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ON STRICT-DOUBLE-BOUND NUMBERS OF COMPLETE GRAPHS WITHOUT EDGES OF STARS AND PANS

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Abstract

For a poset $P = (X, \leq_P)$, the strict-double-bound graph of P is the graph $\text{sDB}(P)$ on $V(\text{sDB}(P)) = X$ for which vertices u and v of $\text{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. The strict-double-bound number $\zeta(G)$ of a graph G is defined as $\min\{n;\ \text{sDB}(P) \cong G \cup \overline{K}_n$ for some poset $P\}$. We obtain that

$$\zeta(K_n - E(K_m)) = \lceil 2\sqrt{m} \rceil \ (2 \leq m \leq n-1),$$

$$\zeta(K_n - E(K_m\text{-pan})) = \lceil 2\sqrt{m-1} \rceil + 1 \ (2 \leq m \leq n-2)$$

and $\zeta(K_n - E(K_{1,m})) = 3 \ (1 \leq m \leq n-2)$.

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1. Introduction

In this paper, we deal with $K_n - E(K_m)$, $K_n - E(K_m\text{-pan})$ and $K_n - E(K_{1,m})$. As we note below, previous work had considered the two types of graphs. One is a typical graph of Graph Theory such as a path, a cycle, a wheel and so on. The other is a complete graph without some edges. In the latter case, we considered complete graphs from a point of view of the number of removing edges. In this paper, we deal with complete graphs without the edges of some typical graphs such as K_m, K_m-pan and $K_{1,m}$.

In this paper, we consider finite graphs without loops and multiple edges. For a graph G and a subgraph H of G, the graph $G - E(H)$ is the graph with the vertex set $V(G - E(H)) = V(G)$ and the edge set $E(G - E(H)) = E(G) - E(H)$.

The *union* $G \cup I$ of two graphs G and I is the graph with the vertex set $V(G \cup I) = V(G) \cup V(I)$ and the edge set $E(G \cup I) = E(G) \cup E(I)$. The *sum* $G + I$ of two graphs G and I is the graph with the vertex set $V(G + I) = V(G) \cup V(I)$ and the edge set $E(G + I) = E(G) \cup E(I) \cup \{uv;\, u \in V(G),\, v \in V(I)\}$.

For a graph G and $S \subseteq V(G)$, $\langle S \rangle_V$ is the induced subgraph of S. The graph $\overline{K}_n$ is a graph with n vertices and no edges.

A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. A family $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_l\}$ is an *edge clique cover* of G if each Q_i is a clique of G, and for each $uv \in E(G)$, there exists $Q_i \in \mathcal{Q}$ such that $u, v \in Q_i$.

For a poset P, let $\mathrm{Max}(P)$ be the set of all maximal elements of P and $\mathrm{Min}(P)$ be the set of all minimal elements of P. For a poset P and elements u and v, $u \| v$ denotes that u is incomparable with v.

McMorris and Zaslavsky [5] introduced concepts of some classes of bound graphs, that is, upper bound graphs, double bound graphs, strict double graphs and so on. Scott [8] dealt with double bound graphs in terms of forbidden subgraphs. Langley et al. [4] considered interval strict upper bound graphs and chordal strict upper bound graphs. From the point of view of algorithms, Cheston and Jap [1] dealt with upper bound graphs.

We consider strict-double-bound graphs and strict-double-bound numbers. For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* (*sDB-graph*) of P is the graph $\mathrm{sDB}(P)$ on $V(\mathrm{sDB}(P)) = X$ for which vertices u and v of $\mathrm{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. We say that a graph G is a *strict-double-bound graph* if there exists a poset whose strict-double-bound graph is isomorphic to G. McMorris and Zaslavsky [5] introduced a concept of strict-double-bound graphs. Note that maximal elements and minimal elements of a poset P are isolated vertices of $\mathrm{sDB}(P)$. So a connected graph with $p \geq 2$ vertices is not a strict-double-bound graph. Scott [9] showed as follows: any graph that is the disjoint union of a non-trivial component and enough number of isolated vertices is a strict-double-bound graph.

