L}_\xi R_\xi = 0$. *Then either M is locally congruent to a tube of radius* $\pi/4$ *over a complex submanifold in* $P_n(\mathbb{C})$ *or to either a geodesic hypersphere or a tube over totally geodesic* $P^k(C)$, $0 < k < m-1$ *with radius* $r \neq \pi/4$.

The purpose of this paper is to generalize Theorem 3 and then to give characterization of real hypersurfaces in a complex space form $M_n(c)$. That is, we shall prove the following theorems:

Theorem A. *Let M be a real hypersurface of* $M_n(c)$, $c \neq 0$. *If it satisfies* (1.1), *then M is a Hopf hypersurface in* $M_n(c)$.

Theorem B. *Let M be a real hypersurface of* $M_n(c)$, $c \neq 0$. *Then it satisfies* $\mathcal{L}_\xi A = 0$ *on M if and only if M is a locally congruent to one of the model spaces of type A.*

In the forthcoming papers, we will give another characterization of a real hypersurface in a non-flat complex space form $M_n(c)$, whose the structure Lie (or Jacobi) operator and the shape operator and the structure tensor ϕ satisfy the anti-commutativity (or commutativity) rule when composed in some order, and in the opposite order, respectively.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2. Preliminaries

Let *M* be a real hypersurface immersed in a complex space form $M_2(c)$, and *N* be a unit normal vector field of *M*.

By $\tilde{\nabla}$, we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\tilde{g}$ of $M_n(c)$.

Then the Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields *X* and *Y* tangent to *M*, where *g* denotes the Riemannian metric tensor of *M* induced from $\tilde{g}$ and *A* is the shape operator of *M* in $M_n(c)$.

For any vector field X on M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.1}$$

for any vector fields X and Y on M.

Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the following:

$$\nabla_X \xi = \phi AX, \tag{2.2}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.3}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations, respectively:

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y$$

$$- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.5}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Let W be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$ and let μ be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish and zero (constant function) if it vanishes. Then it is easily seen from (1.1) that

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where $\alpha = \eta(A\xi)$. We notice here that W is orthogonal to ξ.

We put

$$\Omega = \{p \in M \mid \mu(p) \neq 0\}. \tag{2.7}$$

Then Ω is an open subset of M.

3. Proof of the Theorems

In this section, we shall prove Theorems A and B. Now we prepare without proof the following:

Lemma 3.1 [4, 6]**.** *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Lemma 3.2 [6]**.** *Assume that ξ is a principal curvature vector and the corresponding principal to α. Then*

$$A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - \frac{c}{4} = 0. \tag{3.1}$$

Proof of Theorem A. Suppose that $\mathcal{L}_\xi A = 0$ for any vector field X. Then we see that

$$(\mathcal{L}_\xi A)X = [\xi, AX] - A[\xi, X] = (\nabla_\xi A)X - \phi A^2 X + A\phi AX.$$

Since Lie derivative of shape operator is zero, we obtain

$$(\nabla_\xi A)X = \phi A^2 X - A\phi AX. \tag{3.2}$$

By the virtue of $\nabla_\xi A$ is symmetric, we get

$$(\phi A^2 - 2A\phi A + A^2\phi)X = 0. \tag{3.3}$$

If we put $X = \xi$ into (3.3) and using $\mu \neq 0$, then we have

$$\phi AW - 2A\phi W + \alpha\phi W = 0. \tag{3.4}$$

If we take inner product of W and ϕW, then we have

$$g(A\phi W, W) = 0 \quad \text{and} \quad \alpha + \gamma - 2g(A\phi W, \phi W) = 0, \tag{3.5}$$

by virtue of $\mu \neq 0$ and where $\gamma = g(AW, W)$.

