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CONFIDENCE INTERVALS FOR THE RATIO OF TWO INDEPENDENT COEFFICIENTS OF VARIATION OF NORMAL DISTRIBUTION

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Abstract

This paper proposes two new confidence intervals for the ratio of two independent coefficients of variation of normal distribution based on the concept of the general confidence interval (GCI) and the method of variance estimates recovery (MOVER). A Monte Carlo simulation study was conducted to compare the performance of these confidence intervals with Verrill and Johnson confidence interval and bootstrap confidence intervals in terms of coverage probabilities, upper and lower probabilities and expected lengths.

1. Introduction

The ratio of two independent coefficients of variation (CVs) is a particular problem in the estimation of two independent normally distributed

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random variables. This problem has previously appeared in the literature of Verrill and Johnson [1]. Their simulation results indicated that the performance of Verrill and Johnson confidence interval for the ratio of coefficients of variation by using the normal approximation worked quite well for small samples and the coverage probability of Verrill and Johnson confidence interval is not at least the nominal confidence level. However, a demonstration of the distribution of the ratio of CVs has not previously appeared. This difficulty has been overcome to some extent by an indirect approach. In a recent paper, McKay [2] proposed an approximation of the distribution of a statistic derived from a sample CV based on the Chi-squared distribution and used the results to make the approximate distribution t for cases in which the mean is more than three standard deviations distant from the origin about which the deviations are taken. Moreover, several authors have studied coefficients of variation as in the test of the equality of r coefficients of variation; see, e.g., Bennett [3], Miller [4] and references cited therein.

In this paper, we focus on constructing the confidence interval for the ratio of CVs of two independent normal distributions. One of the new confidence intervals we construct is based on the generalized confidence interval of Weerahandi [5]. Weerahandi [5] argued that a generalization of the definition of interval estimation provides satisfactory solutions in a variety of problems. More recently, Tian [6] used Weerahandi's approach to obtain a generalized confidence interval for a coefficient of variation associated with r populations. By simulation, Tian [6] concluded that the proposed generalized confidence interval provides satisfactory coverage probabilities and good balance between upper and lower error rates. Tian [6] also suggested that the proposed generalized confidence interval can be obtained through a simple simulation procedure. In addition, the confidence interval obtained by using Weerahandi's approach has received considerable attention in the literature; we refer to Tsui and Weerahandi [7], Buntao and Niwitpong [8], Lee and Lin [9] and Lin et al. [10] and the references cited therein. In the recent paper of Roy and Mathew [11], the authors recommended that the confidence interval based on Weerahandi's approach is exact: it provides the intended coverage probability and a short expected length. The second new confidence interval we construct is based on MOVER described by Donner and Zou [12]. Donner and Zou [12] suggested MOVER as the simple approach which uses variance estimates recovered from confidence limits computed for the mean and standard deviation separately. Tang et al. [13] applied this method to construct the confidence intervals for the difference between proportions based on paired data and simulation results. The simulation results showed that the MOVER-type CIs based on the continuity corrected Phi coefficient and the Tango score CI perform satisfactorily in small sample designs and spare data structures. Fagerland and Newcombe [14] suggested that the MOVER-R intervals for ratios of proportions derived from Wilson intervals perform well and can be used more easily than other confidence intervals which are based on a computational approach.

Furthermore, the purpose of this paper is to investigate the coverage probabilities, upper and lower probabilities and expected lengths of the two proposed confidence intervals and compare them to Verrill and Johnson's confidence interval and the bootstrap confidence intervals. The remainder of this paper is organized as follows. Section 2 provides confidence intervals for the ratio of CVs obtained by some existing methods when data are normally distributed. Then the new confidence intervals for the ratio of CVs are proposed in Section 3. Section 4 presents the results of simulation studies and report the performance based on the coverage probability, upper and lower probabilities and expected length for each confidence interval. The proposed confidence intervals are applied to real data in Section 5. Finally, Section 6 gives conclusions from simulation studies and applied real data.