Thus, we introduced the strict-double-bound number of a graph in [7]. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{n;\ \mathrm{sDB}(P) \cong G \cup \overline{K}_n \text{ for some poset } P\}$.

Scott [9] obtained the following result, using a concept of transitive double competition numbers.

Theorem 1.1 (Scott [9]). *For a non-trivial connected graph G and a minimal edge clique cover $\mathcal{Q}$ of G,* $\lceil 2\sqrt{|\mathcal{Q}|}\,\rceil \leq \zeta(G) \leq |\mathcal{Q}| + 1$.

We obtain the following result.

Proposition 1.2. *Let G be a connected graph with $p \geq 2$ vertices and P be a poset with* $\mathrm{sDB}(P) \cong G \cup \overline{K}_{\zeta(G)}$. *Then* $|\mathrm{Max}(P) \cup \mathrm{Min}(P)| = \zeta(G)$.

Proof. Since maximal elements and minimal elements of P are isolated vertices of $\mathrm{sDB}(P)$, $|\mathrm{Max}(P) \cup \mathrm{Min}(P)| \leq \zeta(G)$ and $V(G) \subseteq V(P) - \mathrm{Max}(P) \cup \mathrm{Min}(P)$. We assume that $|\mathrm{Max}(P) \cup \mathrm{Min}(P)| < \zeta(G)$. Then there exists an isolated vertex $x \notin \mathrm{Max}(P) \cup \mathrm{Min}(P)$ of $\mathrm{sDB}(P)$. For an edge uv of G, if $u, v \leq_P x$, then there exists a maximal element $\alpha \neq x$ such that $u, v \leq_P \alpha$, because $x \notin \mathrm{Max}(P)$, and if $x \leq_P u, v$, then there exists a minimal element $\beta \neq x$ such that $\beta \leq_P u, v$, because $x \notin \mathrm{Min}(P)$. We construct the poset $P - x$ such that $V(P - x) = V(P) - \{x\}$ and $u \leq_{P-x} v$ if $u \leq_P v$. Then $\mathrm{sDB}(P - x) \cong G \cup \overline{K}_{\zeta(G)-1}$, which is a contradiction. □

By Theorem 1.1, $\zeta(K_n) = 2$ for $n \geq 2$. Ogawa et al. [7] and Konishi et al. [3] gave strict-double-bound numbers of $K_{1,n}$, P_n, C_n and W_n. Ogawa et al. [7] also gave an upper bound of strict-double-bound numbers of non-trivial trees. These results discuss some graphs with small number of edges. In [6], Ogawa et al. discussed some graphs with large number of edges, that is, complete graphs missing one, two or three edges. In [2], Kanada et al. dealt with strict-double-bound numbers of complete graphs missing four edges. In this paper, we consider complete graphs without the edges of some typical graphs, that is, $K_n - E(K_m)$, $K_n - E(K_m\text{-pan})$ and $K_n - E(K_{1,m})$.

We use the following result. This result is a key result of this paper. Using this theorem, we estimate strict-double-bound numbers of complete graphs without edges of some typical graphs.

Theorem 1.3 (Konishi et al. [3]). *For a graph G with $p \geq 2$ vertices and no isolated vertices, $\zeta(K_n + G) = \zeta(G)$ for $n \geq 1$.*

We also knew the following results.

Theorem 1.4 (Ogawa et al. [7]). *Let G be a graph. If G has a minimal edge clique cover $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_l\}$ such that there exists a non-maximal* *complete subgraph* $H \neq \emptyset$ *satisfying*

(1) $Q_i \cap Q_j = V(H)$ *for each pair* $Q_i, Q_j \in \mathcal{Q}$ *and*

(2) $Q_i - V(H) \neq \emptyset$ *for all* $Q_i \in \mathcal{Q}$,

then $\zeta(G) = \lceil 2\sqrt{|\mathcal{Q}|} \rceil$.

