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DIMENSION OF VERTEX LABELING OF *k*-UNIFORM dcsl PATH

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Abstract

Let an injective function $f : V(G) \to 2^X$, where $V(G)$ is the vertex set of a graph G and 2^X is the power set of a non-empty set X, be given. Consider the induced function $f^{\oplus} : V(G) \times V(G) \to 2^X \setminus \{\phi\}$ defined by $f^{\oplus}(u, v) = f(u) \oplus f(v)$, where $f(u) \oplus f(v)$ denotes the symmetric difference of the two sets. The function f is called a k-uniform dcsl (and X a k-uniform dcsl-set) of the graph G, if there

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exists a constant k such that $|f^{\oplus}(u, v)| = kd_G(u, v)$, where $d_G(u, v)$ is the length of a shortest path between u and v in G. If a graph G admits a k-uniform dcsl, then G is called a k-uniform dcsl graph. The k-uniform dcsl index of a graph G, denoted by $\delta_k(G)$, is the minimum of the cardinalities of X, as X varies over all k-uniform dcsl-sets of G. A linear extension $\mathcal{L}$ of a partial order $\mathcal{P}$ is a linear order on the elements of $\mathcal{P}$, such that $x \preceq y$ in $\mathcal{P}$ implies $x \preceq y$ in $\mathcal{L}$ for all $x, y \in \mathcal{P}$. A set $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_K\}$ of linear extensions of $\mathcal{P}$ is a realizer of $\mathcal{P}$ if, for every incomparable pair $x, y \in \mathcal{P}$, there are $\mathcal{L}_i, \mathcal{L}_j \in \mathcal{R}$ with $x \preceq y$ in $\mathcal{L}_i$ and $x \succeq y$ in $\mathcal{L}_j$. The dimension of $\mathcal{P}$, denoted by $\dim(\mathcal{P})$, is the minimum of the cardinalities of realizers of $\mathcal{P}$. Let $\mathscr{F}$ be the range of a k-uniform dcsl of the path P_n on $n > 2$ vertices. The purpose of this paper is to prove that $\dim(\mathscr{F}) \leq \delta_k(P_n)$, whether or not $\mathscr{F}$ forms a lattice with respect to set inclusion '$\subseteq$'.

1. Introduction

Acharya [1] introduced the notion of vertex *set-valuation* as a set-analogue of number valuation. For a graph $G = (V, E)$ and a non-empty set X, Acharya defined a *set-valuation* of G as an injective *set-valued* function $f : V(G) \to 2^X$, and he defined a *set-indexer* $f^{\oplus} : E(G) \to 2^X \setminus \{\phi\}$ as a *set-valuation* such that the function given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all the subsets of X and '$\oplus$' is the binary operation of taking the symmetric difference of subsets of X.

Acharya and Germina [2], who has been studying topological set-valuation, introduced the particular kind of set-valuation for which a metric, especially the cardinality of the symmetric difference, is associated with each pair of vertices is k (where k is a constant) times that of the distance between them in the graph [2]. In other words, the question is whether one can determine those graphs $G = (V, E)$ that admit an injective set-valued function $f : V(G) \to 2^X$, where 2^X is the power set of a non-empty set *X*, such that, for each pair of distinct vertices *u* and *v* in *G*, the cardinality of the symmetric difference $f(u) \oplus f(v)$ is *k* times of the usual distance $d_G(u, v)$ between *u* and *v* in *G*, where *k* is a constant. They [2] called such a *set-valuation f* of *G* a *k-uniform distance-compatible set-labeling* (*k-uniform dcsl*) of *G*, and the graph *G* which admits *k*-uniform dcsl a *k-uniform distance-compatible set-labeled graph* (*k-uniform dcsl graph*) and the non-empty set *X* corresponding to *f* a *k-uniform dcsl-set*.

The following universal theorem has been established.

Theorem 1 [2]**.** *Every graph admits a dcsl.*

The 1-*uniform dcsl index* of a graph *G*, denoted by $\delta_d(G)$, is the minimum of the cardinalities of *X*, as *X* varies over all 1-uniform dcsl-sets of *G*.

The *k-uniform dcsl index* [4] of a graph *G*, denoted by $\delta_k(G)$, is the minimum of the cardinalities of *X*, as *X* varies over all *k*-uniform dcsl-sets of *G*.

One may recall a *partially ordered set* (or a *poset*, in short) $\mathcal{P}$ as a structure $(P, \preceq)$, where *P* is a non-empty set and '$\preceq$' is a partial order relation on *P* such that '$\preceq$' is *reflexive*, *antisymmetric* and *transitive*. We denote $(x, y) \in \mathcal{P}$ by $x \preceq y$. By standard notation, we usually identify the ground set of a poset with the whole poset.

Two elements of $\mathcal{P}$ standing in the relation of $\mathcal{P}$ are called *comparable*, otherwise they are *incomparable*. A poset is a *chain* if it contains no incomparable pair of elements. In this case, the partial order is a *linear order*. A poset is an *antichain* if all of its pairs are incomparable. The size of a largest chain in a poset $\mathcal{P}$ is called the *height* of the poset, denoted by $\mathrm{height}(\mathcal{P})$ (or $h(\mathcal{P})$), and that of a largest antichain is called its *width*, denoted by $\mathrm{width}(\mathcal{P})$ (or $w(\mathcal{P})$).

We say that z *covers* y if and only if $y \prec z$ and $y \preceq x \preceq z$ implies either $x = y$ or $x = z$. A *Hasse diagram* of a poset $(P, \preceq)$ is a drawing in which the points of P are placed so that if y *covers* x, then y is placed at a higher level than x and joined to x by a line segment. A poset $\mathcal{P}$ is *connected*, if its Hasse diagram is connected as a graph. A *cover graph* of a poset $(P, \preceq)$ is the graph with vertex set P such that $x, y \in P$ are adjacent if and only if one of them covers the other. All posets depicted in this paper are shown by their Hasse diagrams.