2. Confidence Intervals for the Ratio of Two Independent Coefficients of Variation of Normal Distribution

In this section, we review two existing methods for constructing confidence intervals for the ratio of CVs when data are normally distributed. The population CV (denoted as θ) is defined as a ratio of the population standard deviation (σ) to the population mean $(\mu, \mu \neq 0)$, i.e.,

$$\frac{\sigma}{\mu}.$$

Therefore, the ratio of CVs is given by

$$\eta = \frac{\theta_1}{\theta_2},$$

where $\theta_1 = \sigma_1/\mu_1$ and $\theta_2 = \sigma_2/\mu_2$.

Let $X_i \sim N(\mu_1, \sigma_1^2)$, $i = 1, 2, ..., n$, $Y_j \sim N(\mu_2, \sigma_2^2)$, $j = 1, 2, ..., m$ and $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ be, respectively, the population means and population variances of X and Y. The sample means $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$, $\overline{Y} = m^{-1}\sum_{j=1}^{m} Y_i$ and sample variances $S_1^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$, $S_2^2 = (m-1)^{-1}\sum_{j=1}^{m}(Y_j - \overline{Y})^2$ are consistent estimates of the population means and population variances of X and Y, respectively. Therefore, the typical sample estimate of η is given by

$$\hat{\eta} = \frac{\hat{\theta}_1}{\hat{\theta}_2},$$

where $\hat{\theta}_1 = s_1/\bar{x}$ and $\hat{\theta}_2 = s_2/\bar{y}$.

To construct a confidence interval for the ratio of CVs, we consider two existing methods namely Verrill and Johnson's method and bootstrap confidence interval. The $100(1-\alpha)\%$ confidence interval for η from each considered method is obtained by the following.

2.1. Verrill and Johnson's confidence interval

Verrill and Johnson [1] proposed confidence interval for the ratio of CVs by using the normal approximation, its confidence interval for η is given by

$$\hat{\eta} \pm z_{1-\alpha/2}\sqrt{\hat{d}_{44}/o}, \tag{1}$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$th percentile of the standard normal distribution, $o \equiv n + m$, $\hat{d}_{44}$ is the 4th diagonal element of inverse Fisher's information matrix $(I(\hat{\theta})^{-1})$, and $I(\hat{\theta})$ is the $I(\theta)$ with $\sigma_x, \sigma_y, \tau_2, \xi$ replaced by s_1, s_2, $\hat{\tau}_2 = s_2/\bar{y}$, $\hat{\xi} = (s_1/\bar{x})/\hat{\tau}_2$.

2.2. Bootstrap confidence intervals

The idea of the bootstrap method was presented by Efron [15] and his colleagues; see, e.g., Efron and Tibshirani [16, 17], DiCiccio and Efron [18]. Efron [15] described the bootstrap method that takes random re-sampling $X_1^*, X_2^*, ..., X_n^*$ (called *bootstrap sample*) drawn with replacement from the original sample $X_1, X_2, ..., X_n$. From these bootstrap samples, the corresponding bootstrap $\hat{\eta}$ denoted by $\hat{\eta}^*$. Suppose B bootstrap samples are available, then we can compute $\hat{\eta}_1^*, \hat{\eta}_2^*, ..., \hat{\eta}_B^*$ which will be the empirical distribution of $\hat{\eta}$. Consistent with Efron and Tibshirani [17] and Chou et al. [19], in this study, 1,000 bootstrap samples are used to compute bootstrap confidence intervals. In this paper, we construct three bootstrap confidence intervals as follows.