Proposition 1.5 (Ogawa et al. [7]). *For a star graph* $K_{1,n}$ $(n \geq 2)$, $\zeta(K_{1,n}) = \lceil 2\sqrt{n} \rceil$.

2. Star Graphs and Complete Graphs

First, we deal with K_m-pan. The graph K_m-pan $(m \geq 2)$ is a graph as follows: $V(K_m\text{-pan}) = \{w, v_1, v_2, ..., v_{m-1}\} \cup \{u\}$, $E(K_m\text{-pan}) = \{\{v_i, v_j\};\ 1 \leq i < j \leq m-1\} \cup \{\{v_i, w\};\ 1 \leq i \leq m-1\} \cup \{\{w, u\}\}$ (see Figure 1).

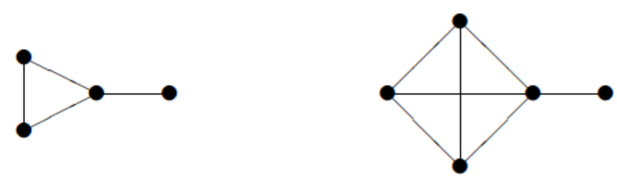

Figure 1. K_3-pan and K_4-pan.

Then $\mathcal{Q} = \{\{w, v_1, ..., v_{m-1}\}, \{w, u\}\}$ is an edge clique cover of K_m-pan satisfying the condition of Theorem 1.4. So we have the following result by Theorem 1.4.

Proposition 2.1. *For a* K_m-pan $(m \geq 2)$, $\zeta(K_m\text{-pan}) = 3$.

Using this result and Theorem 1.3, we have the next result.

Proposition 2.2. $\zeta(K_n - E(K_{1,m})) = 3$, *where* $1 \leq m \leq n-2$.

Proof. Since $m + 2 \leq n$ and $K_{m+2} - E(K_{1,m}) = K_{m+1}$-pan,

$$K_n - E(K_{1,m}) = K_{n-(m+2)} + (K_{m+2} - E(K_{1,m})) = K_{n-(m+2)} + K_{m+1}\text{-pan}.$$

By Proposition 2.1, $\zeta(K_{m+1}\text{-pan}) = 3$. Thus, $\zeta(K_n - E(K_{1,m})) = 3$ by Theorem 1.3. □

We also obtain the next result.

Proposition 2.3. $\zeta(K_n - E(K_m)) = \lceil 2\sqrt{m} \rceil$, *where* $2 \le m \le n - 1$.

Proof. Since $m + 1 \le n$ and $K_{m+1} - E(K_m) = K_{1,m}$,

$$K_n - E(K_m) = K_{n-(m+1)} + (K_{m+1} - E(K_m)) = K_{n-(m+1)} + K_{1,m}.$$

We know that $\zeta(K_{1,m}) = \lceil 2\sqrt{m} \rceil$ by Proposition 1.5. Thus, $\zeta(K_n - E(K_m)) = \lceil 2\sqrt{m} \rceil$ by Theorem 1.3. □

3. K_m-pans

Next, we consider $K_n - E(K_m\text{-pan})$. We obtain the following result.

Proposition 3.1. *For a graph* G *with* $p \ge 2$ *vertices and no isolated vertices*, $\zeta((G \cup \overline{K}_m) + K_n) \le \zeta(G) + m$, *where* $n, m \ge 1$.

Proof. For a graph G, let P_G be a poset such that $\mathrm{sDB}(P_G) \cong G \cup \overline{K}_{\zeta(G)}$. For the poset P_G, we construct the poset P'_G such that (1) $V(P'_G) = V(P_G)$ and (2) $u \le_{P'_G} v$ if (a) $u, v \in V(P_G)$ and $u \le_{P_G} v$ or (b) $u \in \mathrm{Min}(P_G)$ and $v \in \mathrm{Max}(P_G)$. Then $\mathrm{sDB}(P'_G) = \mathrm{sDB}(P_G)$.