Putting $X = W$ and ϕW into (3.3) and using the first equation of (2.1), we obtain

$$\phi A^2 W - 2A\phi AW + A^2\phi W = 0, \tag{3.6}$$

$$\phi A^2\phi W - 2A\phi A\phi W - A^2 W = 0. \tag{3.7}$$

If we substitute (3.7) into (3.6) and making use of (2.6), then we have

$$2(\phi A\phi A\phi W + A\phi AW) = \eta(A^2\phi W)\xi. \tag{3.8}$$

Differentiating the smooth function $\alpha = g(A\xi, \xi)$ along any vector field X on Ω and using (2.2), (2.5) and (2.6), we have

$$X\alpha = g((\nabla_\xi A)\xi, X) - 2\mu g(A\phi W, X). \tag{3.9}$$

Since we have $(\nabla_\xi A)\xi = \nabla_\xi(\alpha\xi + \mu W) - A\phi(\alpha\xi + \mu W)$, we see from (3.9) that the gradient vector field $\nabla\alpha$ of α is given by

$$\nabla\alpha = \mu\nabla_\xi W - 3\mu A\phi W + (\xi\alpha)\xi + (\xi\mu)W + \alpha\mu\phi W. \tag{3.10}$$

On the other hand, putting $X = \xi$ into (3.2) and making use of (2.6), then we have

$$(\nabla_\xi A)\xi = \mu\phi AW - \mu A\phi W + \alpha\mu\phi W. \tag{3.11}$$

If we substitute (3.11) into (3.9), then we have

$$\nabla\alpha = \mu\phi AW - 3\mu A\phi W + \alpha\mu\phi W. \tag{3.12}$$

If we compare (3.10) and (3.12), then we obtain

$$\mu\nabla\xi W = \mu\phi AW - (\xi\alpha)\xi - (\xi\mu)W.$$

Taking inner product of this equation with ξ and W, respectively, then we get

$$\xi\alpha = 0 \quad \text{and} \quad \xi\mu = 0 \tag{3.13}$$

on Ω and hence the initial equation is reduced to

$$\nabla_{\xi}W = \phi AW. \tag{3.14}$$

Since A is symmetric, we can choose a local orthogonal frame field $\{X_0 = \xi, X_1, \ldots, X_{2(n-1)}\}$ on M such that $AX_i = \lambda_i X_i$ for $1 \le i \le 2(n-1)$.

The vector field $\nabla_{\xi}W$ can be expressed as

$$\nabla_{\xi}W = a\xi + bW + c\phi W + \sum_{i=4}^{2n-1} a_i X_i. \tag{3.15}$$

If we take inner product of (3.15) with ξ and W, then we have

$$a = 0 \quad \text{and} \quad b = 0. \tag{3.16}$$

It follows from (3.14), (3.15) and (3.16) that

$$\phi AW = c\phi W + \sum_{i=4}^{2n-1} a_i X_i. \tag{3.17}$$

If we apply ϕ to (3.17) and using the first equation of (2.1), then we get

$$AW = \mu\xi + \gamma W - \sum_{i=4}^{2n-1} a_i \phi X_i. \tag{3.18}$$

If we substitute (3.18) into (3.4), then we can verify that

$$(\alpha + \gamma)\phi W + \sum_{i=4}^{2n-1} a_i X_i - 2A\phi W = 0.$$

If we take inner product of the above equation with X_i, then we have $a_i = 0$ and hence (3.18) is reduced to

$$AW = \mu\xi + \gamma W. \tag{3.19}$$

Also, from above equation, we can verify that

$$2A\phi W = (\alpha + \gamma)\phi W. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.6), then we obtain

$$4\mu^2 + (\alpha - \gamma)^2 = 0. \tag{3.21}$$

If we differentiate $\mu = g(A\xi, W)$ along any vector field X and take account of (2.5), (2.6), (3.19) and (3.21), then we obtain $\nabla\mu = (\nabla_\xi A)W + \frac{1}{2}\left(\alpha^2 - \gamma^2 + \frac{c}{2}\right)\phi W$ and hence

$$\nabla\mu = (\xi\mu)\xi + (\xi\gamma)W + \left\{\mu^2 + \frac{\alpha}{2}(\alpha - \gamma) + \frac{c}{4}\right\}\phi W. \tag{3.22}$$

On the other hand, if we differentiate $\mu = g(AW, \xi)$, then we have

$$\nabla\mu = \mu\nabla_W W + (W\alpha)\xi + (W\mu)W + \frac{1}{2}\{\alpha^2 + \alpha\gamma - 2\gamma^2 + c\}\phi W. \tag{3.23}$$

If we compare (3.22) and (3.23) and using the first equation (3.5), then we obtain

$$\xi\mu = W\alpha \quad \text{and} \quad \xi\gamma = W\mu, \tag{3.24}$$

$$\mu\nabla_W W = \left\{\mu^2 - \alpha\gamma + \gamma^2 - \frac{c}{4}\right\}\phi W. \tag{3.25}$$