A *planar drawing* of a poset $\mathcal{P}$ is a representation of the Hasse diagram of $\mathcal{P}$ such that no edges of the Hasse diagram cross each other. A *planar poset* is a poset that has a planar drawing, otherwise it is called *non-planar poset*.

A poset $\mathcal{Q}$ is a subposet of $\mathcal{P}$ if $Q \subseteq P$, and for each pair $x, y \in \mathcal{Q}$, $x \preceq y$ in $\mathcal{Q}$ exactly if $x \preceq y$ in $\mathcal{P}$. Two posets $\mathcal{P}$ and $\mathcal{Q}$ are called *isomorphic* if there is a one-to-one correspondence $\Phi : P \to Q$ such that $x \preceq y$ in $\mathcal{P}$ if and only if $\Phi(x) \preceq \Phi(y)$ in $\mathcal{Q}$. The poset $\mathcal{Q}$ is said to be *embedded* in $\mathcal{P}$, denoted by $\mathcal{Q} \subseteq \mathcal{P}$, if $\mathcal{Q}$ is isomorphic to a subposet of $\mathcal{P}$.

A *linear extension* $\mathcal{L}$ of $\mathcal{P}$ is a linear order on the elements of $\mathcal{P}$, such that $x \preceq y$ in $\mathcal{P}$ implies $x \preceq y$ in $\mathcal{L}$ for all $x, y \in \mathcal{P}$. We write a *linear extension* as $\mathcal{L} : [x_1, x_2, ..., x_n]$ which stands for $x_1 \preceq x_2 \preceq \cdots \preceq x_n$ in $\mathcal{L}$.

Definition 1 [8]**.** A set $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_K\}$ of linear extensions of $\mathcal{P}$ is a *realizer* of $\mathcal{P}$ if for every incomparable pair $x, y \in \mathcal{P}$, there are $\mathcal{L}_i$, $\mathcal{L}_j \in \mathcal{R}$ with $x \preceq y$ in $\mathcal{L}_i$ and $x \succeq y$ in $\mathcal{L}_j$ for $1 \leq i \neq j \leq k$. The *dimension* of $\mathcal{P}$ (denoted by $\dim(\mathcal{P})$) is the minimum cardinality of a realizer.

Equivalently, $\dim(\mathcal{P})$ can be defined as the minimum k for which there are linear extensions $\mathcal{L}_1, ..., \mathcal{L}_k$ such that $\mathcal{P} = \mathcal{L}_1 \bigcap \mathcal{L}_2 \bigcap \cdots \bigcap \mathcal{L}_k$, where the intersection is taken over the sets of relations of the $\mathcal{L}_i$, for $1 \leq i \leq k$.

Another characterization of dimension, in terms of coordinates, can be obtained by using an embedding of $\mathcal{P}$ into R^t (called *t-dimensional poset*) and it was given by Ore [13]. Let R^t denote the poset of all t-tuples of real numbers, partially ordered by inequality in each coordinate, i.e., $(a_1, a_2, ..., a_t) \leq (b_1, b_2, ..., b_t)$ if and only if $a_i \leq b_i$, for $i = 1, 2, ..., t$. Then the dimension of a poset $\mathcal{P}$ is the minimum number t such that $\mathcal{P}$ has an embedding into R^t.

One should note that, a planar poset $\mathcal{P}$ having a greatest and least element has dimension at most 2 [10]. Trotter and Moore [7] proved that a planar poset having either a greatest or least element has dimension at most 3. Hiraguchi [16] proved that the dimension cannot exceed the width, and antichains show that the dimension can be much less than the width, and also he proved that the dimension cannot exceed half of the number of points. Kelly [15] then constructed planar posets of arbitrary dimension.

A poset $(L, \preceq)$ is a *lattice* if every pair of elements $x, y \in L$; has a *least upper bound* (*lub*, for short), denoted by $x \vee y$ (called *join*), and a *greatest lower bound* (*glb*, for short), denoted by $x \wedge y$ (called *meet*). In general, a lattice is denoted by $(L, \preceq)$.

Throughout this paper, lattice (and poset) means lattice (and poset) under set inclusion $\subseteq$. Unless otherwise mentioned, for all the terminology in graph theory and lattice theory, the reader is referred, respectively to [6, 5].

In this paper, when we speak of the dimension of vertex labeling of k-uniform dcsl graph, we usually mean a dimension of a poset whose elements are the vertex labeling of k-uniform dcsl graph, and also, whenever we consider the vertex labeling of k-uniform dcsl of G by $\mathscr{F}$, it means that $\mathscr{F}$ is a poset, whose elements are vertex labeling of k-uniform dcsl of G. In particular, we consider only planar posets (connected), and we prove that $\dim(\mathscr{F}) \leq \delta_k(P_n)$, where $\mathscr{F}$ is a set of vertex labeling of the k-uniform dcsl path P_n $(n > 2)$ whether or not forms a lattice with respect to set inclusion '$\subseteq$'.

2. Preliminaries

In this section, we provide some basic results of partial orders and k-uniform dcsl index $\delta_k(G)$ of G that are relevant to research. Most of the results discussed here can be found in Dilworth, Mirsky and in Trotter and Moore.

The most classical theorems of posets are given by Dilworth and Mirsky.

Theorem 2 [12]. *Suppose that the largest antichain in the poset $\mathcal{P}$ has size r. Then $\mathcal{P}$ can be partitioned into r chains, but not fewer.*

Theorem 3 [14]. *Suppose that the largest chain in the poset $\mathcal{P}$ has size r. Then $\mathcal{P}$ can be partitioned into r antichains, but not fewer.*

The following theorem is one of an important result which was given by Trotter and Moore.