For the poset P'_G, we construct the poset P_H such that (1) $V(P_H) = V(P'_G) \cup \{w_1, w_2, ..., w_m\} \cup \{\alpha_1, \alpha_2, ..., \alpha_m\} \cup V(K_n)$ and (2-1) $u \le_{P_H} v$ if (a) $u, v \in V(P'_G)$ and $u \le_{P'_G} v$, (b) $u \in V(K_n)$ and $v \in \mathrm{Max}(P'_G)$ or (c) $u \in \mathrm{Min}(P'_G)$ and $u \in V(K_n)$, (2-2) $u \le_{P_H} \alpha_i$ for all $u \in V(K_n)$ and for all $i = 1, 2, ..., m$, (2-3) $w_i \le_{P_H} \alpha_i$ for each $i = 1, 2, ..., m$, (2-4) $u \le_{P_H} w_i$ for some $u \in \mathrm{Min}(P'_G)$ and all w_i.

Then $\mathrm{sDB}(P_H) \cong ((G \cup \overline{K}_m) + K_n) \cup \{\alpha_1, \alpha_2, ..., \alpha_m\} \cup \overline{K}_{\zeta(G)}$. Thus, $\zeta((G \cup \overline{K}_m) + K_n) \leq \zeta(G) + m$. □

$(K_{1, m-1} \cup K_1) + K_1$ $(m \geq 3)$ is a graph as follows: $V((K_{1, m-1} \cup K_1) + K_1) = \{u, v_1, ..., v_{m-1}\} \cup \{w, z\}$, $E((K_{1, m-1} \cup K_1) + K_1) = \{\{v_i, u\}; 1 \leq i \leq m-1\} \cup \{\{v_i, z\}; 1 \leq i \leq m-1\} \cup \{\{u, z\}\} \cup \{\{w, z\}\}$ (see Figure 2).

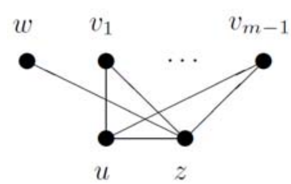

Figure 2. $(K_{1, m-1} \cup K_1) + K_1$.

Proposition 3.2. $\zeta(K_n - E(K_m\text{-pan})) = \lceil 2\sqrt{m-1} \rceil + 1$, *where* $2 \leq m \leq n-2$.

Proof. Since $m + 2 \leq n$ and $K_{m+2} - E(K_m\text{-pan}) = (K_{1, m-1} \cup K_1) + K_1$,

$$\begin{aligned} K_n - E(K_m\text{-pan}) &= K_{n-(m+2)} + (K_{m+2} - E(K_m\text{-pan})) \\ &= K_{n-(m+2)} + ((K_{1, m-1} \cup K_1) + K_1). \end{aligned}$$

By Proposition 3.1, $\zeta((K_{1, m-1} \cup K_1) + K_1) \leq \zeta(K_{1, m-1}) + 1$. By Proposition 1.5, $\zeta(K_{1, m-1}) = \lceil 2\sqrt{m-1} \rceil$. Thus, $\zeta((K_{1, m-1} \cup K_1) + K_1) \leq \lceil 2\sqrt{m-1} \rceil + 1$.