2.2.1. The standard bootstrap confidence interval

From the 1,000 bootstrap estimates, the sample mean can be calculated as

$$\overline{\eta}^* = \frac{1}{1{,}000}\sum_{i=1}^{1{,}000}\hat{\eta}_i^* \tag{2}$$

and the sample standard deviation is given by

$$S_\eta^* = \left[\frac{1}{999}\sum_{i=1}^{1{,}000}(\hat{\eta}_i^* - \overline{\eta}^*)^2\right]. \tag{3}$$

The $100(1-\alpha)\%$ standard bootstrap confidence interval for η is

$$\overline{\eta}^* \pm Z_{1-\alpha/2} S_{\eta}^*. \tag{4}$$

2.2.2. The percentile bootstrap confidence interval

Rank array of $\hat{\eta}^*$ from the smallest to the largest and denoting these bootstrap values as $\hat{\eta}^*(1), \hat{\eta}^*(2), ..., \hat{\eta}^*(1000)$. The 95% bootstrap confidence interval for η is

$$[\hat{\eta}^*(25), \hat{\eta}^*(975)]. \tag{5}$$

2.2.3. The biased-corrected and accelerated (BCa) confidence interval

The bias-corrected and accelerated confidence interval was developed by Efron and Tibshirani [16] in 1998. Efron and Tibshirani [16] adjusted for both bias and the rate of change (called the *acceleration*) in the bootstrap distribution. Hence, the construction of the BCa confidence interval is depended on the bias-correction and the accelerations. The bias-correction can be obtained by

$$z_0 = \Phi^{-1}(p_0),$$

where $p_0 = Pr(\hat{\eta}^* \leq \eta)$.

Chou et al. [19] recommended that a simple expression for the acceleration φ should be considered as the jackknife value of a statistic. The acceleration φ can be calculated from the following equation:

$$\varphi = \frac{\sum_{i=1}^{n} (\hat{\eta}_{(.)} - \hat{\eta}_{(i)})^3}{6\left\{\sum_{i=1}^{n} (\hat{\eta}_{(.)} - \overline{\eta}_{(i)})^2\right\}^{3/2}},$$

where $\hat{\eta}_{(i)}$ is the bootstrap estimate of η computed from the original sample ith point deleted, for $i = 1, ..., n$ and

$$\hat{\eta}_{(.)} = \sum_{i=1}^{n} \hat{\eta}_{(i)}/n.$$

Thus, the $100(1-\alpha)\%$ BCa confidence interval for η is

$$[\hat{\eta}^*(1000 \times P_{\text{AL}}),\ \hat{\eta}^*(1000 \times P_{\text{AU}})], \tag{6}$$

where

$$P_{\text{AL}} = \Phi\left(z_0 + \frac{z_0 - Z_{1-\alpha/2}}{1 - \varphi(z_0 - Z_{1-\alpha/2})}\right),$$

where $\Phi(.)$ is the standard normal cumulative distribution function, and

$$P_{\text{AU}} = \Phi\left(z_0 + \frac{z_0 + Z_{1-\alpha/2}}{1 - \varphi(z_0 + Z_{1-\alpha/2})}\right).$$

3. The Proposed Confidence Intervals for the Ratio of Two Independent Coefficients of Variation of Normal Distribution

In this section, we propose two new confidence intervals for the ratio of CVs based on GCI and the MOVER method. The proposed confidence intervals for η are as follows.

3.1. The generalized confidence interval (GCI)

Weerahandi [5] defined a generalized pivot (GP) as a statistic that has a distribution free of unknown parameters and an observed value of a generalized pivotal quantity does not depend on nuisance parameters. In a general setting, a generalized pivot can be defined as follows: Let X be a random quantity having a density function $f(X, \xi)$, where $\xi = (\theta, \tau)$ are unknown parameters; θ is the parameter of interest and τ is a nuisance parameter. Let x be the observed value of X. In the procedure to construct the confidence interval for θ, we start with a generalized pivotal quantity $R(X, x, \xi)$ which is a function of random variable X, its observed value x and the parameter ξ. Also, $R(X, x, \xi)$ is required to satisfy the following conditions:

A1. For fixing x, a probability distribution of $R(X, x, \xi)$ is free of unknown parameters.

A2. The observed pivot, which is defined as $R(x, x, \xi)$, does not depend on nuisance parameters.