Theorem 4 [7]. *If the Hasse diagram of $\mathcal{P}$ is a tree, then* $\dim(\mathcal{P}) \leq 3$.

In [4, 3] and [11], the following results are established.

Proposition 1 [4]. *For a k-uniform dcsl graph G,* $\delta_k(G) \geq k \;\, diam(G)$.

Theorem 5 [4]. *If G is k-uniform dcsl, and m is a positive integer, then G is mk-uniform dcsl.*

Lemma 1 [3]. $\delta_d(P_n) = n - 1$, *for* $n > 2$.

Proposition 2 [11]. *The set $\mathscr{F}$ of vertex labeling of a* 1*-uniform dcsl path P_n $(n > 2)$ forms a lattice.*

3. Main Results

Since, by Proposition 2, the set $\mathscr{F}$ of vertex labeling of a 1-uniform dcsl path P_n $(n > 2)$ form a lattice, and also, all the members of $\mathscr{F}$ are comparable, so that, $\dim(\mathscr{F}) = 1$. By Lemma 1, $\delta_d(P_n) = n - 1$, for $n > 2$. Hence, we conclude the following:

Theorem 6. *Let $\mathscr{F}$ be the set of vertex labeling of a* 1-*uniform dcsl path* P_n $(n > 2)$ *which form a lattice with respect to set inclusion* '$\subseteq$'. *Then* $dim(\mathscr{F}) < \delta_d(P_n)$.

Remark 1. One may observe that, every vertex labeling of a 1-uniform dcsl path $P_n(n > 2)$ need not form a lattice, they are some vertex labeling, which do not form a lattice, so depending on the poset of different height and width, one may get different dimensions. For, consider $\mathscr{F} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is the vertex labeling of 1-uniform dcsl path P_3, which is a lattice of height$(\mathscr{F}) = 3$, width$(\mathscr{F}) = 1$, and dim$(\mathscr{F}) = 1$. Also, consider $\mathscr{H} = \{\{1, 3\}, \{1\}, \{1, 2\}\}$ is the another vertex labeling of 1-uniform dcsl path P_3, which is a poset but not lattice of height$(\mathscr{F}) = 2$, width$(\mathscr{F}) = 2$, and dim$(\mathscr{F}) = 2$.

Let $\mathscr{F}$ be a set of vertex labeling of a 1-uniform dcsl path P_n $(n > 2)$ which form a poset. If height$(\mathscr{F}) = n$, then dim$(\mathscr{F}) = 1$, and width$(\mathscr{F}) = 1$, which is maximum, now, on words, we call it as *maximum width*$(\mathscr{F})$. However, when height$(\mathscr{F}) \neq n$, then dim$(\mathscr{F}) \neq 1$, and maximum width$(\mathscr{F}) \neq 1$, so calculating maximum width$(\mathscr{F})$ and dim$(\mathscr{F})$ is an interesting problem. We start with height$(\mathscr{F}) = 2$, and calculate maximum width$(\mathscr{F})$ and dim$(\mathscr{F})$. This observation leads us to construct a new definition, which we named as "height-2 poset".

Definition 2. The height-2 poset H_n on $2n$ elements $a_1, ..., a_n, b_1, ..., b_n$ is the poset of height two consisting of two antichains $A = \{a_1, a_n\}$ and $B = \{b_1, ..., b_n\}$ such that $b_i \preceq a_j$ in H_n exactly if $i = j$, and $j = i - 1$ (see Figure 1).

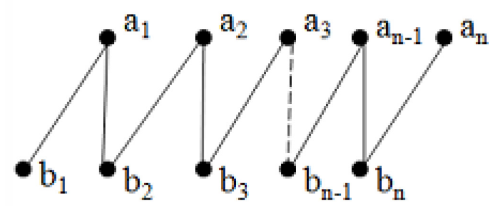

Figure 1. Height-2 poset H_n.

Proposition 3. *For* $n \geq 2$, $dim(H_n) = 2$.

Proof. Consider the set $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$ of linear extensions of the height-2 poset H_n, where

$$\mathcal{L}_1 : [b_1, b_2, a_1, b_3, a_2, a_3, ..., a_{n-1}, a_n, b_4, b_5, ..., b_{n-1}, b_n]$$

and

$$\mathcal{L}_2 : [b_n, b_{n-1}, ..., b_4, a_n, a_{n-1}, ..., a_2, b_3, b_2, b_1, a_1].$$

Then $\mathcal{R}$ is a realizer of H_n (since for every incomparable pair $x, y \in H_n$, there are $\mathcal{L}_i, \mathcal{L}_j \in \mathcal{R}$ with $x \preceq y$ in $\mathcal{L}_i$ and $x \succeq y$ in $\mathcal{L}_j$ for $1 \leq i \neq j \leq 2)$, and hence $\dim(H_n) \leq 2.$ We claim that there is no proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes H_n. If possible, suppose there is a proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes H_n, which means the only one member in $\mathcal{S}$ whose intersection equal to the poset H_n, thus all the elements of H_n are comparable, a contradiction. Hence $\dim(H_n) = 2.$ □

Proposition 4. *There exists a vertex labeling* $\mathscr{F}$ *of a* 1-*uniform dcsl path* $P_n (n > 2)$ *which does not form a lattice of* $width(\mathscr{F}) = \left\lceil \frac{|V(P_n)|}{2} \right\rceil$ *and* $height(\mathscr{F}) = 2$, *and the poset* $\mathscr{F}$ *is embedded in height*-2 *poset* H_n.