Let P be a poset such that

$$\mathrm{sDB}(P) \cong ((K_{1, m-1} \cup K_1) + K_1) \cup \overline{K}_{\zeta((K_{1, m-1} \cup K_1) + K_1)},$$

$$|\,\mathrm{Max}(P) \cup \mathrm{Min}(P)\,| = \zeta((K_{1, m-1} \cup K_1) + K_1)$$

and $S = \{u, v_1, v_2, ..., v_{m-1}\} \subseteq V(\mathrm{sDB}(P))$. Let Q be the subposet of P such that

$$V(Q) = S \cup \{\alpha \in \mathrm{Max}(P);\ \exists x,\ y \in S,\ x,\ y \leq_P \alpha\}$$

$$\cup \{\beta \in \mathrm{Min}(P);\ \exists x,\ y \in S,\ \beta \leq_P x,\ y\}$$

and for $a, b \in V(Q)$, $a \leq_Q b$ if $a \leq_P b$. Then $\mathrm{sDB}(Q) \cong K_{1,m-1} \cup \overline{K}_q$ and $\lceil 2\sqrt{m-1}\,\rceil \leq q \leq \zeta((K_{1,m-1} \cup K_1) + K_1)$, because $\zeta(K_{1,m-1}) = \lceil 2\sqrt{m-1}\,\rceil$. We consider the case $q = \lceil 2\sqrt{m-1}\,\rceil$. Since u is adjacent to all v_i $(i = 1, 2, \ldots, m-1)$, each v_i is not adjacent to other v_j $(j \neq i)$ and $\zeta(K_{1,m-1}) = \lceil 2\sqrt{m-1}\,\rceil$, $u \leq_P \alpha$ for all $\sigma \in \mathrm{Max}(Q)$ and $\beta \leq_P u$ for all $\beta \in \mathrm{Min}(Q)$. Thus, there exists $\gamma \in (\mathrm{Max}(P) \cup \mathrm{Min}(P)) - (\mathrm{Max}(Q) \cup \mathrm{Min}(Q))$, because w is not adjacent to u and $(K_{1,m-1} \cup K_1) + K_1$ is connected. Hence, $\zeta((K_{1,m-1} \cup K_1) + K_1) \geq q + 1 = \lceil 2\sqrt{m-1}\,\rceil + 1$. In the case $q > \lceil 2\sqrt{m-1}\,\rceil$, $\zeta((K_{1,m-1} \cup K_1) + K_1) \geq q \geq \lceil 2\sqrt{m-1}\,\rceil + 1$.

Therefore,

$$\zeta((K_{1,m-1} \cup K_1) + K_1) = \lceil 2\sqrt{m-1}\,\rceil + 1$$

and

$$\zeta(K_n - E(K_m\text{-pan})) = \lceil 2\sqrt{m-1}\,\rceil + 1$$

by Theorem 1.3. □

References

[1] G. A. Cheston and T. S. Jap, A survey of the algorithmic properties of simplicial, upper bound and middle graphs, J. Graph Algorithms Appl. 10 (2006), 159-190.

[2] K. Kanada, K. Kushima, K. Ogawa, S. Tagusari and M. Tsuchiya, On strict-double-bound numbers of complete graphs missing four edges, Bulletin of the ICA 68 (2013), 79-89.

[3] S. Konishi, K. Ogawa, S. Tagusari and M. Tsuchiya, Note on strict-double-bound numbers of paths, cycles, and wheels, JCMCC 83 (2012), 205-210.

[4] L. Langley, S. K. Merz, J. R. Lundgren and C. W. Rasmussen, Posets with interval or chordal strict upper and lower bound graphs, Congr. Numer. 125 (1997), 153-160.

[5] F. R. McMorris and T. Zaslavsky, Bound graphs of a partially ordered set, J. Combin. Inform. System Sci. 7 (1982), 134-138.

[6] K. Ogawa, R. Soejima, S. Tagusari and M. Tsuchiya, Note on strict-double-bound numbers of nearly complete graphs missing some edges, Discrete Math. 312 (2012), 584-587.

[7] K. Ogawa, S. Tagusari and M. Tsuchiya, Note on strict-double-bound graphs and numbers, AKCE Int. J. Graphs Comb. 11 (2014), 127-132.

[8] D. D. Scott, Posets with interval upper bound graphs, Order 3 (1986), 269-281.

[9] D. D. Scott, The competition-common enemy graph of a digraph, Discrete Appl. Math. 17 (1987), 269-280.