As a similar argument as the above, we can verify that the gradient vector field of the smooth function $\gamma = g(AW, W)$ is given by

$$\nabla\gamma = -(A - \gamma I)\nabla_W W + (W\mu)\xi + (W\gamma)W + \mu(\alpha + 2\gamma)\phi W. \tag{3.26}$$

If we apply the smooth function μ to (3.26) and substituting (3.25) into (3.26), then we obtain

$$\mu W\gamma = \mu(W\mu)\xi + \mu(W\gamma)W + \frac{1}{2}\left\{\mu^2(\alpha + \gamma) + \frac{c}{4}(\alpha - \gamma)\right\}\phi W. \tag{3.27}$$

From equation (3.22), we have

$$4\mu\nabla\mu + (\alpha - \gamma)(\nabla\alpha - \nabla\gamma) = 0. \tag{3.28}$$

If we apply the smooth function μ to (3.28) and take inner product of ϕW, then we have

$$4\mu^2(\phi W)\mu + \mu(\alpha - \gamma)((\phi W)\alpha - (\phi W)\gamma) = 0. \tag{3.29}$$

If we take inner product of (3.12), (3.22) and (3.27) with ϕW and substituting (3.29), then we can find

$$4\mu^2 + (\alpha - \gamma)^2 + \frac{3c}{2} = 0. \tag{3.30}$$

By equations (3.22) and (3.30), we have $c = 0$ on Ω and it is a contradiction.

Thus, the set $\Omega = \{p \in M \mid \mu(p) \neq 0\}$ is empty and hence M is a Hopf hypersurface. □

Proof of Theorem B. By Theorem A, the real hypersurface M satisfying $\mathcal{L}_\xi A = 0$ is a Hopf hypersurface in $M_n(c)$, that is, $A\xi = \alpha\xi$. Therefore, our assumption $\mathcal{L}_\xi A = 0$ is given

$$(\phi A^2 - 2A\phi A + A^2\phi)X = 0. \tag{3.31}$$

For any vector field X on M such that $AX = \lambda X$, it follows from (3.1) that

$$\left(\lambda - \frac{\alpha}{2}\right)A\phi X = \frac{1}{2}\left(\alpha\lambda + \frac{\alpha}{2}\right)\phi X. \tag{3.32}$$

We can choose an orthonormal frame field $\{X_0 = \xi, X_1, X_2, ..., X_{2(n-1)}\}$ on M such that $AX_i = \lambda_i X_i$ for $1 \leq i \leq 2(n-1)$.

If $\lambda_i \neq \frac{\alpha}{2}$ for $1 \leq i \leq p \leq 2(n-1)$, then we see from (3.32) that ϕX_i is also a principal direction, say $A\phi X_i = \mu_i \phi X_i$.

From (3.31), we have $\lambda_i = \mu_i$ and hence $A\phi X_i = \phi AX_i$ for $1 \le i \le p$.

If $\lambda_i \ne \frac{\alpha}{2}$ and $\lambda_j = \frac{\alpha}{2}$ for $1 \le i \le p$ and $p + 1 \le j \le 2(n-1)$, respectively, then it follows from (3.31) that

$$A^2\phi X_j - 2\lambda_j A\phi X_j + \lambda_j^2 \phi X_j = 0. \tag{3.33}$$

Taking inner product of (3.33) with X_i, we obtain $(\lambda_i - \lambda_j)^2 \cdot g(\phi X_j, X_i) = 0$ for $1 \le i \le p$. Since $\lambda_i \ne \lambda_j$, we obtain $g(\phi X_j, X_i) = 0$ for $1 \le i \le p$.

Thus, the vector field ϕX_j is expressed by a linear combination of X_j's only, which implies $A\phi X_j = \frac{\alpha}{2}\phi X_j = \phi AX_j$. If $\lambda_j = \frac{\alpha}{2}$ for $1 \le j \le 2(n-1)$, then it is easily seen that $A\phi X_j = \phi AX_j$ for all j.

Therefore, we have $\phi A - A\phi = 0$ on M. The statement of Theorem B follows immediately Theorem 1. □

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