Proof. Let $V(P_n) \doteq \{v_1, v_2, ..., v_n\}$.

Let $X = \{1, 2, ., w, ., n-1\}$, where $w = \left\lceil \frac{|V(P_n)|}{2} \right\rceil$.

Define $f : V(P_n) \to 2^X$ by $f(v_1) = \{1, 2, ..., w-1\}$, $f(v_2) = \{1, 2, ..., w-1, w\}$, $f(v_3) = \{2, ., w-1, w\}$, $f(v_4) = \{2, ., w-1, w, w+1\}$, $f(v_5) = \{3, ., w, w+1\}$.

Hence, in general, for $1 \le i \le n$,

$$f(v_i) = \begin{cases} \left\{\frac{i+1}{2}, \frac{i+1}{2}+1, ..., w+\frac{i+1}{2}-3, w+\frac{i+1}{2}-2\right\}, & \text{if } i \text{ is odd,} \\ \left\{\frac{i}{2}, \frac{i}{2}+1, ..., w+\frac{i}{2}-2, w+\frac{i}{2}-1\right\}, & \text{otherwise.} \end{cases}$$

Then

$|f(v_1) \,\Delta\, f(v_3)| = 2 = d(v_1, v_3)$, $|f(v_1) \,\Delta\, f(v_4)| = 3 = d(v_1, v_4)$,

$|f(v_2) \,\Delta\, f(v_3)| = 1 = d(v_2, v_3)$,

hence $|f(v_i) \Delta f(v_j)| = j - i = 1.d(v_i, v_j)$, for $1 \le i < j \le n$. Thus, f is a 1-uniform dcsl.

Without loss of generality, suppose n is even. For $1 \le i \le \frac{n}{2}$, choose $\mathcal{A} = \{f(v_2), f(v_4), ..., f(v_{2i})\}$ and $\mathcal{B} = \{f(v_1), f(v_3), ..., f(v_{2i-1})\}$. Then $\mathscr{F} = \{f(v)/v \in V(P_n)\} = \mathcal{A} \cup \mathcal{B}$ form a poset of height 2 and width w with respect to $\subseteq$, also $\mathcal{A}$ and $\mathcal{B}$ are antichains of same length $\frac{n}{2}$. Since, for each pair of elements $f(v_i), f(v_j)$ in $\mathscr{F}$ for $1 \le i < j \le n$, which are comparable, both supremum and infimum exist, while for the incomparable elements $f(v_i), f(v_j)$ in $\mathscr{F}$, for $1 \le i < j \le n$, supremum $\{f(v_i), f(v_j)\} = f(v_i) \cup f(v_j)$ when i and j are odd, but infimum does not exist, and when i and j are even, infimum $\{f(v_i), f(v_j)\} = f(v_i) \cap f(v_j)$, but supremum does not exist.

Hence, $(\mathscr{F}, \subseteq)$ is not a lattice.

Finally, we prove that $\mathscr{F}$ is embedded in H_n.

Define $\Phi : \mathscr{F} \to H_{\frac{n}{2}}$ by

$$\Phi(f(v_i)) = \begin{cases} a_{\frac{i}{2}}, & \text{if } i \text{ is even,} \\ b_{\left\lceil \frac{i}{2} \right\rceil}, & \text{otherwise,} \end{cases}$$

where $f(v_i) \in \mathscr{F}$ for $1 \leq i \leq n$, and $H_{\frac{n}{2}}$ is a subposet of H_n on n elements $a_1, ..., a_{\frac{n}{2}}, b_1, ..., b_{\frac{n}{2}}$ with the same partial order of H_n. Since, for $1 \leq l \leq \frac{n}{2}$, $f(v_{2l-1}) \subseteq f(v_{2l})$ if and only if $b_l \preceq a_l$ in $H_{\frac{n}{2}}$, and for $1 \leq l \leq \frac{n}{2} - 1$, $f(v_{2l+1}) \subseteq f(v_{2l})$ in $\mathscr{F}$ if and only if $b_{l+1} \preceq a_l$ in $H_{\frac{n}{2}}$. Hence, $\mathscr{F} \cong H_{\frac{n}{2}}$, and hence $\mathscr{F}$ is embedded in H_n. □

Proposition 5. *Let $\mathscr{F}$ be the set of vertex labeling of a* 1-*uniform dcsl path P_n $(n > 2)$, which is embedded in H_n. Then height$(\mathscr{F}) = 2$ if and only if width$(\mathscr{F}) = \left\lceil \frac{|V(P_n)|}{2} \right\rceil$, and width$(\mathscr{F})$ is maximum.*

Proof. Let $V(P_n) \doteq \{v_1, v_2, ..., v_n\}$.

Let $X = \{1, 2, ., w, ., n-1\}$, where $w = \left\lceil \frac{|V(P_n)|}{2} \right\rceil$, and let f be a 1-uniform dcsl of P_n $(n > 2)$ which is given in Proposition 4, such that $\mathscr{F} = \{f(v)/v \in V(P_n)\}$ is embedded in H_n.

Suppose height$(\mathscr{F}) = 2$, then, by Mirsky's Theorem 3, $\mathscr{F}$ can be partitioned into 2 antichains, but not fewer, say $\hat{W}_1$, and $\hat{W}_2$, and also one of $\hat{W}_1, \hat{W}_2$ is of length $n - w$, and w, respectively. Thus, one of $\hat{W}_1, \hat{W}_2$ is of maximum length w. Hence, $w(\mathscr{F}) = \left\lceil \frac{|V(P_n)|}{2} \right\rceil (= w)$, and hence $w(\mathscr{F})$ is maximum, otherwise, there exists an antichain whose length is greater than w, which is a contradiction. Conversely, suppose $\text{width}(\mathscr{F}) = \left\lceil \frac{|V(P_n)|}{2} \right\rceil$ $(= w)$, then, by Dilworth's Theorem 2, $\mathscr{F}$ can be partitioned into w chains, but not fewer, let it be $L_1, L_2, \ldots,$ and L_w, and also, $|L_i| \leq 2$, for $1 \leq i \leq w$. Hence, $\text{height}(\mathscr{F}) = 2$.