Following the idea presented by Weerahandi [5], we construct the confidence interval for $\eta = \theta_1/\theta_2$ when we set $\theta_1 = \sigma_1/\mu_1$, $\theta_2 = \sigma_2/\mu_2$, It is straight forward to see that

$$\eta = (\sigma_1/\mu_1)/(\sigma_2/\mu_2). \tag{7}$$

From equation (7), it is easy to see that

$$\eta = \frac{\sqrt{\frac{(n-1)S_1^2}{U_1}}}{\overline{X} - T_1 S_1/\sqrt{n}} \times \frac{\overline{Y} - T_2 S_2/\sqrt{m}}{\sqrt{\frac{(m-1)S_2^2}{U_2}}}, \tag{8}$$

where $T_1 \sim t_{(n-1)}$, $T_2 \sim t_{(m-1)}$, $U_1 = \frac{(n-1)s_1^2}{\sigma_1^2} \sim \chi_{n-1}^2$ and $U_2 = \frac{(m-1)s_2^2}{\sigma_2^2} \sim \chi_{m-1}^2$.

Using equation (8), we can define the generalized pivotal quantity as

$$R(X, Y; x, y, \xi) = \frac{\sqrt{\frac{(n-1)s_1^2}{U_1}}}{\bar{x} - T_1 s_1/\sqrt{n}} \times \frac{\bar{y} - T_2 s_2/\sqrt{m}}{\sqrt{\frac{(m-1)s_2^2}{U_2}}}, \tag{9}$$

where $s_1, s_2, \bar{x}, \bar{y}$ are the observed values of $S_1, S_2, \overline{X}, \overline{Y}$, respectively.

It is easy to see that $R(X, Y; x, y, \xi)$ satisfies for A1-A2. Therefore, the $100(1-\alpha)\%$ generalized confidence interval for η is

$$[R_{\alpha/2}, R_{1-\alpha/2}], \tag{10}$$

where $R_{1-\alpha/2}$ is the $(1-\alpha/2)$ th percentile of $R(X, Y; x, y, \xi)$.

3.2. The method of variance estimates recovery (MOVER)

Donner and Zou [12] proposed the confidence interval for a ratio of parameters, θ_1/θ_2, by using the method of variance estimates recovery in 2010. They constructed the confidence interval for θ_1/θ_2 by using the confidence limits (l_i, u_i), $i = 1, 2$, where (l_i, u_i) are the $100(1-\alpha)\%$ two-sided confidence intervals for θ_i. Then, the confidence interval for θ_1/θ_2 is given by $[\eta_L, \eta_U]$, where

$$\eta_L = \frac{\hat{\theta}_1\hat{\theta}_2 - \sqrt{(\hat{\theta}_1\hat{\theta}_2)^2 - l_1u_2(2\hat{\theta}_1 - l_1)(2\hat{\theta}_2 - u_2)}}{u_2(2\hat{\theta}_2 - u_2)} \tag{11}$$

and

$$\eta_U = \frac{\hat{\theta}_1\hat{\theta}_2 + \sqrt{(\hat{\theta}_1\hat{\theta}_2)^2 - u_1l_2(2\hat{\theta}_1 - u_1)(2\hat{\theta}_2 - l_2)}}{l_2(2\hat{\theta}_2 - l_2)}. \tag{12}$$

Following Donner and Zou [12], the confidence intervals for $\theta_1 = \sigma_1/\mu_1$ and $\theta_2 = \sigma_2/\mu_2$ are given by

$$(l_1^*, u_1^*) = \left[\frac{S_1}{c_1}(\bar{X} - \sqrt{\max(0, \bar{X}^2 + a_1c_1(a_1 - 2))}), \right.$$
$$\left. \frac{S_1}{c_1}(\bar{X} + \sqrt{\max(0, \bar{X}^2 + b_1c_1(b_1 - 2))})\right] \tag{13}$$

and

$$(l_2^*, u_2^*) = \left[\frac{S_2}{c_2}(\bar{Y} - \sqrt{\max(0, \bar{Y}^2 + a_2c_2(a_2 - 2))}), \right.$$
$$\left. \frac{S_2}{c_2}(\bar{Y} + \sqrt{\max(0, \bar{Y}^2 + b_2c_2(b_2 - 2))})\right], \tag{14}$$