Proposition 6. *Let $\mathscr{F}$ be the set of vertex labeling of a* 1-*uniform dcsl path $P_n(n > 2)$, which is embedded in H_n. Then $\dim(\mathscr{F}) = 2$.*

Proof. Let f be a 1-uniform dcsl of P_n $(n > 2)$ which is given in Proposition 4, such that $\mathscr{F} = \{f(v)/v \in V(P_n)\}$ is embedded in $H_{\frac{n}{2}}$. Since $H_{\frac{n}{2}}$ is embedded in H_n, and since by Proposition 3, $\dim(H_n) = 2$, hence $\dim(H_{\frac{n}{2}}) = 2$. Since, $\mathscr{F}$ is embedded in $H_{\frac{n}{2}}$, hence $\dim(\mathscr{F}) = 2$. □

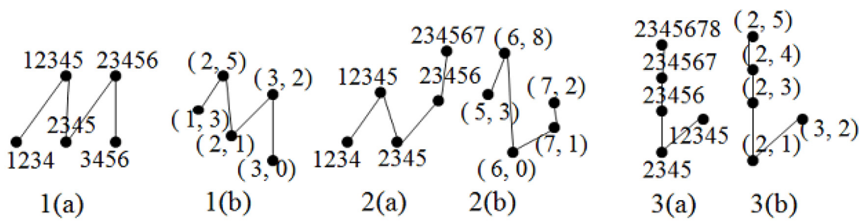

Figure 2. The poset $\mathscr{F}$ of vertex labeling of a 1-uniform dcsl path P_5 of height 2, 3 and 4 are 1(a), 2(a) and 3(a), and its embedding in R^2 are 1(b), 2(b) and 3(b).

Remark 2. It is noticed that, when the height (other than n) of a poset $\mathscr{F}$ of set of vertex labeling of a 1-uniform dcsl path P_n $(n > 2)$ is increasing from 2 to $n - 1$, then the corresponding maximum width is decreasing from $\left\lceil \frac{|V(P_n)|}{2} \right\rceil$ to 2. Hence, it is of interest to find the formula for maximum width of $\mathscr{F}$, when $\mathscr{F}$ has an arbitrary height. We have calculated the maximum width of a poset $\mathscr{F}$ of set of vertex labeling of a 1-uniform dcsl path P_5 of height 2, 3 and 4, and they are 3, 2 and 2, respectively, and $\dim(\mathscr{F}) = 2$ (see Figure 2). However, in general, the calculation of the maximum width of $\mathscr{F}$, when $\mathscr{F}$ has an arbitrary height other than 1 and n is under further investigation.

Remark 3. Let $\mathscr{F}$ be a set of vertex labeling of a 1-uniform dcsl path $P_n (n \geq 2)$ which form a poset of $\text{width}(\mathscr{F}) = 1$. Then $\dim(\mathscr{F}) = 1$, and $\text{height}(\mathscr{F})$ is n. However, when $\text{width}(\mathscr{F}) = 2$, then $\text{height}(\mathscr{F}) \neq n$, but it lies between $\left\lfloor \frac{|V(P_n)|}{2} \right\rfloor + 1$ and $n - 1$. Hence, calculating the $\dim(\mathscr{F})$, when $\text{width}(\mathscr{F}) = 2$, and $\text{height}(\mathscr{F})$ is minimum (we call it as *minimum* $\text{height}(\mathscr{F})$), is an another interesting problem.

Definition 3. A width-2 poset W_n is the poset $(\{a_1, ..., a_n, b_1, ..., b_n\}, \preceq)$ of width two consisting of two chains $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$ such that $a_i \preceq a_{i-1}$ for $2 \leq i \leq n$, $b_i \preceq b_{i+1}$ for $1 \leq i \leq n$, $a_n \preceq b_i$ for $1 \leq i \leq n$, and for $1 \leq i \leq n-1$ and $1 \leq j \leq n$, $a_i \,||\, b_j$ (see Figure 3).

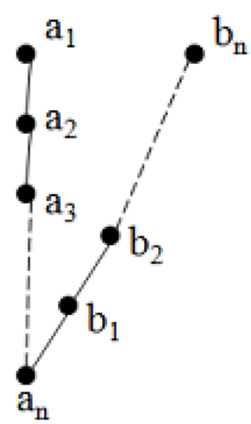

Figure 3. Width-2 poset W_n.

Proposition 7. *For* $n \geq 2$, $\mathit{dim}(W_n) = 2$.

Proof. Consider $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$ of linear extensions of W_n, where $\mathcal{L}_1 : [a_n, a_{n-1}, ..., a_2, a_1, b_1, b_2, ..., b_{n-1}, b_n]$ and $\mathcal{L}_2 : [b_1, b_2, ..., b_n, a_n, a_{n-1}, ..., a_2, a_1]$. Then $\mathcal{R}$ is a realizer of W_n (since for every incomparable pair $x, y \in H_n$, there are $\mathcal{L}_i, \mathcal{L}_j \in \mathcal{R}$ with $x \preceq y$ in $\mathcal{L}_i$ and $x \succeq y$ in $\mathcal{L}_j$ for $1 \le i \ne j \le 2$), hence $\dim(W_n) \le 2$. We prove that there is no proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes W_n. If possible, suppose there is a proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes W_n, which means the only one member in $\mathcal{S}$ whose intersection equal to the poset W_n, thus all the elements of W_n are comparable, a contradiction. Hence $\dim(W_n) = 2$. □

Remark 4. Let $\mathscr{F}$ be a vertex labeling of a 1-uniform dcsl path P_n $(n > 2)$, which form a poset of $\text{width}(\mathscr{F}) = 2$, and if $\mathscr{F}$ is embedded in W_n, then $\text{height}(\mathscr{F})$ lies between $\left\lfloor \frac{|V(P_n)|}{2} \right\rfloor + 1$ and $n - 1$.

The following proposition shows the existence of one such embedding.

Proposition 8. *There exists a vertex labeling $\mathscr{F}$ of a* 1*-uniform dcsl path P_n $(n > 2)$, which does not form a lattice of height$(\mathscr{F}) = \left\lfloor \frac{|V(P_n)|}{2} \right\rfloor + 1$ and width$(\mathscr{F}) = 2$, and the poset $\mathscr{F}$ is embedded in W_n.*

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$.

Let $X = \{1, 2, ., h, ., n-1\}$, where $h = \left\lfloor \frac{|V(P_n)|}{2} \right\rfloor + 1$.

Define $f : V(P_n) \to 2^X$ by

$$f(v_1) = \{1, 2, ..., h-1\},\ f(v_2) = \{1, 2, ..., h-1, h-2\},$$

$$f(v_3) = \{1, 2, ..., h-2, h-3\}, ...,\ f(v_h) = \emptyset,\ f(v_{h+1}) = \{h\},$$

$$f(v_{h+2}) = \{h, h+1\}, ...,\ f(v_n) = \{h, h+1, ..., n-1\}.$$

Then

$$|\, f(v_1)\,\Delta\, f(v_h)\,| = h - 1 = d(v_1,\, v_h),$$

$$|\, f(v_h)\,\Delta\, f(v_n)\,| = n - h = d(v_h,\, v_n),$$

$$|\, f(v_1)\,\Delta\, f(v_n)\,| = n - 1 = d(v_1,\, v_n),$$

hence $|\, f(v_i)\,\Delta\, f(v_j)\,| = j - i = 1.d(v_i,\, v_j),$ for $1 \le i < j \le n.$ Thus, f is a 1-uniform dcsl.

Choose $\mathcal{A} = \{f(v_h),\, f(v_{h-1}), \ldots,\, f(v_1)\}$ and $\mathcal{B} = \{f(v_{h+1}),\, f(v_{h+2}), \ldots,$ $f(v_n)\}.$ Then $\mathscr{F} = \{f(v)/v \in V(P_n)\} = \mathcal{A} \cup \mathcal{B}$ form a poset of height h and width 2 with respect to $\subseteq$, also $\mathcal{A}$ and $\mathcal{B}$ are chains of length h and $n - h$, respectively. Since, for each pair of elements $f(v_i),\, f(v_j)$ in $\mathscr{F}$ for $1 \le i < j \le n,$ which are comparable, both supremum and infimum exist, while for the incomparable elements $f(v_i),\, f(v_j)$ in $\mathscr{F}$, for $1 \le i < j \le n$, infimum $\{f(v_i),\, f(v_j)\} = \varnothing,$ but supremum does not exist. Hence, $(\mathscr{F}, \subseteq)$ is not a lattice.

Finally, we prove that $\mathscr{F}$ is embedded in W_n.

Define $\Phi : \mathscr{F} \to W^\star$ by $\Phi(f(v_i)) = \begin{cases} a_i, & \text{if } i \le i \le h, \\ b_{i-h}, & \text{if } h+1 \le i \le n, \end{cases}$

where $f(v_i) \in \mathscr{F}$ for $1 \le i \le n,$ and $W^\star$ is a subposet of W_n on n elements $a_1, \ldots, a_h, b_1, b_2, \ldots, b_{n-h}$ with the same partial order of W_n. Since, for $2 \le l \le h,$ $f(v_l) \subseteq f(v_{l-1})$ if and only if $a_l \preceq a_{l-1}$ in $W^\star$, also, for $h + 1 \le l \le n,$ $f(v_{l-1}) \subseteq f(v_l)$ in $\mathscr{F}$ if and only if $a_h \preceq b_{l-h}$ in $W^\star$, and for $h + 2 \le l \le n,$ $f(v_{l-1}) \subseteq f(v_l)$ in $\mathscr{F}$ if and only if $b_{l-(h+1)} \preceq b_{l-h}$ in $W^\star$. Furthermore, by definition of Φ, for $1 \le i \le h - 1$ and $1 \le j \le n - h,$ $f(v_i) \,||\, f(v_j)$ if and only if $a_i \,||\, b_j$. Hence, $\mathscr{F} \cong W^\star$. Hence, $\mathscr{F}$ is embedded in W_n. $\square$

Proposition 9. *Let* $\mathscr{F}$ *be the set of vertex labeling of a* 1-*uniform dcsl path* $P_n (n > 2)$, *which is embedded in* $W^\star$, *where* $W^\star$ *is a subposet of* W_n. *Then,* $width(\mathscr{F}) = 2$ *if and only if* $height(\mathscr{F}) = \left\lfloor \frac{|V(P_n)|}{2} \right\rfloor + 1$, *and* $height(\mathscr{F})$ *is minimum, for all embeddings* $\mathscr{F}$ *in* W_n.

Proof. Let $V(P_n) \doteq \{v_1, v_2, ..., v_n\}$.

Let $X = \{1, 2, ., h, ., n-1\}$, where $h = \left\lfloor \frac{V(P_n)}{2} \right\rfloor + 1$, and let f be a 1-uniform dcsl of P_n $(n > 2)$ which is given in Proposition 8, such that $\mathscr{F} = \{f(v)/v \in V(P_n)\}$ is embedded in $W^\star$.