where $a_1 = \sqrt{\dfrac{n-1}{\chi^2_{1-\alpha/2,\, n-1}}}$, $b_1 = \sqrt{\dfrac{n-1}{\chi^2_{\alpha/2,\, n-1}}}$, $c_1 = \overline{X}^2 - z^2_{1-\alpha/2}\dfrac{S_1^2}{n}$, $a_2 = \sqrt{\dfrac{m-1}{\chi^2_{1-\alpha/2,\, m-1}}}$, $b_2 = \sqrt{\dfrac{m-1}{\chi^2_{\alpha/2,\, m-1}}}$ and $c_2 = \overline{Y}^2 - z^2_{1-\alpha/2}\dfrac{S_2^2}{m}$.

We set $\hat{\theta}_1 = s_1/\bar{x}$, $\hat{\theta}_2 = s_2/\bar{y}$, $l_1 = l_1^*$, $l_2 = l_2^*$, $u_1 = u_1^*$ and $u_2 = u_2^*$ replacing into equations (11) and (12). Therefore, the confidence interval for η is given by

$$\left[\frac{\dfrac{s_1}{\bar{x}}\dfrac{s_2}{\bar{y}} - \sqrt{\left(\dfrac{s_1}{\bar{x}}\dfrac{s_2}{\bar{y}}\right)^2 - l_1^* u_2^*\left(2\dfrac{s_1}{\bar{x}} - l_1^*\right)\left(2\dfrac{s_2}{\bar{y}} - u_2^*\right)}}{u_2^*\left(2\dfrac{s_2}{\bar{y}} - u_2^*\right)}, \frac{\dfrac{s_1}{\bar{x}}\dfrac{s_2}{\bar{y}} + \sqrt{\left(\dfrac{s_1}{\bar{x}}\dfrac{s_2}{\bar{y}}\right)^2 - u_1^* l_2^*\left(2\dfrac{s_1}{\bar{x}} - u_1^*\right)\left(2\dfrac{s_2}{\bar{y}} - l_2^*\right)}}{l_2^*\left(2\dfrac{s_2}{\bar{y}} - l_2^*\right)}\right]. \tag{15}$$

In the next section, we evaluate these six confidence intervals based on their coverage probabilities and expected lengths. Typically, we prefer a confidence interval whose estimated coverage probability is at least the nominal confidence level $(1-\alpha)$, with balanced upper and lower probabilities and narrower width. A Monte Carlo simulation will be used to assess these criteria.

4. Simulation Studies

In this section, we evaluated the performance of GCI as compared to the MOVER method, Verrill and Johnson confidence interval and bootstrap confidence intervals via Monte Carlo simulation. The simulation studies are carried out to evaluate coverage probabilities, upper and lower probabilities and expected lengths of each confidence interval. The number of simulation runs is equal to 10,000 and the design values of the nominal confidence level, upper and lower probabilities are 0.95, 0.025 and 0.025, respectively. For the generalized pivotal approach, in each of 10,000 simulations, 500 pivotal quantities are used. All data are generated from normal distribution with means $\mu_1 = \mu_2 = 1$ and standard deviations $\mu_1\theta_1 = \theta_1$, $\mu_2\theta_2 = \theta_2$, $(\theta_1/\theta_2) = 0.15, 0.50, 0.90$ and sample sizes $n, m = 10, 30, 50$ and 100. The results were performed using a program written in the R version 3.1.0 software. The simulation results are shown in Table 1. As seen in Table 1, the two confidence intervals from GCI and the MOVER method have coverage probabilities are approximately nominal confidence level in all cases. On the other hand, the existing confidence intervals are such that they provide coverage probabilities much different from the nominal confidence level especially when the sample sizes are small. For the results of the upper and lower probabilities, we also note that the MOVER method has slightly unbalanced tail probabilities as compared to GCI for small sample sizes while Verrill and Johnson's confidence interval and bootstrap confidence intervals have unbalanced tail probabilities in all cases. In general, the expected length of the confidence interval based on GCI is slightly narrower than that of confidence interval based on the MOVER method. In addition, the expected lengths of these confidence intervals increase upon an increase in the high values of the ratio of coefficients of variation.