Suppose width$(\mathscr{F}) = 2$, then, by Dilworth's Theorem 2, $\mathscr{F}$ can be partitioned in to 2 chains, but not fewer, say L_1, and L_2. Without loss of generality, choose L_1 is of length h, and L_2 is of length $n-h$, so that height$(\mathscr{F}) = h$.

Claim. height$(\mathscr{F})$ is minimum, for the embedding $\mathscr{F}$ in $W^\star$.

Choose two chains L^* and L^{**} in $W^\star$, such that there is exactly one element in common between L^* and L^{**}. Suppose L^* is of maximum length, say h, so that L^{**} is of length $n-h+1$. If suppose, height$(\mathscr{F})$ is not minimum, that is height$(\mathscr{F})$ is less than h, let it be $h-1$, which implies L^* is of length $h-1$, and the other is of length $n-h+2$, which is greater than $h-1$, a contradiction to height$(\mathscr{F})$. Hence, height$(\mathscr{F})$ in $W^\star$, is minimum.

Conversely, suppose height$(\mathscr{F}) = \left\lfloor \frac{V(P_n)}{2} \right\rfloor + 1 (= h)$, then, by Mirsky's Theorem 3, $\mathscr{F}$ can be partitioned in to h antichains, but not fewer, let it be $\hat{W}_1, \hat{W}_2, ...,$ and $\hat{W}_h$, and also $|\hat{W}_i| \leq 2$, for $1 \leq i \leq h$. Hence, width$(\mathscr{F}) =$ 2. $\square$

Remark 5. In the above Proposition 9, the poset $\mathscr{F}$ is embedded in $W^{\star}$, so that the height of poset $\mathscr{F}$ is minimum. However, it is not true, when the poset $\mathscr{F}$ is embedded in W_n, for, consider

$$\mathscr{F} = \{f(v_1),\ f(v_2),\ f(v_3),\ \ldots,\ f(v_n)\}$$

$$= \{\{1,\ 2\},\ \{1\},\ \varnothing,\ \{3\},\ \{3,\ 4\},\ \ldots,\ \{3,\ 4,\ 5,\ \ldots,\ n-1\}\},$$

is the vertex labeling of 1-uniform dcsl path P_n $(n > 2)$, which form a poset of $\text{width}(\mathscr{F}) = 2$, and it is embedded in W_n, but the $\text{height}(\mathscr{F}) = n - 2 > \left\lfloor \frac{|V(P_n)|}{2} \right\rfloor + 1$, which means, it is not minimum.

Proposition 10. *Let $\mathscr{F}$ be the set of vertex labeling of a* 1*-uniform dcsl path P_n $(n > 2)$, which is embedded in W_n, then $\dim(\mathscr{F}) = 2$.*

Proof. Let f be a 1-uniform dcsl of P_n $(n > 2)$ which is given in Proposition 8, such that $\mathscr{F} = \{f(v)/v \in V(P_n)\}$ is embedded in $W^{\star}$. Since $W^{\star}$ is embedded in W_n, and since by Proposition 7, $\dim(W_n) = 2$, hence $\dim(W^{\star}) = 2$. Since, $\mathscr{F}$ is embedded in $W^{\star}$, hence $\dim(\mathscr{F}) = 2$. $\square$

Remark 6. We observed that, when the width (other than 1) of the poset $\mathscr{F}$ of vertex labeling of a 1-uniform dcsl path P_n $(n > 2)$, is increasing from 2 to $\left\lceil \frac{|V(P_n)|}{2} \right\rceil$, then the corresponding minimum height is decreasing from $\left\lfloor \frac{|V(P_n)|}{2} \right\rfloor + 1$ to 2. We calculated the minimum height of the poset $\mathscr{F}$ of set of vertex labeling of a 1-uniform dcsl path P_7 of width 2, 3, and 4, and they are 4, 3, and 2, respectively, and $\dim(\mathscr{F}) = 2$ (see Figure 4). However, the calculation of the minimum height of $\mathscr{F}$, when $\mathscr{F}$ has an arbitrary width other than 1, is under further investigation.

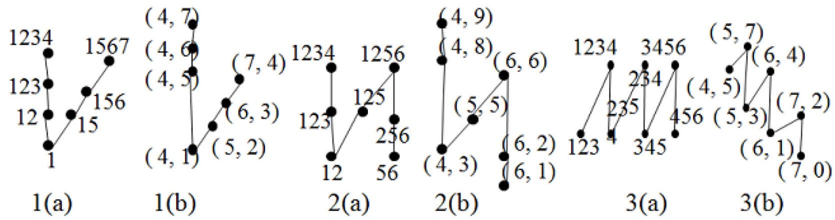

Figure 4. The poset $\mathscr{F}$ of vertex labeling of a 1-uniform dcsl path P_7 of width 2, 3 and 4 are 1(a), 2(a) and 3(a), respectively, and its embedding in R^2 are 1(b), 2(b) and 3(b), respectively.

Theorem 7. *If there exists any vertex labeling $\mathscr{F}$ of a 1-uniform dcsl path P_n $(n > 2)$, which form a poset. Then, $dim(\mathscr{F}) \le 3$.*

Proof. Since the Hasse diagram of a poset $\mathscr{F}$ of vertex labeling of a 1-uniform dcsl path $P_n(n > 2)$ is a tree, hence, by Theorem 4, $\dim(\mathscr{F}) \le 3$.