Table 1. Simulation results of 95% confidence intervals for the ratio of two independent coefficients of variation in normal distribution

(n, m)	(θ_1/θ_2)	Method	CP*	Upper+	Lower++	Length$
(10, 10)	0.15	Verrill and Johnson	0.9151	0.0816	0.0033	0.1983
		Standard bootstrap	0.9158	0.0766	0.0076	0.2504
		Percentile bootstrap	0.9082	0.0458	0.0460	0.2387
		BCa bootstrap	0.9090	0.0429	0.0481	0.2421
		GCI	0.9484	0.0246	0.0270	0.2423
		MOVER	0.9548	0.0221	0.0231	0.2486

	0.50	Verrill and Johnson	0.9149	0.0815	0.0036	0.6627
		Standard bootstrap	0.9180	0.0754	0.0066	0.8514
		Percentile bootstrap	0.9101	0.0449	0.0450	0.8039
		BCa bootstrap	0.9126	0.0430	0.0444	0.8155
		GCI	0.9455	0.0283	0.0262	0.8093
		MOVER	0.9534	0.0242	0.0224	0.8307
	0.90	Verrill and Johnson	0.9156	0.0807	0.0037	1.2048
		Standard bootstrap	0.9218	0.0711	0.0071	1.5248
		Percentile bootstrap	0.9110	0.0444	0.0446	1.4583
		BCa bootstrap	0.9135	0.0421	0.0444	1.4748
		GCI	0.9478	0.0251	0.0271	1.4717
		MOVER	0.9542	0.0216	0.0242	1.5103
(30, 30)	0.15	Verrill and Johnson	0.9376	0.0523	0.0101	0.1087
		Standard bootstrap	0.9254	0.0622	0.0124	0.1109
		Percentile bootstrap	0.9276	0.0381	0.0343	0.1111
		BCa bootstrap	0.9275	0.0379	0.0346	0.1117
		GCI	0.9491	0.0258	0.0251	0.1159
		MOVER	0.9512	0.0249	0.0239	0.1166
	0.50	Verrill and Johnson	0.9378	0.0533	0.0089	0.3646
		Standard bootstrap	0.9281	0.0619	0.0100	0.3717
		Percentile bootstrap	0.9296	0.0347	0.0357	0.3725
		BCa bootstrap	0.9292	0.0335	0.0373	0.3744
		GCI	0.9506	0.0247	0.0247	0.3878
		MOVER	0.9544	0.0229	0.0227	0.3905

	0.90	Verrill and Johnson	0.9398	0.0509	0.0093	0.6634
		Standard bootstrap	0.9264	0.0611	0.0125	0.6768
		Percentile bootstrap	0.9285	0.0337	0.0378	0.6784
		BCa bootstrap	0.9287	0.0337	0.0376	0.6820
		GCI	0.9496	0.0238	0.0266	0.7053
		MOVER	0.9527	0.0224	0.0249	0.7104
(50, 50)	0.15	Verrill and Johnson	0.9403	0.0477	0.0120	0.1397
		Standard bootstrap	0.9313	0.0557	0.0130	0.1406
		Percentile bootstrap	0.9352	0.0351	0.0297	0.1410
		BCa bootstrap	0.9340	0.0349	0.0311	0.1415
		GCI	0.9469	0.0289	0.0242	0.1451
		MOVER	0.9515	0.0271	0.0214	0.1455
	0.50	Verrill and Johnson	0.9432	0.0442	0.0126	0.2803
		Standard bootstrap	0.9350	0.0506	0.0144	0.2817
		Percentile bootstrap	0.9344	0.0334	0.0322	0.2825
		BCa bootstrap	0.9328	0.0341	0.0331	0.2835
		GCI	0.9476	0.0274	0.0250	0.2908
		MOVER	0.9506	0.0254	0.0240	0.2919
	0.90	Verrill and Johnson	0.9403	0.0467	0.0130	0.5096
		Standard bootstrap	0.9304	0.0554	0.0142	0.5125
		Percentile bootstrap	0.9321	0.0342	0.0337	0.5141
		BCa bootstrap	0.9331	0.0339	0.0330	0.5158
		GCI	0.9460	0.0284	0.0256	0.5286
		MOVER	0.9483	0.0262	0.0255	0.5306

(100, 100)	0.15	Verrill and Johnson	0.9491	0.0361	0.0148	0.0589
		Standard bootstrap	0.9427	0.0426	0.0147	0.0589
		Percentile bootstrap	0.9454	0.0268	0.0278	0.0591
		BCa bootstrap	0.9443	0.0272	0.0285	0.0592
		GCI	0.9512	0.0239	0.0249	0.0601
		MOVER	0.9529	0.0233	0.0238	0.0601
	0.50	Verrill and Johnson	0.9467	0.0385	0.0148	0.1968
		Standard bootstrap	0.9396	0.0450	0.0154	0.1970
		Percentile bootstrap	0.9411	0.0306	0.0283	0.1977
		BCa bootstrap	0.9378	0.0312	0.0310	0.1980
		GCI	0.9477	0.0278	0.0245	0.2005
		MOVER	0.9510	0.0256	0.0234	0.2008
	0.90	Verrill and Johnson	0.9487	0.0346	0.0167	0.3588
		Standard bootstrap	0.9457	0.0394	0.0149	0.3590
		Percentile bootstrap	0.9436	0.0268	0.0296	0.3604
		BCa bootstrap	0.9432	0.0270	0.0298	0.3609
		GCI	0.9506	0.0234	0.0260	0.3656
		MOVER	0.9519	0.0228	0.0253	0.3660

*Coverage probability, $^{+}$Upper probability, $^{++}$Lower probability, $^{\$}$Expected length

5. Example

In this example, we use two data sets to exemplify our method for a ratio of two independent coefficients of variation. One of the data sets for this example, we select data which consists of 25 frogs from the length from top to tail in millimeters of a large group of frogs in Chapter 8: the normal distribution [20]. Another data set is presented by Kevin E. Bonine (see Johnson and Wichern [21], p. 17) in which the sample estimates are based on

25 lizards. The two data sets show the length from top to tail in millimeters of the frogs and the snout-vent length in millimeters of the lizards. The length from top to tail of frog data yields $\bar{x} = 86.92$, $s_1 = 11.41242$, while the snout-vent length of lizard data yields $\bar{y} = 68.4$, $s_2 = 7.985664$. The 95% confidence interval for the ratio of CVs is obtained from the GCI as (0.7212, 1.6917) with the expected length equal to 0.9705. By equations (13) and (14), the two confidence limits for the coefficient of variation of frog and lizard data are estimated as (0.1020, 0.1835) and (0.0908, 0.1630), respectively. Thus, the 95% confidence interval for the ratio of CVs is obtained from the MOVER method as (0.7391, 1.7115) with the expected length equal to 0.9724. Note that the two data sets are tested for normality by the Kolmogorov test.

6. Conclusions

In this paper, the confidence intervals for the ratio of coefficients of variation of normal distribution are studied. The performances of these confidence intervals were assessed in terms of coverage probabilities, upper and lower probabilities and expected lengths through simulation studies. The simulation results indicated that the GCI and the MOVER method are better than Verrill and Johnson’s confidence interval and the bootstrap confidence intervals in terms of the closeness of the coverage probability to the nominal confidence level, good balance between upper and lower probabilities and short expected length, especially in the situation in which the sample sizes are small. Therefore, our recommendation for the ratio of CVs is to use GCI or the MOVER method to construct the confidence interval. However, the GCI and the bootstrap confidence intervals are hard to compute and widely available in software packages. On the other hand, the MOVER method can be calculated quite easily. As a result of our investigation, we recommend that the MOVER method is considered as an alternative to the confidence interval for the ratio of coefficients of variation.

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