□

Theorem 8. *Let $\mathscr{F}$ be a set of vertex labeling of 1-uniform dcsl path P_n $(n > 2)$ which does not form a lattice with respect to set inclusion '$\subseteq$'. Then $dim(\mathscr{F}) \le \delta_d(P_n)$.*

Proof. Let f be a 1-uniform dcsl of P_n $(n > 2)$, such that

$$\mathscr{F} = \{f(v)/v \in V(P_n)\}$$

does not form a lattice with respect to set inclusion '$\subseteq$'. We prove this Theorem in two cases.

Case 1. When $3 \le n \le 4$, if we prove that $\mathscr{F}$ is embedded in H_n or W_n, then $\dim(\mathscr{F}) \le \delta_d(P_n)$.

When $n = 3$, the poset $\mathscr{F}$ has $\text{height}(\mathscr{F}) = 2$ and $\text{width}(\mathscr{F}) = 2$, and since by Proposition 8, $\mathscr{F}$ is embedded in W_n, also by Proposition 7, $\dim(W_n) = 2$, hence $\dim(\mathscr{F}) = 2$. Since, by Lemma 1, $\delta_d(P_n) = n - 1$, thus $\dim(\mathscr{F}) = \delta_d(P_n)$.

When $n = 4$, the poset $\mathscr{F}$ has either $\text{height}(\mathscr{F}) = 2$ and $\text{width}(\mathscr{F}) = 2$ or $\text{height}(\mathscr{F}) = 3$ and $\text{width}(\mathscr{F}) = 2$.

Suppose, $\mathscr{F}$ has $\text{height}(\mathscr{F}) = 2$ and $\text{width}(\mathscr{F}) = 2$, then by Proposition 4, $\mathscr{F}$ is embedded in H_n, also by Proposition 3, $\dim(H_n) = 2$, hence $\dim(\mathscr{F}) = 2$. Since, by Lemma 1, $\delta_d(P_n) = n - 1$, thus $\dim(\mathscr{F}) < \delta_d(P_n)$.

Now, suppose $\mathscr{F}$ has $\text{height}(\mathscr{F}) = 3$ and $\text{width}(\mathscr{F}) = 2$, then by Proposition 8, $\mathscr{F}$ is embedded in W_n, also by Proposition 7, $\dim(W_n) = 2$, hence $\dim(\mathscr{F}) = 2$. Since, by Lemma 1, $\delta_d(P_n) = n - 1$, thus $\dim(\mathscr{F}) < \delta_d(P_n)$.

Hence, when $3 \leq n \leq 4$, $\dim(\mathscr{F}) \leq \delta_d(P_n)$.

Case 2. When $n > 4$, if we prove that $\dim(\mathscr{F}) \leq 3$, then $\dim(\mathscr{F}) \leq \delta_d(P_n)$.

When $n > 4$, by Theorem 7, $\dim(\mathscr{F}) \leq 3$, also by Lemma 1, $\delta_d(P_n) = n - 1$, hence $\dim(\mathscr{F}) \leq \delta_d(P_n)$.

The following theorem is obtained as its analogous result of Theorem 6, and Theorem 8.

Theorem 9. *Let* $\mathscr{F}$ *be a set of vertex labeling of* 1-*uniform dcsl path* P_n $(n > 2)$ *whether or not form a lattice with respect to set inclusion* '$\subseteq$'. *Then*, $\dim(\mathscr{F}) \leq \delta_d(P_n)$.

Now, it is of interest to find the dimension of vertex labeling of k-uniform dcsl path P_n $(n > 2)$. Since all paths are 1-uniform dcsl graphs, and by Theorem 5, paths are k-uniform dcsl graphs. So that all the structural properties of 1-uniform dcsl paths holds good for k-uniform dcsl paths, and k-uniform dcsl index of path P_n $(n > 2)$ is k times that of 1-uniform dcsl index.

Lemma 2. $\delta_k(P_n) = k(n-1),$ *for* $n > 2.$

Proof. By Proposition 1, for a k-uniform dcsl graph G, $\delta_k(G) \geq k$ diam(G). Hence, $\delta_k(P_n) \geq k$ diam$(P_n) = k(n-1)$, i.e., $\delta_k(P_n) \geq k(n-1)$.

We claim that there exists k-uniform dcsl path P_n $(n > 2)$ with underlying set X of cardinality $k(n-1)$. Let $X = \{1, 2, ..., k(n-1)\}$. Consider the dcsl labeling $f : V(P_n) \to 2^X$ defined by $f(v_1) = \varnothing$, and $f(v_i) = \{1, 2, ..., k(i-1)\}$, for $2 \leq i \leq n$. Thus, for

$$2 \leq i \leq n, \, | \, f(v_1) \Delta f(v_i) \, | = k(i-1) = kd(v_1, v_i),$$

and $| \, f(v_i) \Delta f(v_j) \, | = (j-i)k = kd(v_i, v_j)$, for $2 \leq i < j \leq n$. Hence, there exists a k-uniform dcsl path P_n $(n > 2)$ with $| \, X \, | = k(n-1)$. Therefore $\delta_k(P_n) = k(n-1)$. □

By Theorem 5, note that every 1-uniform dcsl of P_n $(n > 2)$, also accept a k-uniform dcsl, also, every vertex labeling of a k-uniform dcsl path P_n $(n > 2)$, need not form a poset. However, there always exists a k-uniform dcsl of P_n $(n > 2)$, which form a connected poset. Hence, the Hasse diagram (poset) which embeds the vertex labeling of the 1-uniform dcsl path, could also embeds the vertex labeling of the k-uniform dcsl.

The following theorem is a consequence of Theorem 5, Lemma 2, and Theorem 9.

Theorem 10. *Let* $\mathscr{F}$ *be a set of vertex labeling of the* k*-uniform dcsl path* P_n $(n > 2)$ *whether or not form a lattice with respect to set inclusion* '$\subseteq$'.

Then $dim(\mathscr{F}) \leq \delta_k(P_n)$.

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